Lower Partial Moments for Skew Elliptical Distributions

Gulnaz Shaidolda

School of Sciences and Humanities

Nazarbayev University

A thesis submitted to the Department of Mathematics in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Nur-Sultan 2024

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my advisor, Professor Kerem Uğurlu, for his invaluable guidance, continuous support, and encouragement throughout my PhD journey. His insightful advice, patience, and unwavering belief in my abilities have been instrumental in shaping my research and academic growth. I am truly grateful for the opportunity to work under his mentorship.

I would like to express my deepest gratitude to my husband for his unwavering love, constant support, and the many personal sacrifices he has made throughout my academic journey. His encouragement, patience, and belief in me have been my source of strength during the most challenging times. I am also deeply thankful to my beloved children for their endless love, patience, and for being my greatest source of motivation and joy. This achievement is as much theirs as it is mine.

A special thank you goes to my mother, Altantseseg, for her invaluable support and endless encouragement. Her love, wisdom, and sacrifices have played a significant role in my academic and personal achievements.

Finally, I am deeply grateful to my family, friends, and colleagues for their encouragement, motivation, and support along the way. Their kindness and belief in me have made this journey all the more meaningful.

This dissertation would not have been possible without the love, guidance, and encouragement of those around me. Thank you all from the bottom of my heart.

Supervisory Committee

Lead Supervisor:
Kerem Ugurlu
Assistant Professor, Department of Mathematics
Nazarbayev University
email: kerem.ugurlu@nu.edu.kz

Co-Supervisor:
Eunghyun Lee
Assistant Professor, Department of Mathematics
Nazarbayev University
email: eunghyun.lee@nu.edu.kz

External Supervisor:
Alperen Ozdemir
Research Scientist
KTH Royal Institute of Technology, Department of Mathematics
email: alpereno@kth.se

Abstract

Robust modeling using skewed distributions are essential in risk management, since many real life examples do not accept the hypothesis the randomness can be modeled by symmetric distributions. This work closes the gap of derivation of explicit representations for lower partial moments of arbitrary powers $n \geq 1$ of normal, skew normal and skew-t distributions that are vital in risk analysis. To the best of our experience, there has been no work in lower partial moment representations using skewed family of distributions. Extensive numerical studies are conducted to statistically examine, whether daily stock prices of the prespecified companies from different sectors can be fitted to these families of distributions. It is verified that for short enough time intervals, it can not be rejected that the stock price data is drawn from some or all of these three families. Furthermore, different portfolios are compared by calculating their LPM's, and it is concluded which of the portfolios is less risky than the other. Our findings suggest that this work closes this gap both theoretically in terms of explicit representations of lower partial moments for skewed family of distributions, and practically in terms of calibration of historical data to the derived operators for risk management and robust portfolio formation.

Keywords: Mathematical finance, Risk management, Lower partial moments, Portfolio optimization

Аңдатпа

Мықты модельдеу барысында қисаюы бар үлестірулерді қолдану тәуекелдерді басқаруда маңызды рөл атқарады, себебі көптеген нақты өмірлік мысалдар кездейсоқтықтың симметриялы үлестірулермен сипатталатыны туралы гипотезаны қабылдамайды. Бұл жұмыс тәуекелдерді талдауда маңызды болып табылатын $n \leq 1$ дәрежелі төменгі жартылай моменттердің (LPM) нақты өрнектерін қалыпты, қисық қалыпты (skew-normal) және қисық-t (skew-t) үлестірулері үшін алу мәселесін шешеді.

Біздің білуімізше, осы уақытқа дейін қисық үлестірулер отбасы үшін төменгі жартылай моменттердің нақты өрнектері алынған зерттеулер болмаған. Бұл жұмыста әртүрлі салаларға жататын алдын ала таңдалған компаниялардың күнделікті акция бағаларын аталған үлестірулер отбасына сәйкестігін статистикалық тұрғыдан зерттеу мақсатында кең көлемді сандық зерттеулер жүргізілді. Нәтижесінде, егер уақыт аралығы жеткілікті қысқа болса, акция бағалары осы үш үлестірудің кейбіріне немесе барлығына сәйкес келетінін жоққа шығаруға болмайтыны анықталды.

Сонымен қатар, әртүрлі инвестициялық портфельдер олардың төменгі жартылай моменттерін есептеу арқылы салыстырылып, қай портфельдің тәуекелі төмен екені анықталды. Бұл зерттеу төменгі жартылай моменттердің нақты өрнектерін қисаюы бар үлестірулер үшін алу арқылы теориялық тұрғыдан, сондай-ақ тарихи деректерді тәуекелдерді басқару және тұрақты портфель құру үшін алынған операторларға сәйкестендіру арқылы практикалық тұрғыдан зерттеу саласындағы олқылықтың орнын толтырады.

Contents

List of Figures

Chapter 1

Introduction

Statistical modeling and inference often rely on parametric assumptions about data distributions. Among these, the Gaussian (normal) distribution is the most prominent due to its analytical simplicity, well-established properties such as the Central Limit Theorem, and straightforward multivariate extension, where both marginals and conditionals remain normal. However, the assumption of normality is not always validated in real-world data. Deviations from normality, particularly asymmetry, frequently occur in various applications, necessitating alternative models that extend or modify the normal distribution.The skew normal (SN) distribution, introduced by Azzalini (1985) [3], is one such extension of the normal distribution, allowing the presence of skewness. Building on Azzalini's work, Azzalini and Dalla Valle (1996)[1] further developed the skew-elliptical family by introducing the skew-t distribution, which allows for both skewness and heavy tails. This distribution has become particularly popular in financial modeling due to its ability to capture the asymmetry and kurtosis observed in asset returns. Other members of skew-elliptical families and their variations have been introduced in various works such as Mudholkar and Hutson (2000) [19], Prentice (1975) [23] and Beaver [2] and the references therein.

Gupta and Chen (2004)[15] expanded the univariate skew-normal model into vector skew-normal models. Later, Ning and Gupta (2012)[21] extended the univariate skew-normal distribution further into the matrix variate case, incorporating ideas from the works of Chen and Gupta (2005)[5] and Harrar and Gupta (2008)[6]. Pewsey (2000) [7] examined statistical inference for the Azzalini skew-normal distribution, opting for the method of moments using center parameterization instead of the direct parameterization approach for estimation. Furthermore, Gupta et al.[15] (2004) offered two characterization results based on quadratic statistics for the skew-normal distribution. In risk management, especially within finance, insurance, and economics, decision-makers are often more concerned with the downside potential of an investment or economic variable than with its upside potential. This asymmetric risk aversion is inadequately captured by traditional symmetric risk measures, such as variance and standard deviation, which treat gains and losses symmetrically. Moreover, asset returns and financial variables typically exhibit non-normal characteristics, such as skewness and heavy tails. These limitations motivate the need for risk measures that can account for such asymmetries and focus more on downside risks. One of the most prominent tools for addressing this challenge is lower partial moments (LPMs), which provide a more intuitive and flexible framework for measuring downside risk by focusing exclusively on the lower tail of a distribution.

Lower partial moments, first introduced by Bawa (1975) [4], focus on the expected shortfall below a specified threshold. Bawa's pioneering work demonstrated that LPMs can effectively represent an investor's preference for downside risk, making them more suitable for assessing risk in cases where distributions deviate from normality. The introduction of LPMs offered an alternative to the mean-variance framework (Markowitz, 1952) [18], which treats upside and downside risk symmetrically, assuming that returns are normally distributed. In contrast, LPMs allow for the modeling of downside risk with flexibility, capturing both the magnitude and the probability of adverse outcomes.

Following Bawa's work, Fishburn (1977)[8] introduced a family of risk measures based on lower partial moments that quantified the expected downside risk with varying levels of aversion to risk. Fishburn's approach showed that LPMs can be parameterized to reflect different degrees of risk sensitivity, which made them adaptable to various risk-averse preferences. This adaptability established LPMs as an essential tool in finance, particularly for portfolio optimization, insurance pricing, and the evaluation of financial derivatives. In the decades following these early contributions, LPMs have been widely applied to risk management, where they are used to quantify measures such as Value at Risk (VaR) and Conditional Value at Risk (CVaR), both of which emphasize downside risk.

1.1 Limitations of Traditional Risk Models

Despite the advantages of LPMs, much of the early research assumed normality in asset returns. The normal distribution, while analytically convenient, assumes symmetry, implying that upside and downside risks are treated equally. However, real-world asset returns and economic variables rarely follow this assumption. Instead, they tend to exhibit skewness (asymmetry) and excess kurtosis (heavy tails), features that are not adequately captured by the normal distribution. The limitations of the normality assumption became evident as empirical studies consistently showed that financial returns tend to be negatively skewed, with extreme negative returns occurring more frequently than predicted by the normal model (Mandelbrot, 1963 [17]; Fama, 1965).

This observation highlighted the need for more flexible distributional models that can accommodate the asymmetry and fat tails present in financial data.

Elliptical distributions, a generalization of the normal distribution, have gained popularity as a flexible family of distributions that maintain many of the attractive properties of the normal distribution, such as ease of estimation and the ability to model multivariate data. Members of the Elliptical family, such as the Student-t, Cauchy, and Laplace distributions, exhibit heavy tails, making them suitable for modeling data with large outliers. However, despite their flexibility in modeling heavy-tailed data, Elliptical distributions still assume symmetry, which limits their ability to fully capture the skewness present in financial returns.

1.2 Lower Partial Moments in Skew-Elliptical Distributions

While skew-Elliptical distributions have been widely studied in the context of modeling financial returns, little research has focused on the application of LPMs to this class of distributions. The integration of LPMs with skew-Elliptical distributions presents a promising avenue for improving downside risk measurement in cases where returns exhibit both skewness and heavy tails. Lower partial moments focus exclusively on the lower tail of a distribution, making them ideal for measuring downside risk. By combining LPMs with skew-Elliptical distributions, we can develop risk measures that more accurately reflect the characteristics of financial returns, leading to better risk management decisions.

Nawrocki (1999)[20] was one of the first to explore LPMs in the context of skewed distributions. He applied LPMs to the skewed Student-t distribution, demonstrating that this approach outperformed traditional mean-variance methods in capturing downside risk. His work illustrated the potential of combining LPMs with skewed distributions but left open the question of how these results could be generalized to other members of the skew-Elliptical family. Lisi (2011) further advanced this research by deriving closed-form expressions for LPMs in the skew-normal distribution. Lisi's work provided valuable insights into the properties of LPMs in skew-Elliptical distributions, but much remains to be explored, particularly in extending these results to other members of the skew-Elliptical family, such as the skew-t and skew-Cauchy distributions.

The primary contribution of this thesis is to extend the theory of LPMs to the full family of skew-Elliptical distributions. Specifically, we derive closed-form expressions for LPMs in several skew-Elliptical distributions, including the skew-normal, skew-t, and skew-Cauchy distributions. These closed-form expressions provide practical tools for risk managers and portfolio analysts who need to assess downside risk in the presence of skewness and heavy tails. Additionally, we analyze the properties of these LPMs, comparing them to their counterparts under symmetric distributions, such as the normal and Student-t distributions. Finally, we apply these results to real-world financial data, demonstrating how LPMs under skew-Elliptical distributions can improve risk measurement compared to traditional methods.

1.3 Literature Review

In recent years, the number of studies exploring the connection between uncertainty and financial market dynamics has been increasing. Pastor and Veronesi[22] investigated the relationship between uncertainty of stock returns and volatility, while Guo[14] and Karabulut[16] examined fluctuations in commodity prices. Recent advancements in robust optimization, supply chain modeling, and uncertainty management have demonstrated the importance of flexible, data-driven approaches for decision-making under uncertainty. Several studies have successfully applied advanced optimization techniques to handle complex, real-world uncertainty: Supply chain management plays a critical role in optimizing financial and physical flows, particularly in the dairy industry. A closed-loop supply chain (CLSC) network model has been proposed to simultaneously maximize net cash flow from assets and enhance shareholder payouts [13]. The integration of blockchain technology with robust product portfolio design has improved traceability, transparency, and decision-making in closed-loop supply chain networks, demonstrating how uncertainty-aware models enhance business performance [9]. In Industry 4.0, hybrid machine learning and meta-heuristic algorithms have been successfully implemented to optimize robust project scheduling, enabling firms to manage resource uncertainty efficiently [10]. In healthcare logistics, robust possibilistic programming has been used to optimize organ transplant supply chains, ensuring efficiency despite uncertainties in demand and resource availability [12]. In energy-aware manufacturing, multi-objective meta-heuristic algorithms have been applied to optimize flow-shop scheduling, demonstrating their effectiveness in reducing operational risk [11]. Inspired by these robust optimization methodologies, this study extends their principles to financial risk modeling, developing new theoretical derivations of risk measures under skewed stock price distributions. Unlike traditional symmetric models, our framework captures real-world skewness in financial data, improving risk estimation, portfolio management, and investment decision-making.

1.4 Contributions and Novelty of This Study

This research introduces several key advancements in financial modeling:

1. New Theoretical Derivations of Lower Partial Moments (LPMs) for Skew-Elliptical Distributions

    - We extend the LPM framework to skewed distributions, filling a gap in financial risk modeling.
    - Our derivations provide more realistic downside risk estimation, essential for financial decision-making.

2. Generalization Across Multiple Skewed Distributions Unlike prior research, which primarily focuses on normal and symmetric Student-t distributions, this study expands risk measures to:

    - Skew Student-t distributions
    - Skew normal distributions
    - Scaled skew-t distributions
    - Scaled skew-normal distributions

    This broadens the applicability of LPMs to highly volatile and asymmetric financial markets.

3. Empirical Validation Using Stock Price Data

    - To ensure real-world relevance, we apply the proposed distributions to stock price datasets.
    - We compare the goodness-of-fit of these distributions using the chi-square test, evaluating their performance relative to traditional models such as the normal and Student-t distributions.

4. Bridging Financial Risk Modeling with Optimization and Robust Decision-Making Inspired by recent advances in supply chain optimization, blockchain-enabled networks, and industrial decision-making, our study integrates robust uncertainty modeling into financial applications. We demonstrate that skewed distributions outperform traditional symmetric models, providing a more accurate and robust framework for stock price modeling.

1.5 Thesis Objectives and Contributions

The main objectives of this thesis are as follows:

Derivation of Lower Partial Moments for Skew-Elliptical Distributions: We derive closed-form expressions for lower partial moments in the skew-normal, skew-t, and skew-Cauchy distributions, providing new theoretical tools for the analysis of downside risk in skewed distributions. Comparison of LPMs Across Distributional Families: We compare the properties of LPMs in skew-Elliptical distributions with those in symmetric distributions, such as the normal and Elliptical families, to highlight the advantages of incorporating skewness in downside risk measurement. Applications to Financial Risk Management: We apply the theoretical results to real-world financial data, demonstrating the practical applicability of LPMs in skew-Elliptical distributions for assessing downside risk in portfolios and individual assets. This research makes several key contributions to the literature on risk management and distributional modeling:

It extends the theory of lower partial moments to a broad class of skew-Elliptical distributions, providing new tools for risk measurement in the presence of asymmetry and heavy tails. It bridges the gap between theoretical advancements in skew-Elliptical distributions and their practical application in financial risk management, offering a more comprehensive approach to downside risk assessment. It offers practical insights for risk managers, portfolio analysts, and financial institutions seeking to improve their risk assessment methods by accounting for skewness and fat tails in asset returns. Structure of the Thesis The remainder of this thesis is organized as follows:

Chapter 1 provides a detailed review of the literature on lower partial moments, focusing on their theoretical development and application in risk management. Chapter 2 introduces skew-Elliptical distributions, including their theoretical properties, estimation methods, and relevance in financial modeling. Chapter 3 presents the derivation of lower partial moments for several skew-Elliptical distributions, exploring their properties and comparing them to LPMs in symmetric distributions. In Chapter 4, we conduct an extensive numerical study using historical stock price data. This study involves calibrating the data to the corresponding skew-elliptical family and evaluating how well these distributions capture the characteristics of real-world financial returns. We employ maximum likelihood estimation and goodness-of-fit tests such as the chi-square test to assess the performance of these distributions in modeling stock prices. Our findings demonstrate that skew-elliptical distributions provide a better fit for empirical financial data compared to traditional symmetric models. In Chapter 5, we summarize our theoretical and numerical findings and conclude the thesis. We discuss the implications of our results for financial risk management and highlight potential directions for future research. By integrating lower partial moments with skew-elliptical distributions, this study offers a novel approach to downside risk assessment, bridging theoretical advancements with practical applications in finance and investment.

Chapter 2

Preliminaries

2.1 Preliminaries and Notations

2.1.1 Percentiles and Quantiles

Quantile is a value that divides a probability distribution or dataset into equal-sized intervals. More formally, the p-th quantile of a distribution is the value below which a proportion p of the data falls.

2.1.1.1 Types of quantiles

Quantiles are special cases of percentiles and fractiles:

1. Median ((50%) quantile or Q_2): The middle value that splits the data into two equal halves.

2. Quartiles: Divides data into four equal parts:

   - First quartile $(Q_1, 25^{th} percentile)$: 25% of the data falls below this value.
   - Second quartile $(Q_2, 50^{th} percentile)$: Same as the median.
   - Third quartile $(Q_3, 75^{th} percentile)$: 75% of the data falls below this value.

3. Deciles: Divides data into ten equal parts.

4. Percentiles: Divides data into 100 equal parts.

5. General p-th Quantile Q_p: The value below which $p \times 100\%$ of the data lies.

Definition 2.1.1. For a cumulative distribution function (CDF) F(x), the p-th quantile q_p is defined as

$$q_p = F^{-1}(p) = inf\{x \in \mathbb{R}\}|F(x \geq p)$$

where p is between 0 and 1.

2.1.2 Probability-Probability (P-P) Plot

A Probability-Probability (P-P) plot is a graphical technique used to compare two probability distributions by plotting their cumulative distribution functions (CDFs) against each other. It is often employed to assess how well a theoretical distribution fits empirical data.

Definition 2.1.2. Let $X_1, X_2, \ldots, X_n$ be a dataset with an empirical distribution function $F_n(x)$ and a theoretical cumulative distribution function $F(x)$. The P-P plot is constructed by plotting the points:

$$\left(F(x_{(i)}), F_n(x_{(i)})\right), \quad i = 1, 2, \ldots, n$$

where:

- $x_{(i)}$ are the ordered sample values (order statistics),
- $F_n(x)$ is the empirical CDF, defined as:

  $$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x)$$

  where $I(\cdot)$ is the indicator function,
- $F(x)$ is the theoretical CDF of the assumed distribution.

If the empirical distribution matches the theoretical distribution, the points in the P-P plot should align closely with the 45-degree reference line $y = x$. Significant deviations from this line indicate discrepancies between the two distributions.

Definition 2.1.3. The probability density function (pdf), denoted f, of a continuous random variable X satisfies the following:

1. $f(x) \geq 0$, for all $x \in \mathbb{R}$
2. f is piecewise continuous
3. $\int_{-\infty}^{\infty} f(x) = 1$
4. $P(a \geq X \geq b) = \int_a^b f(x)$

The first three conditions in the definition state the properties necessary for a function to be a valid pdf for a continuous random variable. The fourth condition tells us how to use a pdf to calculate probabilities for continuous random variables, which are given by integrals the continuous analog to sums.

Definition 2.1.4. The cumulative distribution function (cdf) of a random variable X is a function on the real numbers that is denoted as F_X and is given by;

$$F(x) = P(X \leq x), \quad for \quad any \quad x \in \mathbb{R}.$$

In particular,

$$F(x) = \begin{cases} \sum_{k \leq x} p(k), & \text{if } X \text{ discrete} \\ \int_{-\infty}^{x} f(\tau) d\tau, & \text{if } X \text{ is continuous} \end{cases}$$

The CDF $F(x)$ accumulates probability "up to (and including)" the value x.

Property 2.1.5. Let X be a random variable with cdf F. Then F satisfies the following:

1. $0 \leq F(x) \leq 1$
2. F is non-decreasing, i.e., F may be constant, but otherwise it is increasing.
3. $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$
4. $P(X > x) = 1 - F(x)$
5. When X is discrete with integer values,

$$F(X) = \sum_{k \leq x} p(k),$$

$$p(x) = P(X = x) = F(x) - F(x^{-}) = F(x) - F(x - 1)$$

For continuous random variables we can further specify how to calculate the cdf with a formula as follows. Let X have pdf f, then the cdf F is given by

- By definition, the cdf is found by integrating the pdf: $F(x) = \int_{-\infty}^{x} f(t)dt$
- By the Fundamental Theorem of Calculus, the pdf can be found by differentiating the cdf: $f(x) = \frac{d}{dx}[F(x)]$

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt \tag{2.1.1}$$

In other words, the cdf for a continuous random variable is found by integrating the pdf. Note that the Fundamental Theorem of Calculus implies that the pdf of a continuous random variable can be found by differentiating the cdf.

Property 2.1.6. Let X be a continuous random variable with pdf f and cdf F

2.1.3 Expected value

If you have a collection of numbers $a_1, a_2, ..., a_N$, their average is a single number that describes the whole collection. Now, consider a random variable X . We would like to define its average, or as it is called in probability, its expected value or mean. The expected value is defined as the weighted average of the values in the range.

Definition 2.1.7. Let X be a discrete random variable with range $R_X = x_1, x_2, x_3, ...$ (finite or countably infinite). The expected value of X, denoted by E[X] is defined as

$$E[X] = \sum_{x_k \in R_X} x_k P(X = x_k) = \sum_{x_k \in R_X} x_k P_X(x_k) \tag{2.1.2}$$

Then the expected value of a continuous random variable is defined as :

$$E[X] = \int_{-\infty}^{\infty} x f_X dx \tag{2.1.3}$$

Property 2.1.8. Expectation is a linear operation, thus we always have

- $E[aX + b] = aEX + b$ for all $a, b \in \mathbb{R}$
- $E[X_1 + X_2 + ... + X_n] = E[X_1] + E[X_2] + ... + E[X_n]$, for any set of random variables $X_1, X_2, ..., X_n$

Expected Value of a Function of a Continuous Random Variable is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx \tag{2.1.4}$$

2.1.4 Skewness, Kurtosis and Variance

Skewness is a measure of symmetry.For positively skewed data set or distribution, the right tail is longer; the mass of the distribution is concentrated on the left. For negatively skewed data set or distribution, the left tail is longer; the mass of the distribution is concentrated on the right.

Definition 2.1.9. The skewness of X is the third moment of the standard score of X:

$$skew(X) = E\left[\left(\frac{x - \mu}{\sigma}\right)^3\right] \tag{2.1.5}$$

The distribution of X is said to be positively skewed, negatively skewed or unskewed depending on whether skew(X) is positive, negative, or 0.

Property 2.1.10. Suppose that the distribution of X is symmetric about a. Then

1. E[X]=a

2. skew(X)=0

3. skew(X) can be expressed in terms of the first three moments of X.

$$skew(X) = \frac{E[X^3] - 3\mu\sigma^2 - \mu^3]}{\sigma^3}$$

Variance is a measure of dispersion, meaning it is a measure of how far a set of numbers is spread out from their average value.

Definition 2.1.11. The variance of a random variable X , with mean $EX = \mu_X$ is defined as

$$\mathrm{Var}(X) = E\big[(X - \mu_X)^2\big] = E[X^2] - (E[X])^2. \tag{2.1.6}$$

Kurtosis is a statistical measure that defines how heavily the tails of a distribution differ from the tails of a normal distribution. In other words, kurtosis identifies whether the tails of a given distribution contain extreme values.

Definition 2.1.12. The kurtosis of X is the fourth moment of the standard score:

$$Kurt(X) = E\left[\left(\frac{x-\mu}{\sigma}\right)^4\right]$$

$kurt(X)$ can be expressed in terms of the first four moments of X.

$$kurt(X) = \frac{E[X^4] - 4\mu E[X^3] + 6\mu^2\sigma^2 + 3\mu^4]}{\sigma^4}$$

2.1.5 Lower partial moment

Definition 2.1.13. If τ is a chosen reference level, n is the degree of the moment and X is a random variable with cumulative distribution $F(x)$ the Lower Partial Moments (LPM) are given by

$$LPMn, \tau(F) = E[max((\tau - X), 0)^n] = \int_{-\infty}^{\tau} (\tau - x)^n dF(x) \tag{2.1.7}$$

2.2 Normal Distribution

A normal distribution, also known as a Gaussian distribution, is a continuous probability distribution that is symmetric around its mean. It is defined by two parameters:
Mean (μ): The center of the distribution, indicating the average value.
Standard deviation (σ): A measure of the spread of the distribution. A larger σ results in a wider distribution, while a smaller σ makes it more concentrated around the mean.

2.2.1 Standard Normal Distribution

Definition 2.2.1. A continuous random variable Z is said to be standard normal random variable, shown as $Z \sim \mathcal{N}(0,1)$ if its probability density function is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \tag{2.2.1}$$

for all $z \in \mathcal{R}$

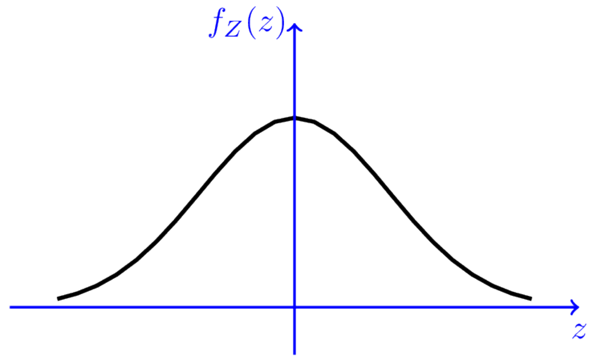

Figure 2.1: PDF of the standard normal random variable

The graph is bell-shaped curve with the distribution is symmetric around the mean. Mean = Median = Mode.

Consider a function $g(u) : \mathcal{R} \to \mathcal{R}$.If $g(u)$ is an odd function then $g(-u) = -g(u)$ and $| \int_0^\infty g(u)du| < \infty$, then $\int_{-\infty}^\infty g(u)du| = 0$
Let

$$g(u) = u^{2k+1} e^{-u^2/2}, \quad k = 0, 1, 2, ...$$

Then $g(u)$ is an odd function and $| \int_0^\infty g(u)du| < \infty$.
Now, let Z be a standard normal random variable. Then, we have

$$E[Z^{2k+1}] = \frac{1}{\sqrt{2\pi}} \int_\infty^{-\infty} u^{2k+1} e^{\frac{-u^2}{2}} du = 0$$

for all $k = 0, 1, 2, ...$ Thus, we have shown that for a standard normal random variable Z, we have

$$E[Z] = E[Z^3] = E[Z^5] = ... = 0$$

In particular, the standard normal distribution has zero mean. This is not surprising as we can see from Figure 2.1 that the PDF is symmetric around the origin, so we expect that $EZ = 0$. Next, let's find EZ^2.

$$E[Z^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty u^2 e^{\frac{-u^2}{2}} du = \frac{1}{\sqrt{2\pi}} (-ue^{\frac{-u^2}{2}}) \Big|_{u=-\infty}^{u=\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{\frac{-u^2}{2}} du = 1$$

Thus, we conclude that for a standard normal random variable Z, we have

$$Var(Z) = 1.$$

Thus ,if $Z \sim \mathcal{N}(0,1)$, then $E[Z] = 0$ and $Var(Z) = 1$

2.2.2 CDf of Standard Normal Distribution

The CDF of the standard normal distribution is denoted by the Φ function:

$$\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-u^2}{2}} du \tag{2.2.2}$$

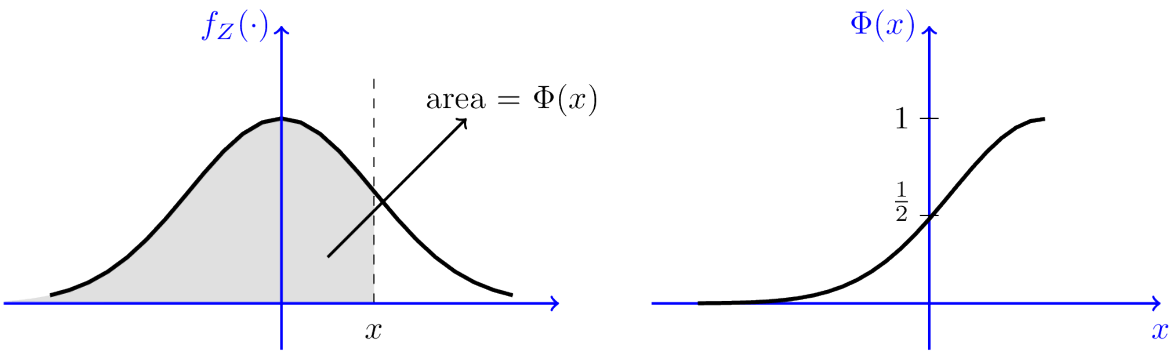

Figure 2.2: CDF of the standard normal

2.2.3 Properties of CDF of Standard Normal Distribution

1. $\lim_{x\to\infty} \Phi(x) = 1, \quad \lim_{x\to-\infty} \Phi(x) = 0$
2. $\Phi(0) = \frac{1}{2}$
3. $\Phi(-x) = 1 - \Phi(x)$ for all $x \in \mathcal{R}$

2.2.4 Normal Random variable

Any normal random variable by shifting and scaling a standard normal random variable. In particular, define

$$X = \sigma Z + \mu \quad where \quad \sigma > 0$$

Then

$$E[X] = \sigma E[Z] + \mu = \mu,$$
$$Var(X) = \sigma^2 Var(Z) = \sigma^2$$

If Z is a standard normal random variable and $X = \sigma Z + \mu$, then X is a normal random variable with mean μ and variance σ^2, i.e,

$$X \sim \mathcal{N}(\mu, \sigma^2).$$

Conversely, if $X \sim \mathcal{N}(\mu, \sigma^2)$, the random variable $Z = \frac{Z-\mu}{\sigma}$ is a standard normal random variable ,in other words, $Z \sim (0, 1)$. To find the CDF of $X \sim \mathcal{N}(\mu, \sigma^2)$, we have

$$\begin{aligned} F(x) = P(X \leq x) = P(\sigma Z + \mu \leq x) \\ = P(Z \leq \frac{x-\mu}{\sigma}) \\ = \Phi(\frac{x-\mu}{\sigma}) \end{aligned}$$

Now to find the PDF , we take the derivative of CDF.

$$\begin{aligned} f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\Phi(\frac{x-\mu}{\sigma}) \\ = \frac{1}{\sigma}\Phi'(\frac{x-\mu}{\sigma}) \\ = \frac{1}{\sigma}F_Z(\frac{x-\mu}{\sigma}) \\ = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \end{aligned}$$

Definition 2.2.2. A random variable X is said to have a normal distribution with mean μ and variance σ^2 if it has the probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{2.2.3}$$

and cumulative distribution function

$$F(x) = \Phi(\frac{x-\mu}{\sigma}) \tag{2.2.4}$$

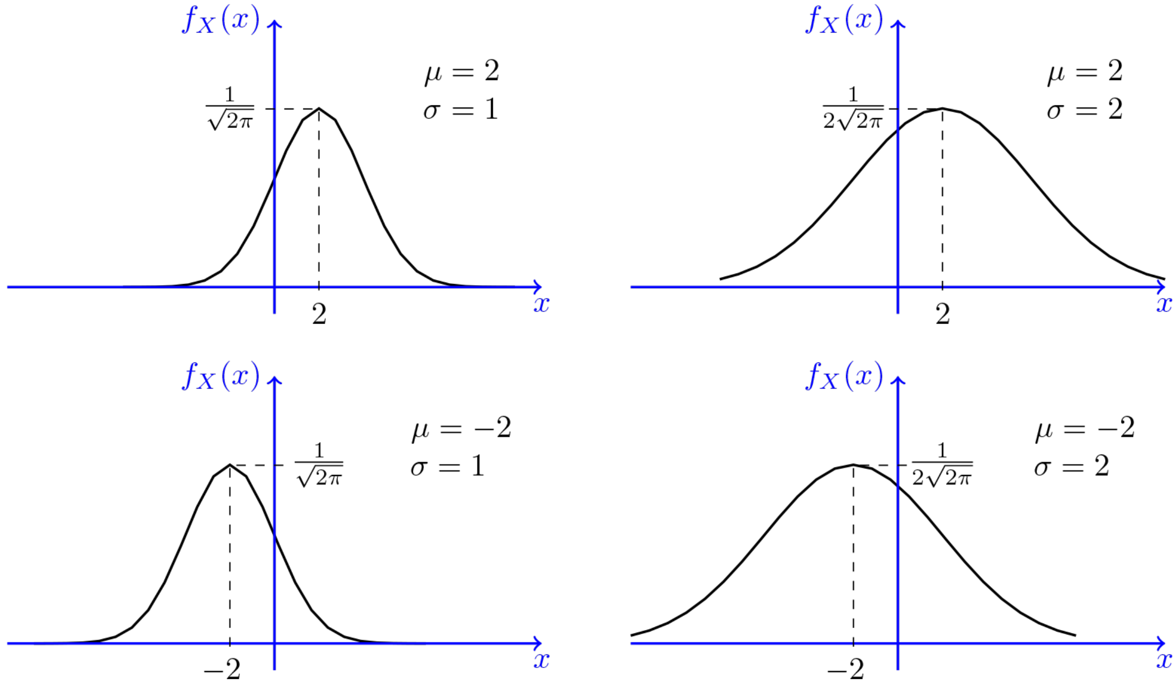

Figure 2.3: PDF for normal distribution

The height of the bell is controlled by the value of σ. As with all normal distribution curves it is symmetrical about the centre and decays as $x \to \pm\infty$. As with any probability density function the area under the curve is equal to 1.

2.3 Skew normal distribution

A skew normal distribution is an extension of the normal distribution that allows for asymmetry (skewness) while retaining some key properties of the normal distribution. Unlike the standard normal distribution, which is perfectly symmetric, a skew normal distribution can have a longer tail on either the left or right side.
It is defined by the shape parameter λ in addition to the mean μ and standard deviation σ. The shape parameter controls the direction and degree of skewness:

If $\lambda = 0$, the distribution reduces to a standard normal distribution (perfectly symmetric). If $\lambda > 0$, the distribution is skewed to the right (positively skewed). If $\lambda < 0$, the distribution is skewed to the left (negatively skewed).

2.3.1 Standard Skew Normal Distribution

Definition 2.3.1. A random variable X is said to have a standard skew normal distribution with shape parameter λ if it has the following density function

$$f(x, \lambda) = 2\phi(x)\Phi(\lambda x), \qquad -\infty \leq x \leq +\infty \tag{2.3.1}$$

where λ, x are real numbers $\phi(.)$ and $\Phi(.)$are the standard normal probability density function (pdf) and cumulative distribution function (cdf) given as (2.2.2), (2.2.1) respectively. We denote it by $X \sim SN(\lambda)$.

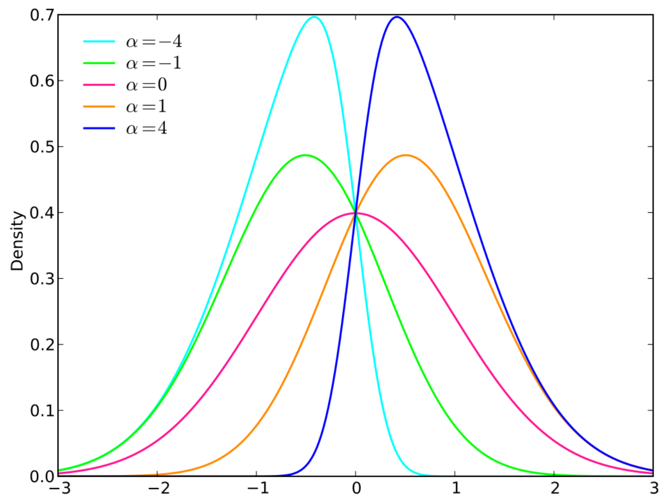

Figure 2.4: Skew normal distribution for different values of skewness parameter.

Figure 2.4 shows the shape of the pdf (2.3.1) for different values of the skewness parameter λ, namely -4,-1,0,1 and 4. It shows that for positive values of λ the skew normal density curve is skewed to the the right, and for negative λ the distribution curve is skewed to the left. When $\lambda = 1$, the shape becomes slightly skewed to the right, and when $\lambda = 4$, the shape becomes close to the pdf of a half normal random variable. Further, when $\lambda = 0$ the skew normal density curve is overlapping with the standard normal density curve.

The cdf of a skew normal distribution is denoted by $\Phi(x; \lambda)$, where

$$\Phi(x; \lambda) = \int_{-\infty}^{x} f(u, \lambda) du \tag{2.3.2}$$

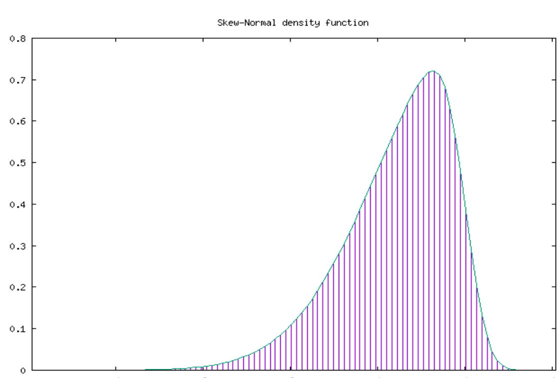

Figure 2.5: Negatively skewed

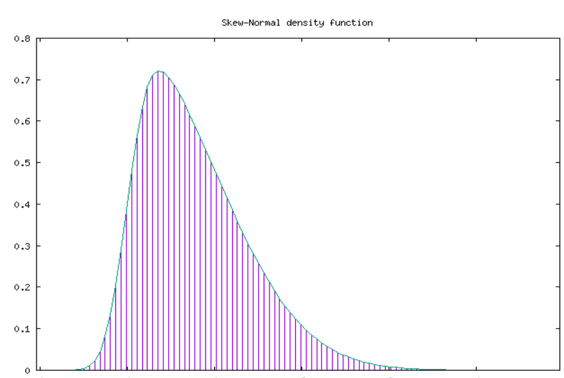

Figure 2.6: Positively skewed

Figure 2.5 is the graph of negatively skewed standard skew normal density function with location $\xi = 0$, scale $\omega = 1$ and skewness $\lambda = -5$. Figure 2.6 is the graph of the standard skew normal density function with $\lambda = 5$. As seen from the figures when skewness λ is negative, then the probability density function $\psi(x; \lambda)$ is negatively skewed, whereas for $\lambda > 0$ the pdf $\psi(x; \lambda)$ is positively skewed.

Definition 2.3.2. Let $X \sim SN(\lambda)$. The moment generating function of X is

$$M(t) = 2e^{\frac{t^2}{2}}\Phi(\frac{\lambda t}{\sqrt{1+\lambda^2}}) \tag{2.3.3}$$

The first moment of a skew normal is given by

$$\begin{aligned} E[X] = M'(0) &= 2\left[e^{\frac{t^2}{2}}\frac{\lambda}{\sqrt{1+\lambda^2}}\phi(\frac{\lambda t}{\sqrt{1+\lambda^2}}) + te^{\frac{t^2}{2}}\Phi(\frac{\lambda t}{\sqrt{1+\lambda^2}})\right]\Bigg|_{t=0} \\ &= 2\frac{\lambda}{\sqrt{1+\lambda^2}}\phi(0) \\ &= \sqrt{\frac{2}{\pi}}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right) \end{aligned}$$

The second moment of a skew normal is given by

$$\begin{aligned} E[X^2] &= M''(0) \\ &= 2[e^{\frac{t^2}{2}}(\frac{\lambda}{\sqrt{1+\lambda^2}})^2\phi'(\frac{\lambda t}{\sqrt{1+\lambda^2}}) \\ &+ 2te^{\frac{t^2}{2}}\frac{\lambda}{\sqrt{1+\lambda^2}}\phi(\frac{\lambda t}{\sqrt{1+\lambda^2}}) \\ &+ (t^2+1)e^{\frac{t^2}{2}}\Phi(\frac{\lambda}{\sqrt{1+\lambda^2}})]\Bigg|_{t=0} \\ &= 2\Phi(0) \\ &= 1 \end{aligned}$$

It follows that

$$E[X] = \sqrt{\frac{2}{\pi}}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)$$

and

$$Var(X) = 1 - \frac{2\lambda^2}{\pi(1+\lambda^2)}$$

2.3.2 Properties of Skew Normal distribution

Let $X \sim SN(\mu, \sigma, \lambda)$. Then,

1. if $\mu = 1$ and $\sigma = 1$ then $X \sim SN(\lambda)$

2. When $\lambda = 0$, the SN becomes the normal distribution. That is $SN(0) = \mathcal{N}(\mu, \sigma)$.

3. As $\lambda \to \infty$, (2.3.1) becomes $f(x) = \phi(x)$, $0 \leq x \leq \infty$which is the half-normal (folded normal) pdf. Then the skew normal density converges to half normal density function.

4. As $|\lambda|$ increases, the skewness of the distribution increases.

5. If $X \sim SN(\lambda)$ then $-X \sim SN(-\lambda)$

6. $1 - F(-x, \lambda) = F(x, -\lambda)$

7. If $X \sim \mathcal{N}(0, 1)$ and $Y \sim SN(\lambda)$, then $|Y|$ and $|X|$ have the same pdf.

8. A measure of skewness of X denoted by $\gamma_1(X)$ ranges from -0.9953 to 0.9953 and a measure of the kurtosis of X denoted by $\gamma_2(X)$ ranges from 0 to 0.869 are defined by;

$$\gamma_1(X) = \frac{4-\pi}{2}\frac{(E[X])^3}{(Var(X))^{3/2}} = \frac{4-\pi}{2}\frac{\left(\sqrt{\frac{2}{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^3}{\left(1-\left(\sqrt{\frac{2}{\pi}\frac{\lambda}{1+\lambda^2}}\right)^2\right)^{3/2}}$$

$$\gamma_2(X) = 2(\pi-3)\frac{E[X]^4}{Var[X]^2} = 2(\pi-3)\frac{\left(\sqrt{\frac{2}{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^4}{\left(1-\left(\sqrt{\frac{2}{\pi}\frac{\lambda}{1+\lambda^2}}\right)^2\right)^2}$$

9. The even moments of the X are equal to the even moments of the standard normal distribution.

10. The odd moments of X are defined as :

$$E[Z^{2k+1}] = \sqrt{\frac{2}{\pi}}\frac{\lambda}{1+\lambda^2}[2(1+\lambda^2)]^{-k}(2k+1)!\sum_{i=0}^{k}\frac{i!(2\lambda)^{2i})}{(2i+1)!(k-i)!} \quad for \quad k = 0, 1, ...$$

2.3.3 Scaled Skew Normal Distribution

Definition 2.3.3. Consider a liner transformation $Y = \mu + \sigma X$ with $\mu \in \mathbb{R}$ and $\sigma > 0$, where $X \sim SN(\lambda)$. Then the random variable Y is said to have the skew normal distribution with location parameter μ, scale parameter σ, and shape parameter λ and denoted by $Y \sim SN(\mu, \sigma, \lambda)$ if it has the pdf given by:

$$f(y; \mu, \sigma, \lambda) = \frac{2}{\sigma}\phi(\frac{y-\mu}{\sigma})\Phi(\lambda\frac{y-\mu}{\sigma}), \quad -\infty \leq y \leq \infty \qquad (2.3.4)$$

where $\phi(.)$ and $\Phi(.)$ are the pdf and cdf of the standard normal distribution respectively and μ, σ and λ are the location, scale, and shape parameters respectively.

The expectation and variance of Y are given by ;

$$E[Y] = \mu + \sigma\sqrt{\frac{2}{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}} \tag{2.3.5}$$

and

$$Var(Y) = \sigma^2(1 - \frac{2\lambda^2}{\pi(1+\lambda^2)}) \tag{2.3.6}$$

Definition 2.3.4. The cumulative density function of X is defined by

$$F(X,\lambda) = 2\int_{-\infty}^{x}\int_{-\infty}^{\lambda x}\phi(t)\phi(u)dudt = \Phi(x) - 2T(x,\lambda) \tag{2.3.7}$$

where T is Owen function defined by

$$T(h,a) = \frac{1}{2\pi}\int_0^a \frac{e^{-h^2(1+x^2)/2}}{1+x^2}dx \tag{2.3.8}$$

This function is studied and tabulated by Owen (1956). For more details about function T, the readers are referred to Young and Minder (1974), Hill (1978) and Thomas (1979).

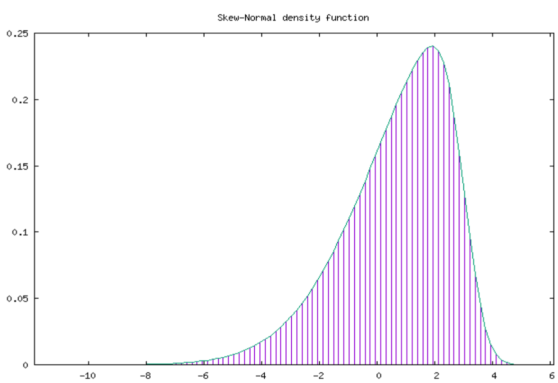

Figure 2.7: Negatively skewed

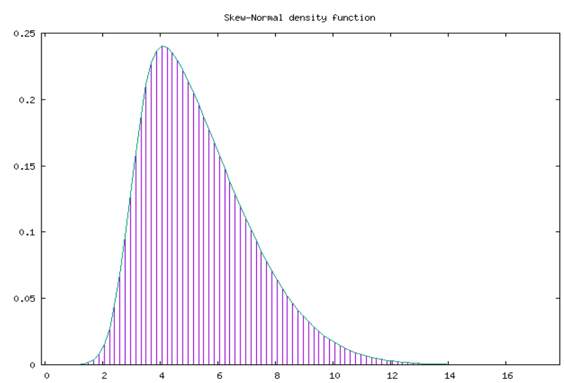

Figure 2.8: Positively skewed

Figure 2.7 is the graph of scaled skew-normal distribution with location $\xi = 3$, scale $\omega = 3$ and skewness $\lambda = -5$. Figure 2.8 is the graph of positively skewed distribution with location $\xi = 3$, scale $\omega = 3$ and skewness $\lambda = 5$.

2.4 The Student-t Distribution

The Student t distribution is symmetric and bell-shaped distribution similar to the normal distribution. However, it has heavier tails than the normal distribution which makes it more prone to producing values that fall far from its mean. The Student t distribution is the second most popular distribution, after the normal distribution, due to its application in estimating the mean of a normally distributed population when the sample size is small and population standard deviation is unknown. It is parametrized by one parameter called the degrees of freedom denoted by r. For finite values of the degrees of freedom r, the tails of the density function decay as an inverse power of order $r + 1$. When the degrees of freedom $r = 1$ the Student t distribution reduces to the Cauchy(0,1) distribution, while as the degrees of freedom tends to infinity the distribution converges to the normal distribution.

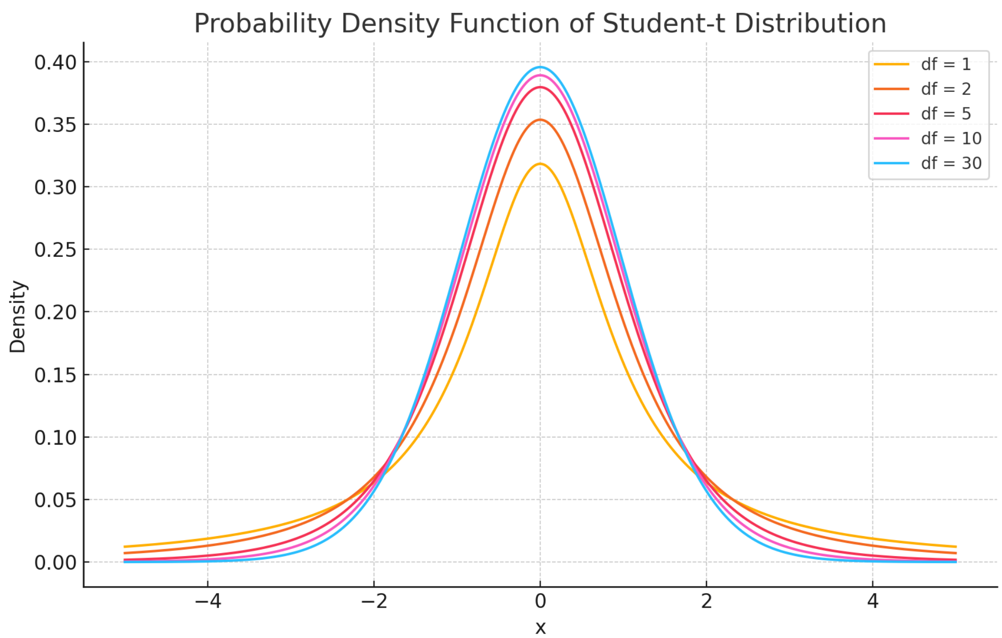

Figure 2.9: Student-t distribution

Definition 2.4.1. A random variable X is said to have the Student t distribution with degrees of freedom r if it has the pdf given by

$$t(x, r) = \frac{1}{\sqrt{r}B(\frac{r}{2}, \frac{1}{2})} \frac{1}{(1 + \frac{x^2}{r})^{\frac{r+1}{2}}}, \quad -\infty < x < \infty \tag{2.4.1}$$

where $B(a, b)$ denotes the Beta function given by

$$B(a, b) = \int_0^\infty x^{a-1}(1 - x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}, \quad where \quad a, b > 0,$$

the degrees of freedom $r > 0$, we say $X \sim t_r$

2.5 The skew-t distribution

Student-t is a symmetric distribution that cannot capture asymmetry. To accommodate asymmetry and long tailed data, Hansen (1994) introduced the so called skewed t distribution while maintaining the property of a zero mean and variance equal to one. The skew-t distribution has four parameters:

- μ - Regulates the location of the distribution
- σ - Controls the spread or dispersion.
- λ - Regulates the assymetry of the distribution.
- ν - Degrees of freedom governs the heaviness of the tails (related to kurtosis).

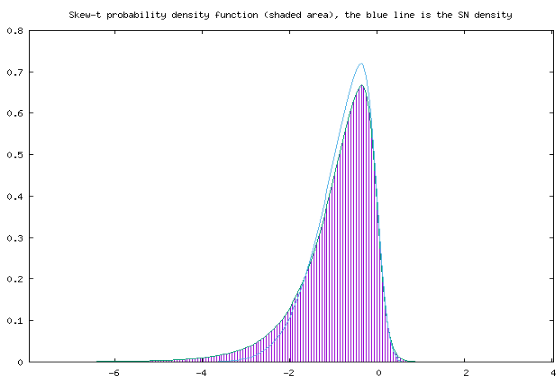

Figure 2.10: Negatively skewed

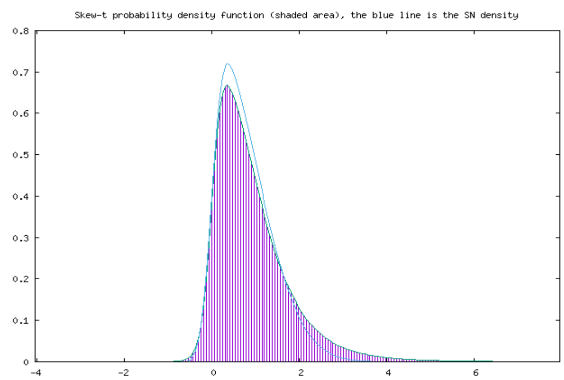

Figure 2.11: Positively skewed

Figure 2.10 shows the graph of probability density function of standard skew-t distribution with location $\xi = 0$, scale $\omega = 1$, skewness $\lambda = -5$ and the degree of freedom $\nu = 5$. Figure 6 illustrates the graph of (PDF) of positively skewed standard skew-t distribution with location $\xi = 0$, scale $\omega = 1$, skewness $\lambda = 5$ and the degree of freedom $\nu = 5$.In both of the figures ,the blue line describes the skew-normal probability density function. As shown, the skew-t has heavier tail than the skew-normal distribution. When the degree of freedom goes to infinity, skew-t distribution converges to skew-normal distribution, since student t-distribution converges to standard normal distribution as $\nu \to \infty$.
Hansen's skew t distribution distribution is derived by introducing a generalization of the Student t distribution as follows;

$$f(x, \lambda, \nu) = b\frac{\Gamma(\frac{\mu+1}{2})}{\sqrt{\pi(\nu-2)}\Gamma(\frac{\nu}{2})}\left(1 + \frac{\xi^2}{\nu-2}\right)^{-\frac{\nu+1}{2}} \tag{2.5.1}$$

where

$$\xi = \begin{cases} (bx+a)/(1-\lambda) & \text{if} \quad x < -a/b \\ (bx+a)/(1+\lambda) & \text{if} \quad x \geq -a/b \end{cases}$$

The constant term a, b are defined by

$$a = 4\lambda c \frac{\mu - 2}{\mu - 1}$$
$$b = 1 + 3\lambda^2 - a^2$$

and

$$c = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi(\nu-2)}\Gamma(\frac{\nu}{2})}$$

In this distribution, $2 < \mu < \infty$ denotes the degrees of freedom parameter and $-1 < \lambda < 1$ is the asymmetry parameter.
Azzalini and Capitanio [2] constructed a skew-t distribution by replacing the normal densities in Definition 2.2.3 (2.3.4) by t densities

Definition 2.5.1. Let $Y \sim SN(\lambda)$ be independent from $Z \sim X_\nu^2$. Let

$$X = \frac{Y}{Z/\nu} \tag{2.5.2}$$

Then the random variable X is said to have the skew t distribution with shape parameter $\lambda \in R$ and degrees of freedom $\nu > 0$ if it has the probability density function given by:

$$f(x; \lambda, \nu) = 2t(x; \nu)T(\lambda x\sqrt{\frac{\nu+1}{x^2+\nu}}; \nu+1), \quad x \in R \tag{2.5.3}$$

, where $t(.)$ and $T(.)$ are the probability density function and the distribution function of the standard Student-t distribution respectively, and we denote this as $X \sim st_\nu(\lambda)$

Definition 2.5.2. Let X be a skew-normal variable with parameters (μ, σ, λ). Let Y be a $\mathcal{X}^2$-variable with ν degrees of freedom. Assume further that X, Y are independent. Let T be the random variable which is constructed using the following transformation:

$$T = \frac{X}{\sqrt{Y/\nu}} \tag{2.5.4}$$

Then T is said to have the skew-t distribution with parameters $(\mu, \sigma, \lambda, \nu)$ denoted by $ST(\mu, \sigma, \lambda, \nu)$.

Definition 2.4.2 incorporates the tuning parameter, ν, to control the rate at which the tail of the distribution decays and, hence, provides a distribution family that is flexible enough to accommodate data with excess kurtosis and long tails.

Theorem 2.5.3. *Let $T \sim ST(\mu, \sigma, \lambda, \nu)$ .Then the PDF of T is given by:*

$$\begin{aligned} f_T(t) &= \frac{2e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{2\pi\nu}\Gamma()\frac{\nu}{2}2^{\nu/2}} \\ &\times \int_0^\infty y^{(\frac{\nu-1}{2})} e^{[-\frac{y}{2}(t^2\nu^{-1}\sigma^{-2}+1)+\frac{t\mu\sqrt{y}}{\sigma^2\sqrt{\nu}}]} \Phi(\lambda\frac{t\sqrt{y/\nu}-\mu}{\sigma})dy \end{aligned} \tag{2.5.5}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function.
Theorem 2.4.3 expresses the PDF of the skew t distribution as an improper integral.

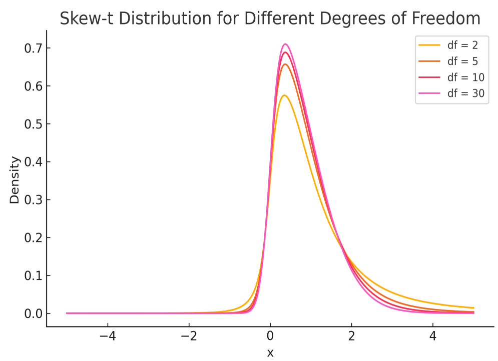

Figure 2.12: Skew-t for different df

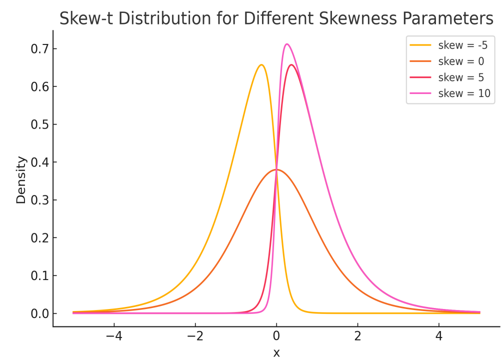

Figure 2.13: Skew-t as α varies

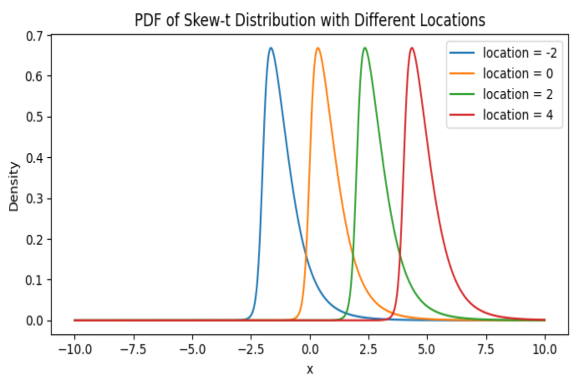

Figure 2.14: Skew-t for different location

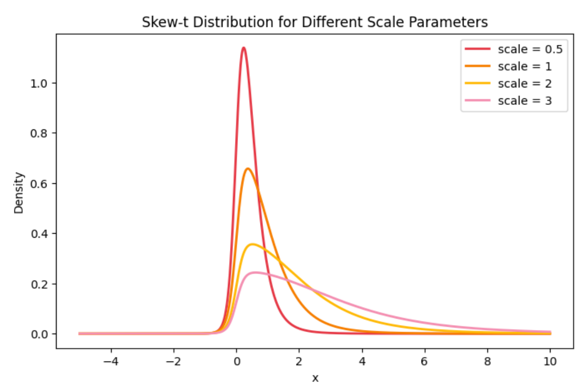

Figure 2.15: Skew-t for different scale

Figure 2.10 illustrates the effect of the parameter α on the shape of the density. The figure 2.11 is shows the pdf of skew-t given for values of $\alpha = -5, 0, 5, 10$. Figure 2.12 illustrates the effect of the location parameter μ on determining the overall shape of the density. Notice that the graphs represented agree on the parameters σ, α, ν yet they are not location shifts of each other. The skew -t family of distributions is not a location-scale family. Figure 2.13 illustrate the effect of the scale parameter σ on determining the overall shape of the density. They show that when the degrees of freedom parameter is fixed, the thickness of the tail of the density can still be controlled by the parameter σ. Thus, the skew -t distribution has two parameters ν and σ that control the thickness of the tail of the density.y. It shows that similar to the skew normal density, for positive shape parameter λ the distribution skewed to the right, and for negative λ the distribution skewed to the left. Because we fixed the location and scale parameters $\mu = 0$ and $\sigma = 1$ we note that when $\lambda = 0$ we get the Student t density. We also see that the skew t density approaches the half t density as λ approaches ∞.

We note that the skew -t density approaches the skew normal density as ν approaches ∞ as presented in Figure 1.5

Theorem 2.5.4. *Let $T \sim ST(\mu, \sigma, \alpha, \nu)$. The k^{th} moment of T is given by;*

$$E[T^k] = \frac{(\nu/2)^{k/2}\Gamma(\frac{\nu-k}{2})}{\Gamma(\frac{\nu}{2})} E[X^k] \tag{2.5.6}$$

where $X \sim SN(\mu, \sigma, \alpha)$ and $\nu > k$

Theorem 2 provides a method to compute the k^{th} moment of the ST random variable in terms of the moments of the skew-normal distribution.
If $T \sim ST(\mu, \sigma, \alpha, \nu)$ then the first two moments of T are given by:

$$E[T] = \frac{(\nu/2)^{1/2}\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})}[\mu + \sqrt{\frac{2}{\pi}}\sigma\delta] \tag{2.5.7}$$

where

$$\delta = \frac{\alpha}{1+\alpha^2} \tag{2.5.8}$$

$$E[T^2] = \frac{\nu}{\nu-2}[\mu^2 + \sigma^2 + \sqrt{\frac{2}{\pi}}\mu(\mu + \delta\sigma)], \quad \nu > 2 \tag{2.5.9}$$

and

$$\begin{aligned} Var(T) &= \frac{\nu}{\nu-2}[\mu^2 + \sigma^2 + \sqrt{\frac{2}{\pi}}\mu(\mu + \delta\sigma)] \\ &- \frac{\nu}{2}([\mu + \sqrt{\frac{2}{\pi}}\sigma\delta]\frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})})]^2, \quad r > 2 \end{aligned} \tag{2.5.10}$$

2.5.1 Scaled skew-t distribution

Consider a liner transformation $Y = \xi + \omega X$, where $X \sim st_\mu(\lambda)$. Then the random variable Y is said to have the skew-t distribution with location parameter ξ, scale parameterω, shape parameter λ, and degrees of freedom $\nu > 0$ if it has the pdf given by:

$$f(y; \xi, \omega, \lambda, \nu) = \frac{2}{\omega} t(\frac{y-\xi}{\omega}, \nu) T(\lambda \frac{y-\xi}{\omega}\sqrt{\frac{\nu+1}{(\frac{y-\xi}{\omega})^2 + \nu}}; \nu+1) \tag{2.5.11}$$

denoted by $Y \sim st_\nu(\xi, \omega, \lambda)$ where $-\infty < y < \infty, \quad \xi, \lambda \in R$ and $\omega, \nu > 0$.

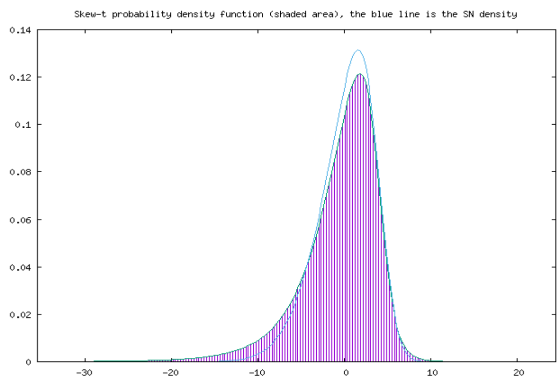

Figure 2.16: Negatively skewed-t

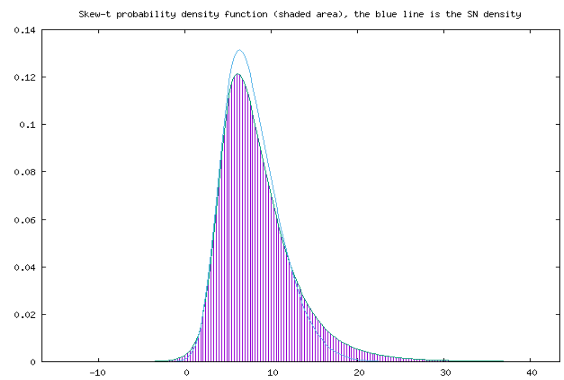

Figure 2.17: Positively skewed-t

Figure 7 shows the graph of probability density function of skew-t distribution with location $\xi = 40$, scale $\omega = 5$, skewness $\lambda = -3$ and the degree of freedom $\nu = 5$. Figure 8 illustrates the graph of PDF of positively skewed standard skew-t distribution with location $\xi = 4$, scale $\omega = 5$,skewness $\lambda = 3$ and the degree of freedom $\nu = 5$. The blue line describes the skew-normal probability density function in both figures.

2.5.2 Properties of Skew-t Distribution

Let $X \sim ST(\mu, \sigma, \lambda, \nu)$. Then

1. If $\mu = 0, \sigma = 1$. Then $X \sim ST(\lambda, \nu)$
2. As $\lambda \to \pm\infty$ then $X \sim |t_r|$ distribution.
3. As $\nu \to \pm\infty$ then $X \sim SN(\mu, \sigma, \lambda)$
4. The skew-t density is unimodal.
5. If $\nu \leq k$, then $E[X^k]$ does not exist.
6. The k^{th} moment is defined as

$$E[X^k] = \frac{(\nu/2)^{k/2}\Gamma(\frac{1}{2}(\nu-1))}{\Gamma(\frac{\nu}{2})} E[Z^k]$$

   where $\nu > 2$ and $Z \sim SN(\lambda)$

7. The moment is $E[X] = \mu + \sigma b_r \delta$ if $\nu > 1$
8. Tha variance is $Var(X) = \sigma^2[\frac{\nu}{\nu-2} - (b_v\delta)^2]$ if $\nu > 2$
9. The measure of skewness of X is

$$\gamma_1(X) = \frac{b_r\delta}{[\frac{\nu}{\nu-2} - (b_r\delta)^2]^3} \left[\frac{\nu(3-\delta^2)}{\nu-3} - \frac{3\nu}{\nu-2} + 2(b_r\delta)^2 \right], \quad if \quad \nu > 3$$

10. $\gamma_1(X)$ ranges between -4 to 4 if $\nu > 4$, but it becomes the whole real line if we consider $\nu > 3$

11. The measure of kurtosis of X is

$$\gamma_2(X) = \frac{1}{[\frac{\nu}{\nu-2} - (b_v\delta)^2]} \left[\frac{3\nu^2}{(\nu-2)(\nu-4)} - \frac{4(b_r\delta)^2\nu(3-\delta^2)}{\nu-3} + \frac{6(b_r\delta)^2\nu}{\nu-2} - 3(b_r\delta)^4\right], \quad if \quad \nu > 4$$

where

$$b_r = \frac{\sqrt{\nu}\Gamma(\frac{1}{2}(\nu-1))}{\sqrt{\pi}\Gamma()\frac{\nu}{2}}, \quad if \quad \nu > 1 \quad and \quad \delta = \frac{\lambda}{\sqrt{1+\lambda^2}}, \quad \delta \in (-1,1) \quad (2.5.12)$$

2.6 Goodness of fit test

The goodness-of-fit tests are used to determine whether a given sample comes from a population with a defined theoretical distribution. In other words, let the observed $x_1, x_2, x_3, ..., x_n$ be a random sample come from a population with a continuous $F(x)$ distribution. This unknown F(x) distribution function should be confirmed with the help of hypothesis by using a goodness-of-fit test. The null hypothesis to be used for this reason is given as follows,

$H_0 : F(x) = F_0(x, \theta)$ or H_0 : *The data follow a specified distribution.*
Where $F_0(x, \theta)$ is the specified distribution function with θ parameter. The alternative hypothesis is given as follows,

$H_1 : F(x) \neq F_0(x, \theta)$ or H_1 : *The data do not follow a specified distribution.*
In order to show whether the data come from the specified distribution, it is first necessary to calculate the test statistics, which its distribution and the critical value are known, according to the null hypothesis. The most basic feature that distinguishes the goodness-of-fit test from each other is the calculation of the test statistics in a different way.

2.6.1 Chi square goodness of fit test

The chi-square goodness-of-fit test which is binning-based method, has taken part in almost all the basic statistical books because it is easy to apply and understand. If the examined data is not binning data, this data can be obtained by calculating a histogram. The Chi-square test is based on the inconsistency between observed and expected frequencies. The Chi-square investigates whether there is a statistically significant difference between observed and expected frequencies. It's preferred over other tests because of its good features such that it can be applied to any uni variate distribution and it can be calculated much easier than other tests.

Definition 2.6.1. The Chi-square goodness of fit test utilizes the null hypothesis to determine whether a sample of data $x_1, x_2, ..., x_n$ is coming from a specified population.

Thi chi -square test statistic is obtained as:

$$\chi^2 = \sum_{j=1}^{k} \frac{O_j - E_j}{E_j} \tag{2.6.1}$$

where O_j is the observed frequency for bin j and E_j is the expected frequency for bin j. The obtained test statistic follows, approximately, a chi-square distribution with (k- 1) degrees of freedom. The calculated test statistic is compared to the critical value from the chi-square critical value with (k - 1) degrees of freedom and $(1-\alpha)$ confidence level so that it can be decided as a result of the test. If the calculated $\chi^2 < critical \chi^2_{k-1,1-\alpha}$, then we cannot reject the null hypothesis.

Theorem 2.6.2. *(Chi-Squared goodness-of-fit test for simple hypothesis)*
Suppose that we observe an independent and identically distributed sample $X_1, ..., X_n$ ofvrandom variables that take a finite number of values $B_1, ..., B_k$ with unknown probabilities $p_1, ..., p_k$
If $np_j \leq 5$ for all j, then :

$$T = \sum_{i=1}^{k} \frac{(v_j - np_j^0)^2}{np_j^0} \to \chi^2_{k-1} \tag{2.6.2}$$

Where $v_j = \#(X_i : X_i = B_j)$ are the observed counts in each category.

To test the hypothesis: $H_0 : p_j = p_j^0$ for all j = 1, ..., k Versus the alternative hypothesis $H_1 : p_j \neq p_j^0$ for some index at the level of significance, reject if

$$T = \sum_{i=1}^{k} \frac{(v_j - np_j^0)^2}{np_j^0} \geq \chi^2_{1-\alpha,k-1}$$

In order to apply the Chi square goodness-of-fit test properly, the expected frequencies in each class should be at least 5. If it is not, the classes must be combined with other classes until assumption is satisfied.

Chapter 3

Explicit Representations for LPM's of Skew-Elliptic Family

3.0.1 LPM's of Standard Normal Distribution

Definition 3.0.1. Let $Z \sim \mathcal{N}(0,1)$ be a standard normal random variable with probability density function ϕ and cumulative distribution function Φ. Let $n \geq 1$ be the prespecified moment and z_α be the α percentile of Z, i.e. $\mathbb{P}(Z \leq z_\alpha) = \alpha$. Then, the lower partial moment of Z is defined by

$$\begin{aligned} \mathrm{LPM}_{(n,0,1,z_\alpha)} &\triangleq \mathbb{E}[\max(z_\alpha - Z, 0)^n] \\ &= \mathbb{E}[(z_\alpha - Z)^n \mathrm{I}_{\{Z \leq z_\alpha\}}] \\ &= \int_{-\infty}^{z_\alpha} (z_\alpha - Z)^n \phi(z) dz \end{aligned} \tag{3.0.1}$$

s

Lemma 3.0.2. *Let $Z \sim \mathcal{N}(0,1)$ be the standard normal random variable with mean $\mu = 0$ and variance $\sigma^2 = 1$ with prespecified z_α threshold, and consider*

$$\mathbb{E}[Z^n \mathrm{I}_{\{Z \leq z_\alpha\}}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_\alpha} z^n e^{-z^2/2} dz. \tag{3.0.2}$$

By repeated integration by parts,

- *(i) for even $n \in \mathbb{N}$,* (3.0.2) *reads as*

$$\begin{aligned} \mathbb{E}[Z^n \mathrm{I}_{Z \leq z_\alpha}] = -\phi(z_\alpha)\Big(&z_\alpha^{n-1} + (n-1)z_\alpha^{n-3} + \ldots + (n-1)(n-3)\ldots z_\alpha\Big) \\ &+ (n-1)(n-3)\ldots\Phi(z_\alpha). \end{aligned} \tag{3.0.3}$$

- *(ii) for odd $n \in \mathbb{N}$,* (3.0.2) *reads as*

$$\begin{aligned} \mathbb{E}[Z^n \mathrm{I}_{Z \leq z_\alpha}] = -\phi(z_\alpha)\Big(&z_\alpha^{n-1} + (n-1)z_\alpha^{n-3} + \ldots \\ &+ (n-1)(n-3)\ldots 4z_\alpha^2 + (n-1)(n-3)\ldots 2\Big) \end{aligned} \tag{3.0.4}$$

Thus,

$$\mathrm{LPM}_{(n,0,1,z_\alpha)}(Z) = \sum_{i=0}^{n} \binom{n}{i} z_\alpha^{n-i}(-1)^i(E)[Z^i \mathrm{I}_{\{Z\leq z_\alpha\}}] \tag{3.0.5}$$

3.0.1.1 LPM's of Scaled Normal Distributions

Lemma 3.0.3. *Let $W \triangleq \mu + \sigma Z$ for mean $\mu \in \mathbb{R}$ and $\sigma > 0$ with $Z \sim \mathcal{N}(0,1)$ such that $W \sim \mathcal{N}(\mu,\sigma)$ with a prespecified percentile w_α and $n \in \mathbb{N}$ with $\mathbb{P}(W \leq w_\alpha) = \alpha$. Denote $\tilde{w}_\alpha \triangleq \frac{w_\alpha - \mu}{\sigma}$. Then,*

$$\mathrm{LPM}_{(n,\mu,\sigma,w_\alpha)}(W) = \mathrm{LPM}_{(n,0,1,\tilde{w}_\alpha)}(Z),$$

where $\mathrm{LPM}_{(n,0,1,\tilde{w}_\alpha)}(Z)$ is as in (3.0.5).

3.0.2 LPM's of Skew Normal Distributions

Definition 3.0.4. A random variable X is called to have the standard skew-normal with location $\xi = 0$, scale $\omega = 1$ and skewness $\lambda \in \mathbb{R}$, denoted by $X \sim SN(0,1,\lambda)$, if X has the probability density function

$$\psi(x;\lambda) \triangleq 2\phi(x)\Phi(\lambda x) \quad \text{for } x \in \mathbb{R}, \tag{3.0.6}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the probability density function (pdf) and the cumulative distribution function (cdf) of a standard normal random variable, respectively.

Remark 3.0.5. Note that by $\Phi(\lambda x) = 1 - \Phi(-\lambda x)$, we have

$$\begin{aligned}\psi(x;\lambda) &= 2\phi(x) - 2\phi(x)\Phi(-\lambda x)\\ &= 2\phi(x) - \psi(x;-\lambda)\end{aligned}$$

Definition 3.0.6. Let $X \sim SN(0,1,\lambda)$ and x_α be the α quantile of X, i.e. $P(X \leq x_\alpha) = \alpha$. Let $n \geq 1$ be the degree of the moment. The lower partial moment of X, denoted by $\mathrm{LPM}_{(n,0,1,\lambda,x_\alpha)}(X)$, is defined as

$$\begin{aligned}\mathrm{LPM}_{(n,0,1,\lambda,x_\alpha)}(X) &\triangleq \mathbb{E}[\max(x_\alpha - X, 0)^n],\\ &= \int_{-\infty}^{z_\alpha} (z_\alpha - x)^n \psi(x;\lambda)dx,\\ &= \mathbb{E}[(x_\alpha - X)^n \mathrm{I}_{\{X\leq x_\alpha\}}].\end{aligned}$$

Thus, by (3.0.6), we have

$$\mathrm{LPM}_{(n,0,1,\lambda,x_\alpha)}(X) = \int_{-\infty}^{x_\alpha} (x_\alpha - x)^n 2\phi(x)\Phi(\lambda x)dx \tag{3.0.7}$$

Next, we give explicit representations of the $\mathrm{LPM}(n,0,1,\lambda,x_\alpha)$ of a standard skew-normal r.v. X for a specified x_α threshold and moment $n \geq 1$.

Lemma 3.0.7. *Let* $X \sim \mathrm{SN}(0,1,\lambda)$ *with* $\lambda > 0$. *First lower partial moment for a prespecified risk averseness level* x_α, $0 < \alpha < 1$ *of* $X \sim SN(0,1,\lambda)$, $\mathrm{LPM}_{(1,0,1,\lambda,x_\alpha)}(X)$, *is given as*

$$\begin{aligned}\mathrm{LPM}_{(1,0,1,\lambda,x_\alpha)}(X) &= x_\alpha \mathbb{P}(X \le x_\alpha) + 2\phi(x_\alpha)\Phi(\lambda x_\alpha) \\ &\quad - \sqrt{\frac{2}{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}}\phi(x_\alpha\sqrt{1+\lambda^2}) \\ &= \alpha x_\alpha + 2\phi(x_\alpha)\Phi(\lambda x_\alpha) - \sqrt{\frac{2}{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}}\phi(x_\alpha\sqrt{1+\lambda^2})\end{aligned} \tag{3.0.8}$$

Proof. The equation (3.0.8) is via integration by parts

$$\begin{aligned}\mathbb{E}[X\mathrm{I}_{\{X\le x_\alpha\}}] &= \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{x_\alpha} xe^{-x^2/2}\Big(\int_{-\infty}^{\lambda x}\frac{1}{\sqrt{2\pi}}e^{-u^2/2}du\Big)dx \\ &= \frac{1}{\pi}\int_{-\infty}^{x_\alpha} xe^{-x^2/2}\Big(\int_{-\infty}^{x_\alpha} e^{-u^2/2}du\Big)dx \\ &= \frac{1}{\pi}\Big[-e^{-x^2/2}\int_{-\infty}^{\lambda x} e^{-u^2/2}du\Big|_{x=-\infty}^{x=x_\alpha} + \int_{-\infty}^{x_\alpha} e^{-x^2/2}\lambda e^{-\lambda^2x^2/2}dx\Big] \\ &= \frac{1}{\pi}\Big[-e^{-x_\alpha^2/2}\Phi(\lambda x_\alpha)\sqrt{2\pi} + \int_{-\infty}^{x_\alpha}\lambda e^{-\frac{x^2}{2}(1+\lambda^2)}dx\Big] \\ &= \frac{1}{\pi}\Big[-e^{-x_\alpha^2/2}\Phi(\lambda x_\alpha)\sqrt{2\pi} + \frac{\lambda\sqrt{2\pi}}{\sqrt{1+\lambda^2}}\int_{-\infty}^{x_\alpha}\frac{1}{\sqrt{2\pi}}\sqrt{(1+\lambda^2)}e^{-\frac{x^2}{2}(1+\lambda^2)}dx\Big] \\ &= \frac{1}{\pi}\Big[-e^{-x_\alpha^2/2}\Phi(\lambda x_\alpha)\sqrt{2\pi} + \frac{\lambda\sqrt{2\pi}}{\sqrt{1+\lambda^2}}\int_{-\infty}^{x_\alpha}\frac{1}{\sqrt{2\pi}}\sqrt{1+\lambda^2}e^{-\frac{x^2}{2}(1+\lambda^2)}dx\Big] \\ &= -2\phi(x)\Phi(\lambda x_\alpha) + \sqrt{\frac{2}{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}}\phi(x_\alpha\sqrt{1+\lambda^2})\end{aligned} \tag{3.0.9}$$

Hence, by x_α being the α quantile, $\mathbb{P}(X \le x_\alpha) = \alpha$, we conclude the result. □

Corollary 3.0.8. *Let* $X \sim \mathrm{SN}(0,1,\lambda)$ *with* $\lambda < 0$, x_α *be the corresponding* α *quantile of* X *with* $\mathbb{P}(X \le x_\alpha) = \alpha$ *and* $Z \sim \mathcal{N}(0,1)$. *Then, for* $n \ge 1$, n^{th} *lower partial first moment for a prespecified risk averseness level* $0 < \alpha < 1$ *of* X, $\mathrm{LPM}_{(n,x_\alpha,\lambda)}(X)$, *is given as:*

$$\mathrm{LPM}_{(n,0,1,\lambda,x_\alpha)}(X) = 2\mathbb{E}[Z^n\mathrm{I}_{\{Z\le x_\alpha\}}] - \mathrm{LPM}_{(n,0,1,-\lambda,x_\alpha)}(X) \tag{3.0.10}$$

Proof. The result follows by $X \sim SN(0,1,-\lambda)$ with $-\lambda > 0$ and by (3.0.3) and (3.0.4). □

Next, we give the representation for the second lower partial moment $\mathrm{LPM}_{(2,0,1,\lambda,x_\alpha)}(X)$ of $X \sim SN(0,1,\lambda)$ for $x_\alpha \in \mathbb{R}$.

Lemma 3.0.9. *Let $X \sim \mathrm{SN}(0,1,\lambda)$ with $\lambda > 0$. Then, second order lower partial moment of $X \sim SN(0,1,\lambda)$ for a prespecified x_α level,* $\mathrm{LPM}_{(2,0,1,\lambda,x_\alpha)}(X)$*, is given as*

$$\mathrm{LPM}_{(2,0,1,\lambda,x_\alpha)}(X) = x_\alpha^2 \mathbb{P}(X \leq x_\alpha) - 2x_\alpha \mathbb{E}[X \mathrm{I}_{\{X \leq x_\alpha\}}] + \mathbb{E}[X^2 \mathrm{I}_{\{X \leq x_\alpha\}}],$$

where

$$\mathbb{E}[X^2 \mathrm{I}_{\{X \leq x_\alpha\}}] = -2x_\alpha \phi(x_\alpha)\Phi(\lambda x_\alpha) + \frac{\lambda}{\pi\sqrt{1+\lambda^2}} e^{-\frac{x_\alpha^2}{2}(1+\lambda^2)} + \alpha.$$

Proof. We have

$$\begin{aligned}
\mathbb{E}[X^2 \mathrm{I}_{\{X \leq x_\alpha\}}] &= 2\int_{-\infty}^{x_\alpha} x^2 e^{-x^2/2} \frac{1}{\sqrt{2\pi}} \Big(\int_{-\infty}^{\lambda x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \Big) dx \\
&= 2\int_{-\infty}^{x_\alpha} x e^{-x^2/2} \frac{1}{\sqrt{2\pi}} \Big(x \int_{-\infty}^{\lambda x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \Big) dx \\
&= 2\Big[-\frac{1}{\sqrt{2\pi}} e^{-x^2/2} x \int_{-\infty}^{\lambda x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \Big|_{x=-\infty}^{x=x_\alpha} \\
&\quad - \int_{-\infty}^{x_\alpha} -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \Big(\int_{-\infty}^{\lambda x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du + \lambda x \frac{1}{\sqrt{2\pi}} e^{-\lambda^2 x^2/2} \Big) dx \Big] \\
&= 2\Big[-x_\alpha \phi(x_\alpha)\Phi(\lambda x_\alpha) + \int_{-\infty}^{x_\alpha} \frac{1}{2\pi} e^{-x^2/2} \lambda x e^{-\lambda^2 x^2/2} dx \\
&\quad + \int_{-\infty}^{x_\alpha} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \Big(\int_{-\infty}^{\lambda x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \Big) dx \Big]
\end{aligned} \tag{3.0.11}$$

Thus, we have

$$\mathbb{E}[X^2 \mathrm{I}_{\{X \leq x_\alpha\}}] = -2x_\alpha \phi(x_\alpha)\Phi(\lambda x_\alpha) + \frac{\lambda}{\pi\sqrt{1+\lambda^2}} e^{-\frac{x_\alpha^2}{2}(1+\lambda^2)} + \mathbb{P}[X \leq x_\alpha].$$

Hence, by $\mathbb{P}(X \leq x_\alpha) = \alpha$ and

$$\begin{aligned}
\mathrm{LPM}_{(2,0,1,\lambda,x_\alpha)}(X) &= \mathbb{E}[(x_\alpha - X)^2 \mathrm{I}_{\{X \leq x_\alpha\}}] \\
&= x_\alpha^2 \mathbb{P}(X \leq x_\alpha) - 2x_\alpha \mathbb{E}[X \mathrm{I}_{\{X \leq x_\alpha\}}] + \mathbb{E}[X^2 \mathrm{I}_{\{X \leq x_\alpha\}}],
\end{aligned}$$

we conclude the result. □

Analogously, we define LPM's for $n \geq 3$ of a standard skew normal r.v. $X \sim \mathrm{SN}(0,1,\lambda)$ with fixed $x_\alpha \in \mathbb{R}$ as

$$\mathrm{LPM}_{(n,0,1,\lambda,x_\alpha)}(X) \triangleq \mathbb{E}[(x_\alpha - X)^n \mathrm{I}_{(X \leq x_\alpha)}]$$

Lemma 3.0.10. *Let $X \sim SN(0,1,\lambda)$ with $0<\alpha<1$ and $n \geq 3$. Then,*

$$\begin{aligned}
\mathrm{LPM}_{(n,0,1,\lambda,x_\alpha)}(X) &= x_\alpha^n \mathbb{P}(X \leq x_\alpha) - n x_\alpha^{n-1}\mathbb{E}[X\mathrm{I}_{\{X\leq x_\alpha\}}] + \binom{n}{2} x_\alpha^{n-2}\mathbb{E}[X^2\mathrm{I}_{\{X\leq x_\alpha\}}] \\
&\quad + \sum_{i=3}^{n}\binom{n}{i} x_\alpha^{n-i}(-1)^i \frac{1}{\pi}\Big(T_1(i,x_\alpha,\lambda)+T_2(i,x_\alpha,\lambda)+T_3(i,x_\alpha,\lambda)\Big), \\
&= x_\alpha^n \alpha - n x_\alpha^{n-1}\mathbb{E}[X\mathrm{I}_{\{X\leq x_\alpha\}}] + \binom{n}{2} x_\alpha^{n-2}\mathbb{E}[X^2\mathrm{I}_{\{X\leq x_\alpha\}}] \\
&\quad + \sum_{i=3}^{n}\binom{n}{i} x_\alpha^{n-i}(-1)^i \frac{1}{\pi}\Big(T_1(i,x_\alpha,\lambda)+T_2(i,x_\alpha,\lambda)+T_3(i,x_\alpha,\lambda)\Big),
\end{aligned}$$

where

$$\begin{aligned}
T_1(n,x_\alpha,\lambda) &\triangleq -e^{-x_\alpha^2/2} x_\alpha^{n-1}\int_{-\infty}^{\lambda x_\alpha} e^{-u^2/2}du \\
&= -e^{-x_\alpha^2/2} x_\alpha^{n-1}\Phi(\lambda x_\alpha)\sqrt{2\pi}
\end{aligned} \tag{3.0.12}$$

$$\begin{aligned}
T_2(n,x_\alpha,\lambda) &\triangleq (n-1)\int_{-\infty}^{x_\alpha}\left(e^{-x^2/2}x^{n-2}\int_{-\infty}^{\lambda x} e^{-u^2/2}du\right)dx \\
&= \frac{\sqrt{2\pi}(n-1)}{2}\mathbb{E}[X^{n-2}\mathrm{I}_{X\leq x_\alpha}]
\end{aligned} \tag{3.0.13}$$

$$\begin{aligned}
T_3(n,x_\alpha,\lambda) &\triangleq \int_{-\infty}^{x_\alpha} \lambda e^{-\frac{x^2(1+\lambda)}{2}} x^{n-1}dx \\
&= \frac{\lambda\sqrt{2\pi}}{(\lambda+1)^{n/2}}\mathbb{E}[Z^{n-1}\mathrm{I}_{Z\leq\sqrt{\lambda+1}x_\alpha}], \quad \textit{where } Z\sim\mathcal{N}(0,1).
\end{aligned} \tag{3.0.14}$$

Proof. By integration by parts, the expression reads as

$$\begin{aligned}
\mathbb{E}[X^n\mathrm{I}_{\{X\leq x_\alpha\}}] &= 2\int_{-\infty}^{x_\alpha} x^n e^{-x^2/2}\frac{1}{\sqrt{2\pi}}\Big(\int_{-\infty}^{\lambda x}\frac{1}{\sqrt{2\pi}}e^{-u^2/2}du\Big)dx \\
&= \frac{1}{\pi}\int_{-\infty}^{x_\alpha} x e^{-x^2/2}\Big(x^{n-1}\int_{-\infty}^{\lambda x} e^{-u^2/2}du\Big)dx \\
&= \frac{1}{\pi}\Big[-e^{-x^2/2}\Big(x^{n-1}\int_{-\infty}^{\lambda x}e^{-u^2/2}du\Big)\Big|_{-\infty}^{x_\alpha} \\
&\quad + \int_{-\infty}^{x_\alpha} e^{-x^2/2}\Big((n-1)x^{n-2}\int_{-\infty}^{\lambda x}e^{-u^2/2}du + x^{n-1}e^{-\lambda^2x^2/2}\lambda\Big)dx\Big]
\end{aligned}$$

We separate the above integral into three pieces

$$
\begin{aligned}
T_1(n, x_\alpha, \lambda) &= -e^{-x_\alpha^2/2} x_\alpha^{n-1} \int_{-\infty}^{\lambda x_\alpha} e^{-u^2/2} du \\
&= -e^{-x_\alpha^2/2} x_\alpha^{n-1} \Phi(\lambda x_\alpha) \sqrt{2\pi} \\
T_2(n, x_\alpha, \lambda) &= (n-1) \int_{-\infty}^{x_\alpha} \left(e^{-x^2/2} x^{n-2} \int_{-\infty}^{\lambda x} e^{-u^2/2} du \right) dx \\
&= \frac{\sqrt{2\pi}(n-1)}{2} \mathbb{E}[X^{n-2} \mathrm{I}_{X \le x_\alpha}] \\
T_3(n, x_\alpha, \lambda) &= \int_{-\infty}^{x_\alpha} \lambda e^{-\frac{x^2(1+\lambda)}{2}} x^{n-1} dx \\
&= \frac{\lambda\sqrt{2\pi}}{(\lambda+1)^{n/2}} \mathbb{E}[Z^{n-1} \mathrm{I}_{Z \le \sqrt{\lambda+1} x_\alpha}], \quad \text{where } Z \sim \mathcal{N}(0,1).
\end{aligned}
$$

Hence, for $n \geq 3$,

$$
\mathbb{E}[X^n \mathrm{I}_{\{X \le x_\alpha\}}] = \frac{1}{\pi} \Big(T_1(n, x_\alpha, \lambda) + T_2(n, x_\alpha, \lambda) + T_3(n, x_\alpha, \lambda) \Big)
$$

Thus, for $X \sim SN(0, 1, \lambda)$, $\mathbb{P}(X \le x_\alpha) = \alpha$ and for $n \geq 3$,

$$
\begin{aligned}
\mathbb{E}\Big[(x_\alpha - X)^n \mathrm{I}_{\{X \le x_\alpha\}}\Big] &= \mathbb{E}\Big[\sum_{i=0}^{n} \binom{n}{i} x_\alpha^{n-i} (-1)^i X^i \mathrm{I}_{\{X \le x_\alpha\}} \Big] \\
&= \sum_{i=0}^{n} \binom{n}{i} x_\alpha^{n-i} (-1)^i \mathbb{E}[X^i \mathrm{I}_{\{X \le x_\alpha\}}] \\
&= \mathbb{P}(X \le x_\alpha) x_\alpha^n - n x_\alpha^{n-1} \mathbb{E}[X \mathrm{I}_{\{X \le x_\alpha\}}] + \binom{n}{2} x_\alpha^{n-2} \mathbb{E}[X^2 \mathrm{I}_{\{X \le x_\alpha\}}] \\
&\quad + \sum_{i=3}^{n} \binom{n}{i} x_\alpha^{n-i} (-1)^i \frac{1}{\pi} \Big(T_1(i, x_\alpha, \lambda) + T_2(i, x_\alpha, \lambda) + T_3(i, x_\alpha, \lambda) \Big) \\
&= \alpha x_\alpha^n - n x_\alpha^{n-1} \mathbb{E}[X \mathrm{I}_{\{X \le x_\alpha\}}] + \binom{n}{2} x_\alpha^{n-2} \mathbb{E}[X^2 \mathrm{I}_{\{X \le x_\alpha\}}] \\
&\quad + \sum_{i=3}^{n} \binom{n}{i} x_\alpha^{n-i} (-1)^i \frac{1}{\pi} \Big(T_1(i, x_\alpha, \lambda) + T_2(i, x_\alpha, \lambda) + T_3(i, x_\alpha, \lambda) \Big)
\end{aligned}
$$

Hence, we conclude the proof. □

3.0.2.1 LPM's of Scaled Skew-Normal

We extend the above derivations to the LPM's of the scaled skew-normal random variables that are not standard; i.e. with scale parameter $\omega > 0$ and location parameter $\xi \in \mathbb{R}$ such that

$$
\begin{aligned}
Y &\triangleq \xi + \omega X \\
\tilde{y}_\alpha &\triangleq \frac{y_\alpha - \xi}{\omega},
\end{aligned}
$$

where $X \sim SN(0,1,\lambda)$. Here, y_α is the α percentile of Y satisfying $\mathbb{P}(Y \leq y_\alpha) = \alpha$. Below, we list the $\mathrm{LPM}_{(n,\xi,\omega,\lambda,y_\alpha)}(Y)$ for $n \geq 1$.

By (3.0.7), we have immediately the following lemma.

Lemma 3.0.11. *Let $Y \sim SN(\xi,\omega,\lambda)$ with y_α be the prespecified risk averseness level, $n \geq 0$ be an integer and $\lambda \in \mathbb{R}$. Denote $\tilde{y}_\alpha \triangleq \frac{y_\alpha - \xi}{\omega}$ Then,*

$$\begin{aligned}\mathrm{LPM}_{(n,\xi,\omega,\lambda,y_\alpha)}(Y) &= \omega \int_{-\infty}^{\tilde{y}_\alpha} (y_\alpha - \omega x - \xi)^n \psi(\omega x + \xi; \lambda) dx \\ &= \omega^{n+1} \int_{-\infty}^{\tilde{y}_\alpha} (\tilde{y}_\alpha - x)^n \psi(\omega x + \xi; \lambda) dx\end{aligned}$$

Next, we give the analog representation of LPM's of the standard skew normal r.v's.

Corollary 3.0.12. *Let $Y \sim SN(\xi,\omega,\lambda)$ with y_α being the prespecified risk averseness level, $\tilde{y}_\alpha = \frac{y_\alpha - \xi}{\omega}$ and $\lambda > 0$. Then,* LPM *for $n \in \mathbb{N}$, $\mathrm{LPM}_{(n,\xi,\omega,\lambda,y_\alpha)}$, is given as follows.*

- *(i) For $n = 0$, we have*

$$\begin{aligned}\mathrm{LPM}_{(0,\xi,\omega,\lambda,y_\alpha)}(Y) &= \mathbb{P}(Y \leq y_\alpha) \\ &= \alpha \\ &= \mathbb{P}(X \leq \tilde{y}_\alpha) \\ &= \mathbb{P}(X \leq x_\alpha)\end{aligned}$$

- *(ii) For $n = 1$, we have*

$$\begin{aligned}\mathrm{LPM}_{(1,\xi,\omega,\lambda,y_\alpha)}(Y) &= \mathbb{E}[(y_\alpha - Y)\mathrm{I}_{\{Y \leq y_\alpha\}}] \\ &= \omega \mathbb{E}[(\tilde{y}_\alpha - X)\mathrm{I}_{X \leq \tilde{y}_\alpha}] \\ &= \omega \frac{\sqrt{2}}{\sqrt{\pi}} \Big[\alpha \tilde{y}_\alpha - \exp(-\frac{\tilde{y}_\alpha^2}{2}) \Phi(\lambda \tilde{y}_\alpha) + \frac{\lambda}{\sqrt{1+\lambda^2}} \Phi(\sqrt{1+\lambda^2}\tilde{y}_\alpha) \Big]\end{aligned}$$

- *(iii) For $n = 2$, we have*

$$\begin{aligned}\mathrm{LPM}_{(2,\xi,\omega,\lambda,y_\alpha)} =& \mathbb{E}[(y_\alpha - Y)^2 \mathrm{I}_{\{Y \leq y_\alpha\}}] \\ &= \mathbb{E}[(y_\alpha - (\xi + \omega X))^2 \mathrm{I}_{\{Y \leq y_\alpha\}})] \\ &= \mathbb{E}[\omega^2 (\tilde{y}_\alpha - X)^2 \mathrm{I}_{\{X \leq \tilde{y}_\alpha\}}] \\ &= \omega^2 \big(\mathbb{P}(X \leq \tilde{y}_\alpha) \tilde{y}_\alpha^2 - 2\tilde{y}_\alpha \mathbb{E}[X \mathrm{I}_{\{Y \leq \tilde{y}_\alpha\}}] + \mathbb{E}[X^2 \mathrm{I}_{\{Y \leq \tilde{y}_\alpha\}}] \big),\end{aligned}$$

  where we use the explicit representations of (3.0.9) *and* (3.0.11) *above.*

- *(iv) For $n \geq 3$, we have*

$$
\begin{aligned}
\mathrm{LPM}_{(3,\xi,\omega,\lambda,y_\alpha)} &= \mathbb{E}[(y_\alpha - Y)^n \mathrm{I}_{\{Y \leq y_\alpha\}}] \\
&= \mathbb{E}[(y_\alpha - (\xi + \omega X))^n \mathrm{I}_{\{Y \leq y_\alpha\}})] \\
&= \mathbb{E}[\omega^n (\tilde{y}_\alpha - X)^n \mathrm{I}_{\{X \leq \frac{y_\alpha - \xi}{\omega}\}}] \\
&= \omega^n \mathbb{E}[(\tilde{y}_\alpha - X)^n \mathrm{I}_{\{X \leq \tilde{y}_\alpha\}}] \\
&= \omega^n \Big[\mathbb{P}(X \leq \tilde{y}_\alpha)\tilde{y}_\alpha^n - n\tilde{y}_\alpha^{n-1}\mathbb{E}[X\mathrm{I}_{\{X \leq \tilde{y}_\alpha\}}] + \binom{n}{2}\tilde{y}_\alpha^{n-2}\mathbb{E}[X^2\mathrm{I}_{\{X \leq \tilde{y}_\alpha\}}] \\
&\quad + \sum_{i=3}^{n} \binom{n}{i} \tilde{y}_\alpha^{n-i}(-1)^i \frac{1}{\pi}\Big(T_1(i,\tilde{y}_\alpha,\lambda) + T_2(i,\tilde{y}_\alpha,\lambda) + T_3(i,\tilde{y}_\alpha,\lambda)\Big)\Big],
\end{aligned}
$$

where T_1, T_2 and T_3 are as in (3.0.12)*,* (3.0.13) *and* (3.0.14)*, respectively.*

Remark 3.0.13. Take any skew normal random variable $X \sim SN(\xi,\omega,\lambda)$. Denote $\mu \in \mathbb{R}$ and $\sigma > 0$ as the expected value and standard deviation of X, respectively. Then, skewness of X, denoted by skew(X), is

$$
\begin{aligned}
\mathrm{skew}(X) &\triangleq \mathbb{E}\Big[\Big(\frac{X-\mu}{\sigma}\Big)^3\Big] \\
&= \frac{4-\pi}{2}\frac{\Big(\delta\sqrt{\frac{2}{\pi}}\Big)^3}{\Big(1-\delta^2\frac{2}{\pi}\Big)^{3/2}},
\end{aligned}
\tag{3.0.15}
$$

where $\delta \triangleq \frac{\lambda}{\sqrt{1+\lambda^2}}$. Similarly, the kurtosis of $X \sim SN(\xi,\omega,\lambda)$, denoted by kurt$(X)$, is

$$
\begin{aligned}
\mathrm{kurt}(X) &\triangleq \Big[\Big(\frac{X-\mu}{\sigma}\Big)^4\Big] \\
&= 2(\pi-3)\frac{(\delta\sqrt{2/\pi})^4}{(1-2\delta^2/\pi)^2}.
\end{aligned}
\tag{3.0.16}
$$

In particular, both the skewness and kurtosis of a skew-normal random variable are bounded from above and below. Thus, it is necessary to enlarge the family of skew normal family to skew elliptical distributions, using centered student-t distribution instead of normal distribution in (3.0.6), in case one can not conclude statistically the data is drawn from a skew normal distribution.

3.0.3 LPM's of Skew-t Distributions

Recall that a random variable T having standard student-t distribution with ν degrees of freedom is defined as

$$
T \triangleq \frac{Z}{\sqrt{\mathcal{X}_\nu^2}}. \tag{3.0.17}
$$

with the PDF and CDF of T denoted by $f_\nu(\cdot)$ and $F_\nu(\cdot)$, respectively. In (3.0.17), $Z \sim \mathcal{N}(0,1)$ is the standard normal random variable and $G \triangleq \sqrt{\mathcal{X}_\nu^2}$ has Gamma distribution, $\Gamma(\nu, 1)$, independent of Z, and density function

$$f_G(g) = \frac{\nu^{\nu/2}}{2^{\nu/2}\Gamma(\nu/2)} g^{\nu/2-1} \exp(-\frac{\nu}{2}g), \text{ for } g > 0. \tag{3.0.18}$$

In particular, conditioned on $G = g$, T_g is a normal r.v. with mean 0 and variance $\frac{1}{g}$, i.e. $T_g = \frac{1}{\sqrt{g}}Z$, i.e. $T_g \sim \mathcal{N}(0, 1/g)$.

Definition 3.0.14. Let f_ν and F_ν be the centered student-t density and cumulative distribution functions with location $\xi = 0$, scale $\omega = 1$, $\nu \in \mathbb{N}$ degrees of freedom of T in (3.0.17). Then, the probability density function of skewed-t random variable V, denoted by $V \sim ST(0, 1, \lambda, \nu)$, is defined as

$$f^V_{\nu,\lambda}(t) \triangleq 2 f_\nu(t) F_\nu(\lambda t), \text{ for } t \in \mathbb{R}. \tag{3.0.19}$$

3.0.3.1 LPM's of Scaled Skew-t Distribution

Lemma 3.0.15. *Let $U \sim \mathrm{ST}(\xi, \omega, \lambda, \nu)$ be the scaled skewed-t distribution with $\xi \in \mathbb{R}$ location and $\omega > 0$ scale parameter*

$$U \triangleq \xi + \omega V, \tag{3.0.20}$$

where $V \sim \mathrm{ST}(0, 1, \lambda, \nu)$ as in Definition 3.0.14. Let u_α be the corresponding α-quantile of U for a prespecified α value, i.e. $\mathbb{P}(U \le u_\alpha) = \alpha$. Denote $\tilde{u}_\alpha \triangleq \frac{u_\alpha\sqrt{g}-\xi}{\omega}$. Then, LPM's of U for $n \in \mathbb{N}$, denoted by $\mathrm{LPM}_{(n,\xi,\omega,\lambda,\nu,u_\alpha)}(U)$, are given below.
LPM For $n = 0$, we have,

$$\begin{aligned} \mathrm{LPM}(U \le u_\alpha) &= \int_0^\infty \mathbb{P}\Big(Z \le \tilde{u}_\alpha\Big) f_G(g)dg \\ &= \alpha. \end{aligned} \tag{3.0.21}$$

For $n = 1$,

$$\begin{aligned} \mathrm{LPM}_{(1,\xi,\omega,\lambda,\nu,u_\alpha)}(U) &= \mathbb{E}[(u_\alpha - U)\mathrm{I}_{\{U \le u_\alpha\}}] \\ &= \alpha u_\alpha - \mathbb{E}[U\mathrm{I}_{\{U \le u_\alpha\}}] \\ &= \alpha u_\alpha + \int_0^\infty \frac{1}{\sqrt{g}} \phi(\tilde{u}_\alpha) f_G(g)dg. \end{aligned} \tag{3.0.22}$$

For $n = 2$,

$$\begin{aligned} \mathrm{LPM}_{(2,\xi,\omega,\lambda,\nu,u_\alpha)}(U) &= \mathbb{E}[(u_\alpha - U)^2\mathrm{I}_{\{U \le u_\alpha\}}] \\ &= \alpha u_\alpha^2 - 2u_\alpha \mathbb{E}[U\mathrm{I}_{U \le u_\alpha}] + \mathbb{E}[U^2\mathrm{I}_{U \le u_\alpha}], \end{aligned} \tag{3.0.23}$$

where by (3.0.3) *and* (3.0.4)

$$\begin{aligned}\mathbb{E}[U^2 \mathrm{I}_{U\le u_\alpha}] &= \int_0^\infty \frac{1}{g}\mathbb{E}\Big[Z^2 \mathrm{I}_{Z\le \tilde{u}_\alpha}|G=g\Big] f_g dg \\ &= \int_0^\infty \frac{1}{g}\Big(-\phi(\tilde{u}_\alpha)+\Phi(\tilde{u}_\alpha)\Big) f_G(g)dg.\end{aligned} \tag{3.0.24}$$

Similarly, for $n \geq 3$ *we have*

$$\mathbb{E}[(u_\alpha - U)^n \mathrm{I}_{\{U\le u_\alpha\}}] = \omega^n \int_0^\infty \frac{1}{g^{n/2}}\left(\mathbb{E}\Big[\big(\tilde{u}_\alpha - Z\big)^n \mathrm{I}_{\{Z\le\tilde{u}_\alpha\}}|G=g\Big]\right) f_G(g)dg \tag{3.0.25}$$

The term inside the integral reads as

$$\mathbb{E}\Big[\big(\tilde{u}_\alpha - Z\big)^n \mathrm{I}_{\{Z\le\tilde{u}_\alpha\}}|G=g\Big] = \sum_{i=0}^n \binom{n}{i}\tilde{u}_\alpha^i(-1)^{n-i}\mathbb{E}[Z^{n-i}\mathrm{I}_{\{Z\le\tilde{u}_\alpha\}}], \tag{3.0.26}$$

by (3.0.3) *and* (3.0.4) *giving the explicit representations. Alternatively, denoting* $\tilde{u}_\alpha = \frac{u_\alpha - \xi}{\omega}$ *and using* (3.0.19) *directly,*

$$\begin{aligned}\mathrm{LPM}_{(n,\xi,\omega,\lambda,\nu,u_\alpha)}(U) &= \omega \int_{-\infty}^{\frac{u_\alpha-\xi}{\omega}} (u_\alpha - \omega x - \xi)^n f^V_{(\lambda,\nu)}\Big(\omega x + \xi\Big)dx \\ \mathrm{LPM}_{(n,\xi,\omega,\lambda,\nu,u_\alpha)}(U) &= \omega^{n+1}\int_{-\infty}^{\tilde{u}_\alpha} (\tilde{u}_\alpha - x)^n f^V_{(\lambda,\nu)}(\omega x + \xi;\lambda)dx\end{aligned} \tag{3.0.27}$$

Chapter 4

Simulation Study

In this section, we will check statistically which of skew normal, skew-t and normal distribution is appropriate for modeling the data of stock prices. This determination is made by applying procedure through simulations and comparing results by chi-square goodness-of-fit test. In simulations below, for skew-t distribution, degree of freedom is taken as $\nu = 5$.

4.0.1 Methodology

The data set covers historical stock prices of Amazon, Apple, Exxon Mobile, Pfizer and Warner Bros within the period from 11 April 2022 to 22 June 2022, and comprises 50 returns of each assets and daily historical stock prices of Volkswagen from 11 April to 9 August 2022 consists of 100 elements.

Remark 4.0.1. Majority of the selected datasets has limited sample size because by increasing the sample size namely increasing time interval, it is observed that only skew-t model fits the data, and eventually, by increasing time interval enough, none of the three distributions fits the data, anymore. In particular, for time horizon $T = 200$, we observe that none of the three distributions can be statistically justified that they are drawn from the skew elliptical family, thus LPM as a risk evaluation operator is not reliable anymore.

To conduct numerical simulations, we used R programming language and sn package. The methodology is as follows. First, logarithmic differences of stock prices are taken in the form

$$G_t \triangleq \ln(P_{t+1}) - \ln(P_t) \tag{4.0.1}$$

where P_{t+1} and P_t are the closing prices for a given asset at days $t+1$ and t, respectively, for $t \geq 0$. Next, we fit the data set of the log difference of the stock prices G_t to the skew normal $SN(\xi, \omega, \lambda)$ and skew- t $ST(\bar{\xi}, \bar{\omega}, \bar{\lambda})$ distribution by the sn R-package and obtain (ξ, ω, λ) and $(\bar{\xi}, \bar{\omega}, \bar{\lambda})$, respectively. Here, we emphasize that we do not calibrate the optimal $\bar{\nu}$ for skew-t distribution, since we will calculate lower partial moments up to $n = 4$. Hence, we take $\nu = 5$, such that

$$\begin{aligned}
f_5(x) &= \frac{3}{8(1+\frac{x^2}{5})^3} \\
F_5(x) &= \frac{1}{2} + \frac{1}{\pi}\Big[\frac{x}{\sqrt{5}(1+\frac{x^2}{5})}\Big(1+\frac{2}{3(1+\frac{x^2}{5})}\Big) + \arctan\Big(\frac{x}{\sqrt{5}}\Big)\Big] \\
f^V_{\lambda,5}(x) &= 2f_5(x)F_5(\lambda x), \text{ for } x \in \mathbb{R}.
\end{aligned}$$

Furthermore, we fit the data set G_t to the normal distributions by fitdist function of the R to find the estimators for mean and standard deviation, μ and σ, respectively. Then, empirical CDF of the data and the theoretical CDF of the corresponding distribution are used in χ^2 goodness of fit test. To summarise, parameters of the distributions to be estimated are as follows:

- **Normal**: μ - mean, σ - variance
- **Skew-normal**: ξ - location, ω - scale and λ - shape
- **Skew-t**: ξ - location, ω - scale, λ - shape.

  Remark 4.0.2. Recall the first four moments of the random variable are mean, variance, skewness, and kurtosis. Each of these four moments gives information about the dataset, such as central location, dispersion, asymmetry, and outliers. Thus, we want to find these first 4 moments. Hence, due to integrability condition we need degrees of freedom more than 4.Thus, we chose $\nu = 5$ for explicit representations of the first 4 moments. Note that the χ^2 test does not reject the hypothesis the data is drawn under $\nu = 5$.

4.0.2 Chi square goodness of fit test

To verify statistically, χ^2 goodness of fit test is conducted to determine, whether the data is drawn from a specified distribution.

Definition 4.0.3. The test statistic for the χ^2 is defined as

$$\chi^2 \triangleq \sum_{i=1}^{K} \frac{(O_i - E_i)^2}{E_i} \tag{4.0.2}$$

where K is the number of bins, O_i is observed value for the bin i, and E_i is expected value for that bin $i.E_i$ denotes the number of observations expected to fall in bin i based on the probability model under test.

If this model is specified in terms of its cumulative distribution function F_θ, then the expected counts are computed as $n \cdot (F(y_{i+1}) - F(y_i))$ where n denotes the sample size, and under the convention that $F(-\infty) = 0$ and $F(\infty) = 1$.

Property 4.0.4. For sufficiently large values of n, Pearson's chi-square test statistic has approximately a chi-square distribution with k–1 degrees of freedom, i.e. $\chi 2(k\breve{}1)$.

Property 1 is used to perform what is called goodness of fit testing, where we check to see whether the observed data correspond sufficiently well to the expected values. In order to apply such tests, the following assumptions must be met.

- Random sample: Data must come from a random sampling of a population.
- Independence: The observations must be independent of each other. This means that the chi-square test cannot be used to test correlated data.
- Bin size: $k \geq 5$ and the expected frequencies $E_i \geq 5$. If the expected frequency for one or more binss is less than 5, it may be beneficial to combine one or more contiguous binss so that this condition can be met .

If the statistic $\chi 2$ is less than the prespecied p-value, then we do not reject the hypothesis; otherwise we reject the null hypothesis that the sample data is drawn from the claimed distribution. In order to perform the χ^2 test and specify the p-value, the α level is fixed. For large values of k, a small percentage of cells with an expected frequency of less than 5 can be acceptable. In any event, no cell should have an expected frequency of less than 1.

4.0.3 The interquartile range (IQR)

The interquartile range (IQR) as a measure of statistical dispersion is found as the difference between the $75th$ and $25th$ percentiles of the data, and the Freedman-Diaconis Rule is

$$B_w = 2\frac{\text{IQR}}{N^{1/3}}, \tag{4.0.3}$$

where IQR is the interquartile range of the data, B_w is the bin width. N is the sample size.

```r
#read excel file
vowg <- read_excel("C:/Users/Asus/Desktop/Gulnaz/Port_2023/Port_5_nonuni.xlsx")

#omit the NaN values
vowg_cln <- na.omit(vowg$ln)
#minimum and maximum of the values
start = min(vowg_cln)
start
end = max(vowg_cln)
end

#find interquartile
iqr_vowg_ln <- IQR(vowg_cln)

#find bin width
bin_width = (2 * iqr_vowg_ln) / (length(vowg_cln) ** (1 / 3))
bin_width
```

Figure 4.1: Estimation of bin width

First, we estimate the bins with B_w to find frequencies and create histogram according to the formula by ordering G_t's and using B_w as in (4.0.3). As an example, in Table 1,bin estimation of the below portfolio X is given as

$$X = \frac{1}{5}S_{AAPL} + \frac{1}{5}S_{AMZN} + \frac{1}{5}S_{PFE} + \frac{1}{5}S_{XOM} + \frac{1}{5}S_{WBD}.$$

Here, 0.01729449 is the bin width calculated according to IQR.

Bin	Frequency
-0.0066	1
-0.0066+0.0172	1
-0.0319	6
-0.0146	7
0.0026	11
0.0199	13
0.0372	10
More	0

Table 4.1: Estimation of bins

In Table 2, we summarise the abbreviations used in numerical examples below.

Name	*Explanation*
S_{AAPL}	Stock price of Apple
S_{AMZN}	Stock price of Amazon
S_{PFE}	Stock price of Pfizer
S_{XOM}	Stock price of Exxon Mobile
S_{WBD}	Stock price of Warner Bros
S_{VWAGY}	Stock price of Volkswagen
$\mathbb{P}(a_i < X \leq b_i)$	Probability that the portfolio X is between start value a_i and end value b_i of bin i.
E_i	Number of elements $\cdot\ \mathbb{P}(a < x < b) = np_i$ = Expected in category i
M_Freq	Frequencies less than 5 are merged by rule of thumb
M_Exp	Expected values are merged along with the frequency column
χ^2	$(M\_Freq - M\_Exp)^2/M\_Exp$

Table 4.2: Abbreviations

4.0.4 Examples

In this subsection, we now give below specific portfolios composed of the stocks in Table 2.

4.0.4.1 Portfolio 1

The portfolio X is composed of the following convex combination of stock prices of Apple, Amazon, Pfizer, Exxon Mobile and Warner Bros.

$$X = \frac{1}{4}S_{AAPL} + \frac{1}{3}S_{AMZN} + \frac{1}{6}S_{PFE} + \frac{1}{8}S_{XOM} + \frac{1}{8}S_{WBD}$$

We begin by calculating the probability that $x < b_i$ for $b_i = 0, 1, \ldots, n$ where n is number of elements assuming a skew-normal distribution, skew-t distribution and a normal distribution respectively. For skew-normal the location, scale and skewness parameters are estimated by following code by applying "SN" package of R programming language.

```
#find scale, location and skeness parameter of skew normal
vowg_sn <- selm(ln ~ 1, family="SN", vowg)
vowg_sn
summary(vowg_sn)
coef(vowg_sn)
params_sn <- vowg_sn@param$dp
xi = params_sn[1]
omega = params_sn[2]
alpha = params_sn[3]
```

Figure 4.2: Skew-normal distribution parameters

For skew-t the parameters are estimated by the "SN" package as:

```
vowg_st <- selm(ln~ 1, family="ST", data=vowg, fixed.param=list(nu=5))
coef(vowg_st)
summary(vowg_st)
vowg_st@param$dp
params <- vowg_st@param$dp
xi_st = params[1]
xi_st
omega_st = params[2]
omega_st
alpha_st = params[3]
alpha_st
```

Figure 4.3: Skew-t distribution parameters

For normal distribution the parameters are estimated by "fitdistr" function of R as:

```
paraw <- fitdistr(vowg_cln, densfun="normal")
paraw
vowg_norm_params <- paraw$estimate
mu = vowg_norm_params[1]
mu
sigma = vowg_norm_params[2]
sigma
```

Figure 4.4: Normal-Distribution parameters

Then the probability $P(X \leq b_i)$ is the CDF of each fitted distribution and for the skew normal case estimated as :

```
#find cdf of skew normal
cdf_sn<-psn(x=seq(start,end+bin_width, by=bin_width),xi= xi, omega=omega
,alpha=alpha)
seq(start,end+bin_width, by=bin_width)
df<-data.frame(cdf_sn)
cdf_sn
```

Figure 4.5: Estimation od CDF of skew-normal

The probability that x is in the interval $(a_i, b_i]$ is then

$$P(a_i < X \leq b_i) = P(X \leq b_{i+1}) - P(X \leq b_i) \tag{4.0.4}$$

The tabulated values belong to the cumulative distribution $P(a_i < x < b_i)$ is used for expected frequency.

$\mathbb{P}(X \leq b_i)$	$\mathbb{P}(a_i < X \leq b_i)$	E_i	M_Freq	**M**_Exp	χ^2-test statistics
0.0108	0.0108	0.5305			
0.0351	0.0242	1.1901			
0.0957	0.0606	2.9711	6	4.1612	0.8124
0.2207	0.1249	6.1238	5	6.1238	0.2062
0.4333	0.2126	10.4207	8	10.4207	0.5623
0.7277	0.2944	14.4257	18	14.4257	0.8855
0.9626	0.2348	11.5059	12	13.3191	0.1306
0.9996	0.03701	1.8132			
				χ^2	2.5972
				p-value	3.8414

Table 4.3: Chi square test for skew-normal for Portfolio 1

Test statistic is $\chi^2 = 2.5972$, which is less than $p = 3.8414$. Thus, we do not reject the hypothesis that the skew-normal model fits the data.

$\mathbb{P}(X \leq b_i)$	$\mathbb{P}(a_i < X \leq b_i)$	E_i	M_Freq	M_Exp	χ^2-test statistics
0.0271	0.0271	1.3297			
0.0521	0.0249	1.2224			
0.1041	0.0520	2.5501	6	3.7726	1.3149
0.2120	0.1079	5.2876	5	5.2876	0.0156
0.4191	0.2070	10.1434	8	10.1434	0.4529
0.7332	0.3141	15.3955	18	15.3955	0.4405
0.9588	0.2255	11.0539	12	12.8920	0.0617
0.9963	0.0375	1.8381			
				χ^2	2.2858
				p-value	3.8414

Table 4.4: Chi square test for skew-t for Portfolio 1

The Chi-square statistic χ^2 is 2.2858. It is less than $p = 3.8414$. Thus,we do not reject the hypothesis skew-t model fits the data. Chi-square statistic $\chi^2 = 6.5508$ is greater than the p-value. Thus, normal model does not fit the data. According to the chi-square goodness of fit test χ^2 values for skew normal is 2.5972, for skew-t is 2.2858 and for normal is 6.5508 where skew-t model is the best model which fits the data. The statistics suggest the data is skewed enough to reject the hypothesis that it is drawn from normal distribution, whereas skew-normal and skew-t distribution hypothesis can not be rejected. Moreover, from the P-P plot it is clearly seen that skew-t model better fits the portfolio than skew-normal and normal model.

]

$\mathbb{P}(X \leq b_i)$	$\mathbb{P}(a_i < X \leq b_i)$	E_i	M_Freq	M_Exp	χ^2-test statistics
0.0022	0.0022	0.1088			
0.0164	0.0142	0.6973			
0.0775	0.0611	2.9959	6	3.6933	1.4406
0.2389	0.1613	7.9069	5	7.9069	1.0687
0.5008	0.2618	12.8327	8	12.8327	1.8199
0.7623	0.2615	12.8140	18	12.8140	2.0988
0.9230	0.1606	7.8724	12	10.8465	0.1226
0.9837	0.0606	2.9741			
				χ^2	6.5508
				p-value	5.9914

Table 4.5: Chi square test for normal for Portfolio 1

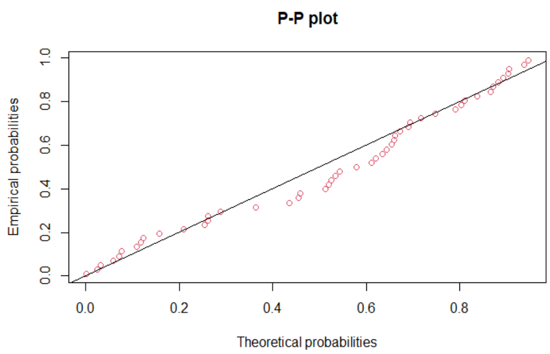

Figure 4.6: Portfolio 1:P-P plot of normal

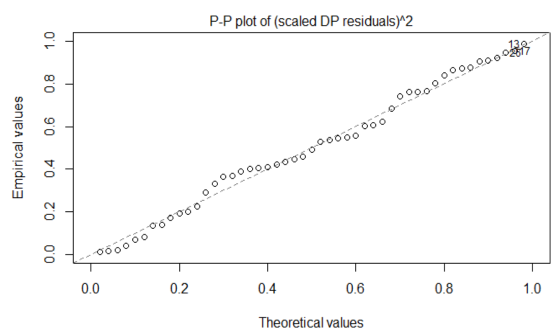

Figure 4.7: Portfolio 1: P-P plot of skew-normal

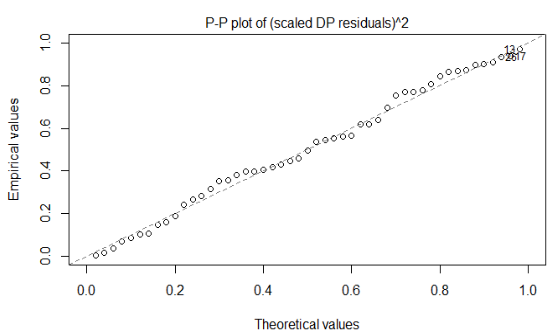

Figure 4.8: Portfolio 1: P-P plot of skew-t

The calibrated model parameters are to be seen in Table 6 below. The degree of freedom parameter of skew-t model is taken as $\nu = 5$. Lower partial moments for $n = 1$, $n = 2$, $n = 3$ and $n = 4$ are calculated with the calibrated parameters.

α-value	Skew-normal	Skew-t
$\alpha = 0.05$	$\xi = 0.0301$	$\xi = 0.0262$
	$\omega = 0.0441$	$\omega = 0.0352$
	$\lambda = -4.4752$	$\lambda = -2.9182$
	$LPM_{(n,\xi,\omega,\lambda,\alpha)}$	$LPM_{(n,\xi,\omega,\lambda,\nu=5,\alpha)}$
$n = 1$	0.0325	0.0301
$n = 2$	0.0402	0.0534
$n = 3$	0.0633	0.1540
$n = 4$	0.1174	0.8282

Table 4.6: Lower partial moments for Portfolio 1

4.0.5 Portfolio 2

The portfolio X is composed of the following convex combination of Apple, Amazon, Pfizer, Exxon Mobile and Warner Bros.

$$\frac{1}{5}S_{AAPL} + \frac{1}{5}S_{AMZN} + \frac{1}{5}S_{PFE} + \frac{1}{5}S_{XOM} + \frac{1}{5}S_{WBD}$$

The data suggests that the data is skewed enough to reject the hypothesis that it is drawn from normal distribution, whereas skew-normal and skew-t distribution hypothesis can not be rejected.

$\mathbb{P}(X \leq b_i)$	$\mathbb{P}(a_i < X \leq b_i)$	E_i	M_Freq	M_Exp	χ^2-test statistics
0.0171	0.0171	0.8395			
0.0524	0.0353	1.7315			
0.1349	0.0825	4.0438	8	6.6149	0.2901
0.2936	0.1586	7.7749	7	7.7749	0.0772
0.5440	0.2503	12.2674	11	12.2674	0.1309
0.8365	0.2924	14.3314	13	14.3314	0.1237
0.9844	0.1479	7.2497	10	7.2497	1.0433
				χ^2	1.6652
				p-value	5.9914

Table 4.7: Chi square test for skew-normal for Portfolio 2

Chi-square test statistic $\chi^2 = 1.6652$ is less than p- value $= 5.9914$. Thus, we do not reject the hypothesis so that the skew-normal model fits the data.

$\mathbb{P}(X \leq b_i)$	$\mathbb{P}(a_i < X \leq b_i)$	E_i	M_Freq	M_Exp	χ^2-test statistics
0.0339	0.0339	1.6635			
0.0669	0.0330	1.6188			
0.1369	0.0699	3.4285	8	6.7109	0.2476
0.2805	0.1436	7.0383	7	7.0383	0.0002
0.5374	0.2568	12.5852	11	12.5852	0.1996
0.8429	0.3054	14.9687	13	14.9687	0.2589
0.9775	0.1346	6.5962	10	6.5962	1.7563
				χ^2	2.4628
				p-value	3.8414

Table 4.8: Chi square test for skew-t for Portfolio 2

Chi-square test statistic $\chi^2 = 2.4628$ is less than p - value $= 3.8414$. Thus we do not reject the hypothesis so that the skew-t model fits the data

$\mathbb{P}(X \leq b_i)$	$\mathbb{P}(a_i < X \leq b_i)$	E_i	M_Freq	M_Exp	χ^2-test statistics
0.0054	0.0054	0.2688			
0.0328	0.0273	1.3390			
0.1274	0.0946	4.6386	8	6.2465	0.4922
0.3314	0.2040	9.9964	7	9.9964	0.8982
0.6051	0.2736	13.4107	11	13.4107	0.4333
0.8338	0.22864	11.2032	13	11.2032	0.2881
0.9527	0.11895	5.8268	10	5.8268	2.9887
				χ^2	5.1007
				p-value	3.8414

Table 4.9: Chi square test for normal for Portfolio 2

Chi-square statistic $\chi^2 = 5.1007$ is greater than the p-value. Thus, normal model does not fit the data. We provide below LPM's for Portfolio in Table 10. The parameters after model calibration is estimated as below table. The degree of freedom parameter of skew-t model is taken as $\nu = 5$. Lower partial moments for $n = 1$, $n = 2$,$n = 3$ and $n = 4$ are calculated with the calibrated parameters.

$\alpha - value$	Skew-normal	Skew-t
$\alpha = 0.05$	$\xi = 0.0261$	$\xi = 0.0222$
	$\omega = 0.0389$	$\omega = 0.0306$
	$\lambda = -3.6549$	$\lambda = -2.4006$
	$LPM_{(n,\xi,\omega,\lambda,\alpha)}$	$LPM_{(n,\xi,\omega,\lambda,\nu=5,\alpha)}$
$n = 1$	0.0287	0.0261
$n = 2$	0.0357	0.0468
$n = 3$	0.0561	0.1351
$n = 4$	0.1048	0.7257

Table 4.10: Lower partial moments for Portfolio 2

Moreover, from the P-P plot and χ^2 goodness-of-fit test, it is clearly seen that skew-t model better fits the portfolio than skew-normal and normal model.

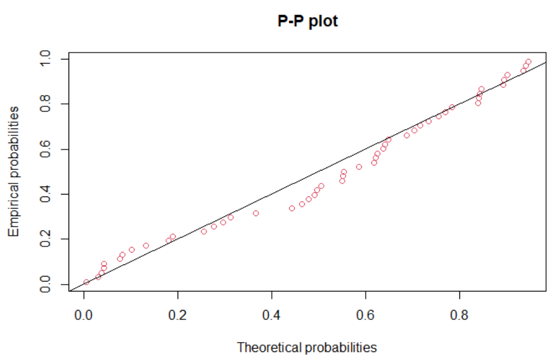

Figure 4.9: Portfolio 2:P-P plot of normal

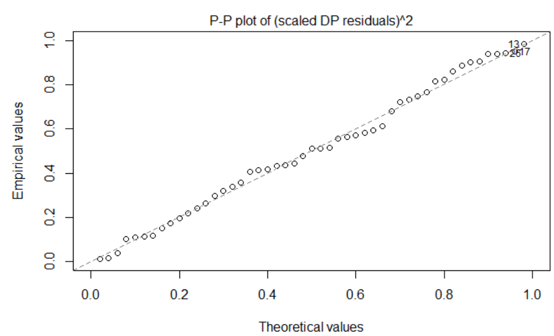

Figure 4.10: Portfolio 2:P-P plot of skew-normal

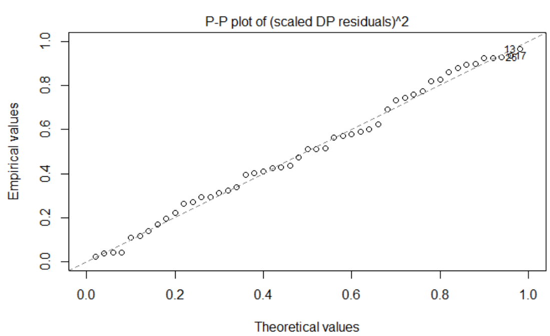

Figure 4.11: Portfolio 2: P-P plot of skew-t

4.0.6 Case 3

In this case, LPM's of Portfolio 1 and Portfolio 2 are compared with respect to different α values for skew normal and skew-t distributions. The comparison of LPM with

$n = 1, 2, 3, 4$ of the and $\alpha = 0.01, 0.05, 0.10, 0.90, 0.95$ for skew-t and skew-normal model fitting are illustrated in Table 11 and Table 12, respectively.

	Portfolio 1	Portfolio 2
	$\xi = 0.0262$ $\omega = 0.0352$ $\lambda = -2.9182$	$\xi = 0.0222$ $\omega = 0.0306$ $\lambda = -2.4006$
	$LPM_{(n,\xi,\omega,\lambda,\alpha)}$	$LPM_{(n,\xi,\omega,\lambda,\nu=5,\alpha)}$
$\alpha = 0.01$		
$n = 1$	0.0285	0.0249
$n = 2$	0.0504	0.0445
$n = 3$	0.1459	0.1289
$n = 4$	0.7974	0.7021
$\alpha = 0.05$		
$n = 1$	0.0301	0.0261
$n = 2$	0.0534	0.0468
$n = 3$	0.1540	0.1351
$n = 4$	0.8282	0.7257
$\alpha = 0.10$		
$n = 1$	0.0306	0.0265
$n = 2$	0.0546	0.04775
$n = 3$	0.1571	0.1375
$n = 4$	0.8404	0.7351
$\alpha = 0.90$		
$n = 1$	0.0328	0.0282
$n = 2$	0.0592	0.0512
$n = 3$	0.1693	0.1470
$n = 4$	0.8871	0.7715
$\alpha = 0.95$		
$n = 1$	0.0330	0.0283
$n = 2$	0.0596	0.0515
$n = 3$	0.1704	0.1479
$n = 4$	0.8912	0.7751

Table 4.11: Comparison of LPMs for skew-t calibration with different α - values

According to Table 11, for the increasing α values, Portfolio 1 is more risky than Portfolio 2. Thus, Portfolio 2 is a better choice as an investment with respect to LPM operator.

	Portfolio 1	Portfolio 2
	$\xi = 0.0301$	$\xi = 0.0261$
	$\omega = 0.0441$	$\omega = 0.0389$
	$\lambda = -4.4752$	$\lambda = -3.6549$
	$LPM_{(n,\xi,\omega,\lambda,\alpha)}$	$LPM_{(n,\xi,\omega,\lambda,\nu=5,\alpha)}$
$\alpha = 0.01$		
$n = 1$	0.0314	0.0278
$n = 2$	0.0385	0.0344
$n = 3$	0.0601	0.0539
$n = 4$	0.1107	0.0995
$\alpha = 0.05$		
$n = 1$	0.0325	0.0287
$n = 2$	0.0402	0.0357
$n = 3$	0.0633	0.0564
$n = 4$	0.1174	0.1048
$\alpha = 0.10$		
$n = 1$	0.0331	0.0291
$n = 2$	0.0412	0.0364
$n = 3$	0.0650	0.0577
$n = 4$	0.1210	0.1076
$\alpha = 0.90$		
$n = 1$	0.0359	0.0313
$n = 2$	0.0460	0.0402
$n = 3$	0.0741	0.0649
$n = 4$	0.1404	0.1229
$\alpha = 0.95$		
$n = 1$	0.0361	0.0315
$n = 2$	0.0464	0.0406
$n = 3$	0.0749	0.0656
$n = 4$	0.1420	0.1243

Table 4.12: Comparison of LPM's for skew-normal calibration with different α-values

4.0.7 Portfolio 4

For some data samples all three distributions fit the data. For example, consider stock price of Amazon data set - S_{AMZN}. The chi-square test results are as below:

$\mathbb{P}(X \leq b_i)$	$\mathbb{P}(a_i < X \leq b_i)$	E_i	M_Freq	M_Exp	χ^2-test statistics
0.0110	0.0110	0.5418			
0.0202	0.0091	0.4494			
0.0389	0.0187	0.9196			
0.0789	0.0399	1.9571			
0.1649	0.0859	4.2126	8	6.71165	0.2472
0.3404	0.1755	8.6020	9	8.6020	0.0184
0.63601	0.2955	14.4816	15	14.4816	0.0185
0.9139	0.2779	13.6197	11	13.6197	0.5038
0.9903	0.0763	3.7418	6	3.7418	1.3628
				χ^2	2.1509
				p-value	5.9914

Table 4.13: χ^2-test for skew-normal for Portfolio 4

$\chi^2 = 2.1509$, thus it is less than 5.9914, and we do not reject the hypothesis the skew-normal model fits the data.

$\mathbb{P}(X \leq b_i)$	$\mathbb{P}(a_i < X \leq b_i)$	E_i	M_Freq	M_Exp	χ^2-test statistics
0.0024	0.0024	0.1202			
0.0091	0.0066	0.3248			
0.0286	0.0195	0.9569			
0.0769	0.0483	2.3685			
0.1774	0.1005	4.9260	8	7.4148	0.0461
0.3531	0.1756	8.6091	9	8.6091	0.0177
0.6112	0.2581	12.6422	15	12.6422	0.4397
0.9053	0.2941	14.4153	11	14.4153	0.8091
0.9989	0.0936	4.5866	6	4.5866	0.4354
				χ^2	1.7482
				p-value	3.8414

Table 4.14: χ^2-test for skew-t for Portfolio 4

χ^2 test statistic is 1.7482 which is less than $p = 5.9914$. Thus, we do not reject the hypothesis skew-t model fits the data.

$\mathbb{P}(X \leq b_i)$	$\mathbb{P}(a_i < X \leq b_i)$	E_i	M_Freq	M_Exp	χ^2-test statistics
0.0001	0.0001	0.0052			
0.0013	0.0012	0.0597			
0.0104	0.0091	0.4492			
0.0536	0.0431	2.1122			
0.1804	0.1268	6.2165	8	8.3340	0.0133
0.4143	0.2339	11.4622	9	11.4622	0.5289
0.6847	0.2703	13.2481	15	13.2481	0.2316
0.8806	0.1959	9.5997	11	9.5997	0.2042
0.9696	0.0889	4.3597	6	4.3597	0.6171
				χ^2	1.9719
				p-value	3.8414

Table 4.15: χ^2-test for normal for Portfolio 4

$\chi = 1.9719$ is less than the p-value with $p = 3.8414$. Thus, we do not reject the hypothesis that the data is drawn from the normal distribution. Also, from the the PP plots it is clearly seen that all three models fit the data.

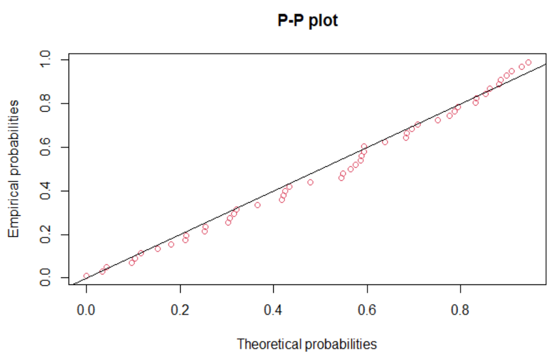

Figure 4.12: Portfolio 4: P-P plot of normal

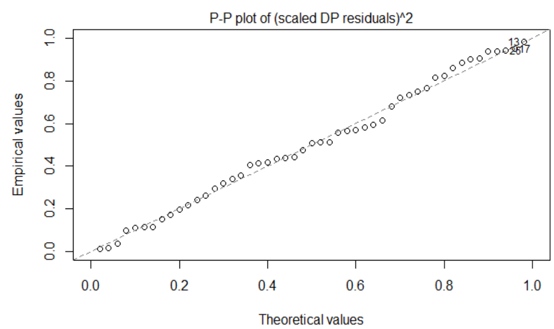

Figure 4.13: Portfolio4:P-P plot of skew-normal

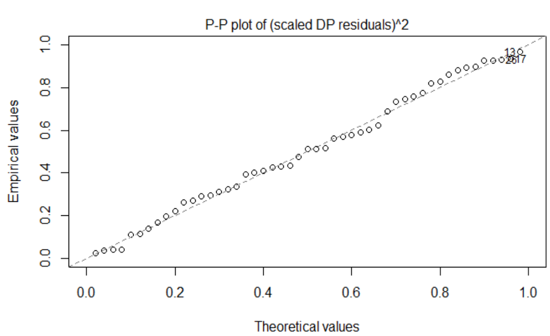

Figure 4.14: Portfolio 4: P-P plot of skew-t

The parameters of normal, skew-normal and skew-t models are given below. Moreover, the LPM of degree 1,2,3 and 4 are provided. For skew-t model degree of freedom is taken as $\nu = 5$.

$\alpha - value$	Normal	Skew-normal	Skew-t
$\alpha = 0.05$	$\mu = -0.0066$	$\xi = 0.0451$	$\xi = 0.0343$
	$\sigma = 0.0391$	$\omega = 0.0648$	$\omega = 0.0472$
		$\lambda = -6.0471$	$\lambda = -2.4811$
	$\text{LPM}_{(n,\xi,\omega,\alpha)}$	$\text{LPM}_{(n,\xi,\omega,\lambda,\alpha)}$	$\text{LPM}_{(n,\xi,\omega,\lambda,\nu=5,\alpha)}$
$n = 1$	0.3404	0.0464	0.0391
$n = 2$	0.4572	0.0567	0.0697
$n = 3$	0.7149	0.0885	0.2016
$n = 4$	1.3252	0.1631	1.0928

Table 4.16: LPM's for Portfolio 4

4.0.8 Case 5

The portfolio is composed of the following non-uniform convex combination of 4 datasets Amazon ,Pfizer, Exxon and Warner Bros.

$$\frac{1}{6}S_{AMZN} + \frac{1}{6}S_{PFE} + \frac{1}{3}S_{XOM} + \frac{1}{3}S_{WBD}$$

The data suggest that it is skewed enough to reject the hypothesis that it is drawn from a normal distribution, whereas the hypotheses of skew-normal and skew-t distributions cannot be rejected.

$\mathbb{P}(X \leq b_i)$	$\mathbb{P}(a_i < X \leq b_i)$	E_i	M_Freq	M_Exp	χ^2-test statistics
0.0132	0.0132	0.6467			
0.0391	0.0259	1.2738			
0.0997	0.0605	2.9691	6	4.8897	0.2521
0.2188	0.1191	5.8335	6	5.8335	0.0047
0.4158	0.1971	9.6554	5	9.6554	3.3127
0.6833	0.2674	13.1030	17	13.1030	1.1589
0.9174	0.2341	11.4707	12	11.4707	0.0244
0.9941	0.0767	3.7604	5	4.76.4	0.0152
				χ^2	4.7682
				p-value	5.9914

Table 4.17: Chi square test for skew-normal for Portfolio 5

Chi-square test statistic $\chi^2 = 4.7682$ is less than p- value = 5.9914. Thus, we do not reject the hypothesis so that the skew-normal model fits the data.

$\mathbb{P}(X \leq b_i)$	$\mathbb{P}(a_i < X \leq b_i)$	E_i	M_Freq	M_Exp	χ^2-test statistics
0.0275	0.0275	1.3505			
0.0519	0.0243	1.1927			
0.1015	0.0496	2.4333	6	4.9765	0.2104
0.2023	0.1005	4.9368	6	4.9368	0.2289
0.3925	0.1902	9.3199	5	8.3199	2.0366
0.6819	0.2894	14.1840	17	14.18403	0.5590
0.9239	0.2419	11.8576	12	11.48576	0.0017
0.9899	0.0659	3.2301	5	4.2310	0.0183
				χ^2	3.2198
				p-value	3.8414

Table 4.18: Chi square test for skew-t for Portfolio 5

χ^2 test statistic is 3.2198 which is less than p = 3.8414. Thus, we do not reject the hypothesis skew-t model fits the data.

$\mathbb{P}(X \leq b_i)$	$\mathbb{P}(a_i < X \leq b_i)$	E_i	M_Freq	M_Exp	χ^2-test statistics
0.0040	0.0040	0.1981			
0.02241	0.0183	1.9000			
0.0862	0.0638	3.1276	6	4.2258	0.7448
0.2351	0.1488	7.2926	6	7.2926	0.2291
0.4680	0.2329	11.4148	5	11.4148	4.8165
0.7128	0.2448	11.9979	17	11.9979	2/0860
0.8856	0.1727	8.4671	12	8.4671	1.4741
0.9675	0.0818	4.0118	5	4.0118	0.0048
				χ^2	8.3506
				p-value	3.8414

Table 4.19: Chi square test for normal for Portfolio 5

Chi-square statistic $\chi^2 = 8.3506$ is greater than the p-value. Thus, normal model does not fit the data. Also, from the the PP plots it is clearly seen that all three models fit the data.

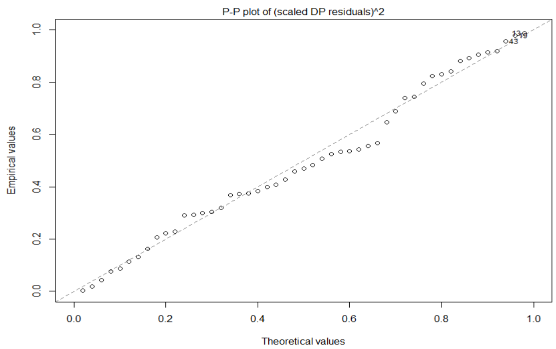

Figure 4.15: Portfolio 5: P-P plot of skew-norm

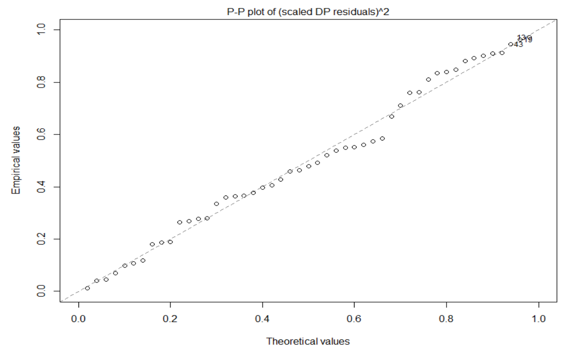

Figure 4.16: Portfolio 5: P-P plot of skew-t

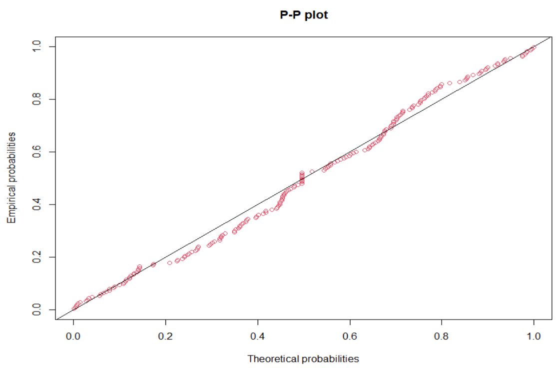

Figure 4.17: Portfolio 5: P-P plot of normal

We provide below LPM's for Portfolio in Table 10. The parameters after model calibration is estimated as below table. The degree of freedom parameter of skew-t model is taken as $\nu = 5$. Lower partial moments for $n = 1$, $n = 2$,$n = 3$ and $n = 4$ are calculated with the calibrated parameters.

$\alpha - value$	Skew-normal	Skew-t
$\alpha = 0.05$	$\xi = 0.0261$	$\xi = 0.0222$
	$\omega = 0.0389$	$\omega = 0.0306$
	$\lambda = -3.6549$	$\lambda = -2.4006$
	$LPM_{(n,\xi,\omega,\lambda,\alpha)}$	$LPM_{(n,\xi,\omega,\lambda,\nu=5,\alpha)}$
$n = 1$	0.0287	0.0261
$n = 2$	0.0357	0.0468
$n = 3$	0.0561	0.1351
$n = 4$	0.1048	0.7257

Table 4.20: Lower partial moments for Portfolio 2

4.0.9 Case 6

This portfolio is composed of a single stock of Volkswagen.

$$X = S_{\text{VWAGY}} \tag{4.0.5}$$

$\mathbb{P}(X \leq b_i)$	$\mathbb{P}(a_i < X \leq b_i)$	E_i	M_Freq	M_Exp	χ^2-test statistics
1.2E-07	1.2E-07	1.2E-05			
1.03E-06	9.1E-07	9.04E-05			
7.2E-06	6.2E-06	0.0006			
4.2E-05	3.5E-05	0.0035			
0.0002	0.0001	0.0169			
0.0009	0.0006	0.0683			
0.00321	0.0023	0.2329			
0.0101	0.0067	0.6701			
0.0265	0.0164	1.6333			
0.0607	0.0341	3.3833			
0.1211	0.0604	5.9817	7	11.9910	2.0774
0.2127	0.0916	9.0741	18	20.9616	0.4184
0.3328	0.1201	11.8874	21	13.5533	4.0913
0.4697	0.1369	13.5533	17	13.5709	0.8664
0.6068	0.1371	13.5709	21	12.0545	6.6383
0.7286	0.1217	12.0545	5	9.5989	2.2034
0.8255	0.0969	9.5989	5	11.4816	3.6590
0.8954	0.0699	6.9217	5	5.7877	0.1072
0.9415	0.0461	4.5598			
0.9694	0.0279	2.7631			
0.9851	0.0156	1.5473			
0.9931	0.0081	0.8030			
0.9971	0.0039	0.3868			
0.9988	0.0017	0.1730			
0.9995	0.0007	0.0719			
0.9998	0.0002	0.0277			
0.9999	0.0001	0.0099			
0.9999	3.3E-05	0.0033			
0.9999	1.0E-05	0.0010			
0.9999	2.9E-06	0.0002			
				χ^2	20.0617
				p-value	11.0704

Table 4.21: Chi square test for skew-normal

Chi-square test statistic is 20.0617 which is greater than 11.0704. Thus, we reject the hypothesis data is drawn from skew normal distribution.

$\mathbb{P}(X \le b_i)$	$\mathbb{P}(a_i < X \le b_i)$	E_i	M_Freq	M_Exp	χ^2-test statistics
0.0001	0.0001	0.0180			
0.0002	8.6E-05	0.0085			
0.0004	0.0001	0.0137			
0.0006	0.0002	0.0231			
0.0010	0.0004	0.0402			
0.0017	0.0007	0.0734			
0.0032	0.0014	0.1414			
0.0061	0.0029	0.2882			
0.0124	0.0062	0.6223			
0.0266	0.0142	1.4116			
0.0597	0.0330	3.2708	7	5.9117	0.2003
0.1330	0.0732	7.2556	18	21.0495	0.4418
0.2723	0.1393	13.7939	21	19.7708	0.0764
0.4720	0.1997	19.7708	17	19.8771	0.4164
0.6728	0.2007	19.8771	21	14.4836	2.9317
0.8191	0.1462	14.4836	5	8.4941	1.4372
0.9049	0.0857	8.4941	5	6.7486	0.4531
0.9501	0.0451	4.4704	5	2.6433	2.1010
0.9731	0.0230	2.2781			
0.9849	0.0118	1.1725			
0.9912	0.0062	0.6217			
0.9946	0.0034	0.3421			
0.9966	0.0019	0.1956			
0.9978	0.0011	0.1161			
0.9985	0.0007	0.0713			
0.9989	0.0004	0.0451			
0.9992	0.0002	0.0294			
0.9994	0.0001	0.0196			
0.9996	0.0001	0.0134			
0.9997	9.5E-05	0.0094			
				χ^2	8.0582
				p-value	9.4877

Table 4.22: Chi square test for skew-t

Chi-square test statistic is 8.0582 which is less than the p-value 9.4877. Thus, we do not reject the hypothesis data is drawn from skew-t distribution.

$\mathbb{P}(X \leq b_i)$	$\mathbb{P}(a_i < X \leq b_i)$	E_i	M_Freq	M_Exp	χ^2-test statistics
3.1E-06	3.1E-06	0.0003			
1.4E-05	1.1E-05	0.0011			
6.2E-05	4.7E-05	0.0047			
0.0002	0.0001	0.0172			
0.0008	0.0005	0.0561			
0.0024	0.0016	0.1627			
0.0066	0.0042	0.4204			
0.0164	0.0097	0.9683			
0.0365	0.0201	1.9881			
0.0733	0.0367	3.6389			
0.1332	0.0599	5.9376	7	13.1958	2.9091
0.2205	0.0872	8.6369	18	19.8368	0.1701
0.3336	0.1131	11.1998	21	12.9472	5.0085
0.4644	0.1307	12.9472	17	13.3428	1.0023
0.5992	0.1347	13.3428	21	12.2583	6.2337
0.7230	0.1238	12.2583	5	10.0398	2.5299
0.8244	0.1014	10.0398	5	12.1018	4.1676
0.8984	0.0741	7.3304	5	5.2771	0.0145
0.9466	0.0481	4.7713			
0.9746	0.0279	2.7685			
0.9891	0.0144	1.4321			
0.9957	0.0066	0.6603			
0.9985	0.0027	0.2714			
0.9995	0.0010	0.0994			
0.9998	0.0003	0.0324			
0.9999	9.5E-05	0.0094			
0.9999	2.4E-05	0.0024			
0.9999	5.7E-06	0.0005			
0.9999	1.1E-06	0.0001			
				χ^2	22.0361
				p-value	12.5915

Table 4.23: Chi square test for normal for Portfolio 5

$\chi^2 = 22.0361$, whereas $p = 12.5915$. Hence, we reject the hypothesis that the data is drawn from normal distribution.

Thus, the data suggests that only skew-t model fits the data, whereas we reject the hypothesis the data is drawn from the normal or skew-normal distribution.

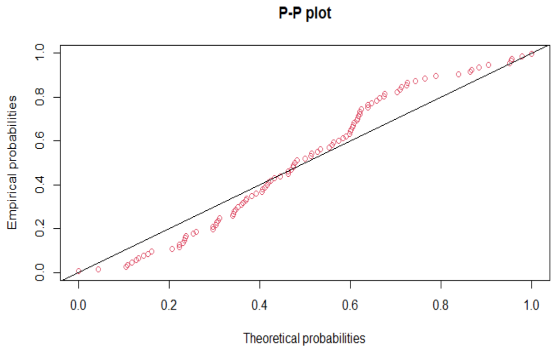

Figure 4.18: Portfolio 6: normal

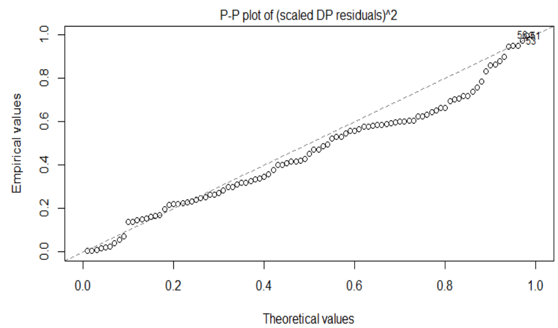

Figure 4.19: Portfolio 6: skew-normal

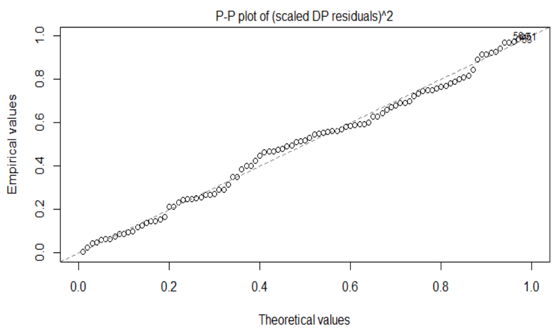

Figure 4.20: Portfolio 6: skew-t

The parameters after skew- t model data fitting and the corresponding Lower partial moments for $n = 1, 2, 3, 4$ are provided below in Table 20.

$\alpha - value$	Skew-t
$\alpha = 0.05$	$\xi = -0.0098$
	$\omega = 0.0245$
	$\lambda = 0.4858$
	$LPM_{(n,\xi,\omega,\lambda,\nu,\alpha)}$
$n = 1$	0.0054
$n = 2$	0.0064
$n = 3$	0.0109
$n = 4$	0.0243

Table 4.24: LPM's for Portfolio 5 for $\alpha = 0.05$

The performance of the proposed methodology is influenced by the sample size, and our empirical observations help identify a suitable range for reliable application. When the sample size is approximately 50, statistically we cannot reject the hypothesis that the data is drawn from skew-t or skew-normal distribution based on goodness of fit test, although, from P-P plot, we read that skew-t distribution fits data better. In many cases, on the other hand we reject the hypothesis that the data is drawn from normal distribution. Hence, we see that financial return data are typically asymmetric and exhibit heavy tails, making skewed and heavy-tailed distributions more appropriate and realistic. However, when we increase the sample size to significantly larger sizes like 200 or significantly smaller sizes like 20, we reject the hypothesis that the data is drawn from skew-elliptical family. Based on these findings along with goodness-of-fit test and visual P-P plots, we suggest that our methodology is applicable for moderate data sizes approximately 50. The proposed methodology is particularly effective when applied to small to moderate datasets. As emphasized above, our empirical results show that for sample size of 30 to 50 data sets, the methodology performs well especially with the skew-t and skew- normal distributions, capturing the asymmetry and heavy tails commonly observed in financial return data. Our findings suggest that the methodology performs best when the sample size is in the range of 30 to 50 observations. In some cases of dataset of 30 sample size all three distributions namely normal, skew-normal and skew-t fit the data. When the data size is approximately 50 normal distributions does not capture the data, but we do not reject the hypothesis that data is drawn from skew-normal or skew-t distribution. Although a skew-t distribution has better fit than skew-normal as evidenced by P-P plot. On the other hand, the distribution fitting procedures and lower partial moment calculations are not computationally intensive. Using R and Python, we can efficiently apply our methodology even to large datasets without encountering significant performance issues, although statistically we reject the hypothesis that the data is drawn from skew-elliptical family.

Chapter 5

Summary and Conclusion

This study presents novel theoretical derivations of lower partial moments (LPMs) for a class of skew-elliptical distributions, including the normal, skew-normal, skew-t, scaled skew-normal, and scaled skew-t distributions. These derivations are formulated for arbitrary higher moments $n \in \mathbb{N}$, thereby generalizing the concept of downside risk to a broader spectrum of moment-based risk measures. Our primary motivation was to enhance the realism and flexibility of risk modeling by incorporating skewness and heavy-tailed behavior frequently observed in financial return distributions—features often inadequately captured by traditional symmetric models.

Through extensive statistical testing, particularly using the chi-square goodness-of-fit test, we verified that for specific time intervals and sample sizes (notably $n = 50$), some or all of the three distributions—normal, skew-normal, and skew-t—exhibited excellent fit to the empirical data. Among them, the skew-t distribution with degrees of freedom $\nu = 5$ emerged as especially effective in modeling financial datasets of small to moderate size. It is worth emphasizing that both the normal and skew-normal distributions can be regarded as special or limiting cases of the skew-t distribution, achieved by letting $\nu \to \infty$ and $\lambda = 0$, respectively. This mathematical generality renders the skew-t distribution a powerful and unifying tool in financial risk modeling.

The analysis further investigated the behavior of two portfolios composed of identical stocks but with different weightings. Utilizing the derived LPMs, we were able to distinguish which portfolio configuration entailed lower downside risk, and was therefore more favorable under varying market conditions. This portfolio comparison underscores a major contribution of our research: the practical applicability of skew-elliptical-based LPMs to real-world financial decision-making. The proposed distributions effectively capture the asymmetry and tail behavior of asset returns, providing a significantly more precise evaluation of downside risk compared to conventional models.

Additionally, our empirical results show that, in most experiments, especially with small sample sizes, the skew-normal and skew-t distributions fit the data better than the normal distribution. However, in fewer cases, all three distributions performed comparably. The ability to select the most appropriate model based on empirical evidence, and to quantify risk using more accurate statistical tools, is essential for investors, risk managers, and analysts seeking to make informed decisions in volatile financial markets.

Despite these advancements, some limitations remain. The scope of the study was limited to a selected subset of skew-elliptical distributions. Expanding this framework to include additional flexible distributions—such as skew-generalized error or skew-Laplace models—could further enhance its robustness and adaptability to varying financial contexts. Moreover, while the chi-square test provided a reliable method for evaluating the goodness-of-fit, incorporating additional model selection criteria such as the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) would offer a more comprehensive comparative assessment across different models.

Another important direction for future research lies in examining **convex combinations** of portfolios. While this study focused on LPM-based risk assessments for two specific portfolios composed of the same assets, it would be valuable to explore which convex combination of these or other portfolios yields the *highest expected return for a given level of downside risk*, or alternatively, the *lowest risk for a given return*. This could be formulated as an optimization problem where the objective function incorporates LPMs, potentially yielding more investor-relevant outcomes than variance-based models. Identifying the optimal convex combination under skew-elliptical assumptions could bridge the gap between theoretical model performance and practical portfolio construction.

Moreover, future work could explore the multivariate extensions of the skew-elliptical distributions considered in this paper. Modeling the joint distribution of returns from multiple financial assets using multivariate skew-normal and skew-t distributions would provide a richer framework for understanding portfolio behavior under realistic market conditions. Such an extension would allow for the computation of multivariate LPMs, which can capture joint downside risk, inter-asset dependencies, and tail comovements—crucial factors in modern risk management.

Additionally, incorporating machine learning techniques to refine parameter estimation and distribution selection could significantly enhance the predictive performance and scalability of the proposed framework. Techniques such as Bayesian optimization, ensemble learning, or deep learning-based distribution fitting may improve estimation accuracy, particularly in high-dimensional or nonstationary environments.

In conclusion, this research contributes to the advancement of financial risk modeling by extending the theoretical foundation and practical applications of LPMs using skew-elliptical distributions. By accurately modeling asymmetry and tail risk, our framework offers a more realistic and actionable approach to downside risk assessment. The future directions highlighted—such as optimal convex portfolio combinations, multivariate extensions, and integration with modern computational tools—lay the groundwork for continued innovation in this important area of quantitative finance.

Appendix A

Appendix

```
// p#read excel file
vowg ← read_excel("C:/Users/Asus/Desktop/Gulnaz/Port_2023/Port_5_nonuni.xlsx")
#omit the NaN values
vowg_cln ← na.omit(vowg$ln)
#minimum and maximum of the values
start = min(vowg_cln)
start
end = max(vowg_cln)
end

#find interquartile
iqr_vowg_ln ← IQR(vowg_cln)

#find bin width
bin_width = (2 * iqr_vowg_ln) / (length(vowg_cln) ** (1 / 3))
bin_width

#skewness and kurtosis
skewness(vowg_cln)
kurtosis(vowg_cln)ut your code here
```

Figure A.1: Interquartile Estimation

```r
#find scale, location and skeness parameter of skew normal
vowg_sn <- selm(ln ~ 1, family="SN", vowg)
vowg_sn
summary(vowg_sn)
coef(vowg_sn)
params_sn <- vowg_sn@param$dp
xi = params_sn[1]
omega = params_sn[2]
alpha = params_sn[3]
xi
omega
alpha
```

Figure A.2: Skew- Normal calibration

```r
#find cdf of skew normal
cdf_sn<-psn(x=seq(start,end+bin_width, by=bin_width),xi= xi, omega=omega ,alpha=alpha)
seq(start,end+bin_width, by=bin_width)
df<-data.frame(cdf_sn)
cdf_sn
#write to excel file
write_xlsx(df,"C:/Users/Asus/Desktop/Gulnaz/Port_2023/port_sn_out.xlsx")
```

Figure A.3: Skew- Normal CDF

```r
#find cdf of skew normal
cdf_sn<-psn(x=seq(start,end+bin_width, by=bin_width),xi= xi, omega=omega ,alpha=alpha)
seq(start,end+bin_width, by=bin_width)
df<-data.frame(cdf_sn)
cdf_sn
#write to excel file
write_xlsx(df,"C:/Users/Asus/Desktop/Gulnaz/Port_2023/port_sn_out.xlsx")
#find cdf of skew t
vowg_st <- selm(ln~ 1, family="ST", data=vowg, fixed.param=list(nu=5))
coef(vowg_st)
summary(vowg_st)
vowg_st@param$dp
params <- vowg_st@param$dp
xi_st = params[1]
xi_st
omega_st = params[2]
omega_st
alpha_st = params[3]
alpha_st
```

Figure A.4: Skew-t calibration

```r
#cdf_st<-pst(x=seq(-0.17,0.222, by=0.007),xi=-0.02034457, omega= 0.02673838 ,alpha=1.90053229  , nu=10)
cdf_st<-pst(x=seq(start,end+bin_width, by=bin_width),xi=xi, omega=omega, alpha=alpha, nu=5)
df_st<-data.frame(cdf_st)
cdf_st
write_xlsx(df_st,"C:/Users/Asus/Desktop/Gulnaz/Port_2023/port_st_out.xlsx")
#find cdf of Normal
```

Figure A.5: Skew- t CDF

```r
#find cdf of Normal
paraw ← fitdistr(vowg_cln, densfun="normal")
paraw
vowg_norm_params ← paraw$estimate
mu = vowg_norm_params[1]
mu
sigma = vowg_norm_params[2]
sigma
```

Figure A.6: Normal calibration

```r
cdf_n = pnorm(seq(start,end+bin_width, by=bin_width), mu, sigma)
cdf_n
df_n←data.frame(cdf_n)
write_xlsx(df_n,"C:/Users/Asus/Desktop/Gulnaz/Port_2023/port_n_out.xlsx")
summary(vowg_st)
aic_sn←AIC(vowg_sn)
aic_st←AIC(vowg_st)
aic_sn
aic_st
aic_n←AIC(paraw)
aic_n
delta_aic ← aic1 - aic2
plot(vowg_st)
plot(vowg_sn)
plot(paraw)
```

Figure A.7: Normal CDF

```r
q ← qsn(0.05,xi=0.04510145, omega=0.06487349,alpha=-6.04709309)
q
qt12 ← qsn(0.05,xi=0, omega=1,alpha=-6.04709309)
qt12
qt12*0.06487349+0.04510145
qnorm(0.05,mean=0,sd=1)
qt1 ← qst(0.05, xi=0, omega=1, alpha=-4, nu=5)
qt1
qt2 ← qst(0.05, xi=1, omega=2, alpha=-4, nu=5)
(qt2-1)/2
```

Figure A.8: Quantile for case 1

```r
qt3 ← qsn(0.05,xi=0.0346, omega=0.0560,alpha=1)
qt3
qt4 ← qsn(0.05,xi=0.05, omega=0.02,alpha=2)
qt4
qt5 ← qst(0.05, xi=0.02621471, omega= 0.03521096, alpha=-2.9182649, nu=5)
qt5
qt6 ← qst(0.05, xi=0, omega= 1, alpha=-2.9182649, nu=5)
qt6
(qt5-0.02621471)/0.03521096
```

Figure A.9: Quantile for $\alpha = 0.05$

```
qt_prtf2 <- qst(0.05, xi=0.02228072 , omega= 0.03068706, alpha=-2.40060773, nu=5)
qt_prtf2
qt_amazon <- qst(0.05, xi=0.03434355 , omega= 0.04725874, alpha=-2.48106940, nu=5)
qt_amazon
qt_vowg <- qst(0.05, xi=-0.009815758 , omega= 0.024590877, alpha=0.485874743, nu=5)
qt_vowg
qt_prt1_10 <- qst(0.95, xi=0.02621471, omega= 0.03521096, alpha=-2.9182649, nu=5)
qt_prt1_10
qt_prt1_11<- qst(0.95, xi=0, omega= 1, alpha=-2.9182649, nu=5)
qt_prt1_11
(qt_prt1_10-0.02621471)/0.03521096
qt_case4 <- qsn(0.05,xi=0.04510145 , omega=0.06487349,alpha=-6.04709309 )
qt_case4
```

Figure A.10: Quantile for Case 4

```
qn_prt1_10 <- qsn(0.05, xi=0.03003038 , omega=0.04419881, alpha=-4.47524767)
qn_prt1_10

qn_prtf2 <- qsn(0.05, xi=0.02617914 , omega= 0.03890090, alpha=-3.6549265)
qn_prtf2
```

Figure A.11: Quantile Case 1 and Case 2

```
from scipy import pi
from scipy import linspace
from scipy import pi, exp
from scipy.special import erf
from scipy.stats import norm
import matplotlib
import matplotlib.pyplot as plt
import numpy as np
import math
from scipy.integrate import quad
from numpy import inf, exp
def f5(x):
  return 3/(8*(1+x**2/5)**3)
def F5(x):
  return 1/2+1/pi*(x/(np.sqrt(5)*(1+x**2/5))*(1+2/(3*(1+x**2/5)))+np.arctan(x/np.sqrt(5)))
def integrand1(x):
    return 2* (-0.04471338-x)*f5(x)*F5(-2.9182649*x)
I1 = quad(integrand1, -inf,-0.04471338)
def LPM1_port1(w):
  return I1[0]*w
def integrand2(x):
    return 2* (-0.04471338-x)**2*f5(x)*F5(-2.9182649*x)
I2 = quad(integrand2, -inf,-0.04471338)
def LPM2_port1(w):
  return I2[0]*w
def integrand3(x):
    return 2* (-0.04471338-x)**3*f5(x)*F5(-2.9182649*x)
I3 = quad(integrand3, -inf,-0.04471338 )
def LPM3_port1(w):
  return I3[0]*w
def integrand4(x):
    return 2* (-0.04471338-x)**4*f5(x)*F5(-2.9182649*x)
I4 = quad(integrand4, -inf,-0.04471338)
def LPM4_port1(w):
  return I4[0]*w
print(I1)
print (LPM1_port1(0.03521096))
print (LPM2_port1(0.03521096))
print (LPM3_port1(0.03521096))
print (LPM4_port1(0.03521096))
```

Figure A.12: LPM for skew normal case 1

```python
from scipy import pi
from scipy import linspace
from scipy import pi, exp
from scipy.special import erf
from scipy.stats import norm
import matplotlib
import matplotlib.pyplot as plt
import numpy as np
import math
from scipy.integrate import quad
from numpy import inf, exp
def f5(x):
  return 3/(8*(1+x**2/5)**3)
def F5(x):
  return 1/2+1/pi*(x/(np.sqrt(5)*(1+x**2/5))*(1+2/(3*(1+x**2/5)))+np.arctan(x/np.sqrt(5)))
def integrand(x):
    return 2* (-0.06428189-x)*f5(x)*F5(-2.9182649*x)
I = quad(integrand, -inf,-0.06428189 )
def LPM1(w):
  return I[0]*w
print(I)
print (LPM1(0.03521096))
x=-0.06428189
```

Figure A.13: LPM for skew-t case 1

```
from scipy import pi
from scipy import linspace
from scipy import pi, exp
from scipy.special import erf
from scipy.stats import norm
import matplotlib
import matplotlib.pyplot as plt
import numpy as np
import math
def Z1(z):
  return -norm.pdf(z)
def Z2(z):
  return -1/np.sqrt(2*pi)*(np.exp(-z**2/2))*z  +norm.cdf(z)
def Z3(z):
  return -norm.pdf(z)*(z**2+2)
def Z4(z):
  return -norm.pdf(z)*(z**3+3*z)+3*norm.cdf(z)
def LPM1(z):
  return z-Z1(z)
def LPM2(z):
  return z**2-2*z*Z1(z)+Z2(z)
def LPM3(z):
  return z**3-3*z**2*Z1(z)+3*z*Z2(z)-Z3(z)
def LPM4(z):
  return z**4-4*z**3*Z1(z)+6*z**2*Z2(z)-4*z*Z3(z)+Z4(z)
print(LPM1(-1.65*0.03908336+0.0066775185))
print(LPM2(-1.65*0.03908336+0.0066775185))
print(LPM3(-1.65*0.03908336+0.0066775185))
print(LPM4(-1.65*0.03908336+0.0066775185))
```

Figure A.14: LPM for normal case 2

```python
def integrand1(x):
    return 2* (-0.08204825-x)*norm.pdf(x)*norm.cdf(-6.04709309*x)
I1 = quad(integrand1, -inf,-0.08204825)
def LPM1_port1(w):
  return I1[0]*w
def integrand2(x):
    return 2* (-0.08204825-x)**2*norm.pdf(x)*norm.cdf(-6.04709309*x)
I2 = quad(integrand2, -inf,-0.08204825)
def LPM2_port1(w):
  return I2[0]*w
def integrand3(x):
    return 2* (-0.08204825-x)**3*norm.pdf(x)*norm.cdf(-6.04709309*x)
I3 = quad(integrand3, -inf,-0.08204825)
def LPM3_port1(w):
  return I3[0]*w
def integrand4(x):
    return 2*(-0.08204825-x)**4*norm.pdf(x)*norm.cdf(-6.04709309*x)
I4 = quad(integrand4, -inf,-0.08204825)
def LPM4_port1(w):
  return I4[0]*w
print(I1)
print (LPM1_port1(0.06487349))
print (LPM2_port1(0.06487349))
print (LPM3_port1(0.06487349))
print (LPM4_port1(0.06487349))
```

Figure A.15: LPM for Skew- Normal Case 2

```
from scipy import pi
from scipy import linspace
from scipy import pi, exp
from scipy.special import erf
from scipy.stats import norm
import matplotlib
import matplotlib.pyplot as plt
import numpy as np
import math
from scipy.integrate import quad
from numpy import inf, exp
def f5(x):
  return 3/(8*(1+x**2/5)**3)
def F5(x):
  return 1/2+1/pi*(x/(np.sqrt(5)*(1+x**2/5))*(1+2/(3*(1+x**2/5)))+np.arctan(x/np.sqrt(5)))
def integrand(x):
    return 2* (-0.08708974-x)**4*f5(x)*F5(-2.48106940*x)
I = quad(integrand, -inf,-0.08708974)
def LPM4(w):
  return I[0]*w
print(I)
print (LPM4(0.04725874))
```

Figure A.16: LPM for skew-t Case 4

Bibliography

[1] A. Dalla Valle A. Azzalini. The multivariate skew-normal distribution. *Biornetrika*, 83(4):715–726, 1996.

[2] Barry C. Arnold and Robert J. Beaver. The skew-cauchy distribution. *Statistics Probability Letters*, 49(3):285–290, 2000.

[3] A. Azzalini. A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics*, 12(2):171–178, 1985.

[4] Vijay S. Bawa. Optimal rules for ordering uncertain prospects. *Journal of Financial Economics*, 2(1):95–121, 1975.

[5] J. T. Chen and A. K. Gupta. Matrix variate skew normal distributions. *Statistics*, 39(3):247–253, 2005.

[6] J. T. Chen and A. K. Gupta. On matrix variate skew normal distributions. *Statistics*, 42(2):247–253, 2005.

[7] Glòria Figueras, Pedro Puig, and Arthur Pewsey. Goodness-of-fit tests for the skew-normal distribution when the parameters are estimated from the data. *Communications in Statistics-theory and Methods - COMMUN STATIST-THEOR METHOD*, 36:1735–1755, 07 2007.

[8] Peter C Fishburn. Mean-Risk Analysis with Risk Associated with Below-Target Returns. *American Economic Review*, 67(2):116–126, March 1977.

[9] Alireza Goli. Integration of blockchain-enabled closed-loop supply chain and robust product portfolio design. *Computers Industrial Engineering*, 179:109211, 2023.

[10] Alireza Goli. Efficient optimization of robust project scheduling for industry 4.0: A hybrid approach based on machine learning and meta-heuristic algorithms. *International Journal of Production Economics*, 278:109427, 2024.

[11] Alireza Goli, Ali Ala, and Mostafa Hajiaghaei-Keshteli. Efficient multi-objective meta-heuristic algorithms for energy-aware non-permutation flow-shop scheduling problem. *Expert Systems with Applications*, 213:119077, 2023.

[12] Alireza Goli, Ali Ala, and Seyedali Mirjalili. A robust possibilistic programming framework for designing an organ transplant supply chain under uncertainty. *Annals of Operations Research*, 328(1):493–530, September 2023.

[13] Alireza Goli and Erfan Babaee Tirkolaee. Designing a portfolio-based closed-loop supply chain network for dairy products with a financial approach: Accelerated benders decomposition algorithm. *Computers Operations Research*, 155:106244, 2023.

[14] Yangli Guo, Feng He, Chao Liang, and Feng Ma. Oil price volatility predictability: new evidence from a scaled pca approach. *Energy Economics*, 105:105714, 2022.

[15] J. T. Chen Gupta, A. K. A class of multivariate skew-normal models. *Annals of the Institute of Statistical Mathematics*, 56(2):305–315, 2004.

[16] Gokhan Karabulut, Mehmet Huseyin Bilgin, and Asli Cansin Doker. The relationship between commodity prices and world trade uncertainty. *Economic Analysis and Policy*, 66:276–281, 2020.

[17] Benoit Mandelbrot. The Variation of Certain Speculative Prices. *The Journal of Business*, 36:394–394, 1963.

[18] Harry Markowitz. Portfolio Selection. *Journal of Finance*, 7(1):77–91, March 1952.

[19] Govind S. Mudholkar and Alan D. Hutson. The epsilon–skew–normal distribution for analyzing near-normal data. *Journal of Statistical Planning and Inference*, 83(2):291–309, 2000.

[20] David Nawrocki. A brief history of downside risk measures. *The Journal of Investing*, 8, 03 2000.

[21] W. Ning and A. K. Gupta. . matrix variate extended skew normal distributions. *Random Operators and Stochastic Equations*, 20:299–310, 2012.

[22] Lubos Pastor and Pietro Veronesi. Uncertainty about government policy and stock prices. *The journal of Finance*, 67(4):1219–1264, 2012.

[23] R. L. Prentice. Discrimination among some parametric models. *Biometrika*, 62(3):607–614, 1975.