Spectral theory of the subelliptic p-Laplacian for Hörmander vector fields

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Declaration

I, Mukhtar Karazym, certify that, with the exception of properly referenced quotes and citations, this thesis titled *"Spectral theory of the subelliptic p-Laplacian for Hörmander vector fields"* is entirely my original work. No part of this thesis has been submitted for any other degree at Nazarbayev University or any other institution, nor does it contain material previously submitted for a degree.

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Abstract

Subject of this thesis is the spectral theory of the subelliptic p-Laplacian in the context of Hörmander vector fields. In Chapter 2, we determine the first eigenvalue λ_1 through the minimization of the Rayleigh quotient, which also leads to finding the best constant in the L^p Poincaré-Friedrichs inequality for Hörmander vector fields. Also, we prove Hölder continuity of eigenfunctions with respect to the Carnot-Carathéodory metric and positivity of the first eigenfunction, which are applied to obtain the simplicity of the first eigenvalue λ_1. By the end of Chapter 2, we show that all eigenfunctions corresponding to any eigenvalue $\lambda \neq \lambda_1$ change sign in the given domain and the first eigenvalue λ_1 is isolated in the set of all eigenvalues.

In Chapter 3, we apply the Lusternik-Schnirelman theory to establish the existence of a sequence of variational eigenvalues for the subelliptic p-Laplacian eigenvalue problem. We use two different kinds of compact, symmetric subsets of some manifold to derive variational eigenvalues of the Lusternik-Schnirelman type.

As applications in the context of partial differential equations, we demonstrate blow-up and extinction behavior of solutions to some parabolic equations with the subelliptic p-Laplacian in Chapter 4.

Keywords: Hörmander vector fields, subelliptic p-Laplacian, spectral problem, Lusternik-Schnirelman theory, variational eigenvalues.

Аңдатпа

Бұл жұмыс субэллиптикалық p-Лапласиан операторының спектрлік теориясын Хёрмандер вектор өрістері аясында зерттейді. Екінші тарауда Рэлей қатынасын минимизациялау арқылы бірінші меншікті мәнді λ_1 таба отырып, L^p Пуанкаре-Фридрихс теңсіздігінің ең кіші тұрақтысын анықтаймыз. Сонымен қатар, меншікті функциялардың Карно-Каратеодори метрикасына қатысты Гёльдер үзіліссіздігі мен бірінші меншікті функцияның берілген облыста оң екендігін көрсетеміз, бұл нәтижелер бірінші меншікті мәннің қарапайымдылығын дәлелдеуге қолданылады. Осы тараудың соңында бірінші меншікті мәннен басқа меншікті мәндерге сәйкес келетін меншікті функциялардың таңбасы берілген облыста өзгеретіндігін және бірінші меншікті мәннің меншікті мәндер жиынында оңашаланған нүкте екендігін дәлелдейміз.

Үшінші тарауда Люстерник-Шнирельман теориясын қолдана отырып, вариациялық меншікті мәндердің тізбегін табамыз. Люстерник-Шнирельман типтегі меншікті мәндерді анықтауда белгілі бір көпбейнедегі компакт, симметриялық жиындардың екі түрін қолданамыз.

Төртінші тарауда алынған нәтижелердің дербес туындылы дифференциалдық теңдеулердегі қолданыстары ретінде субэллиптикалық p-Лапласиан операторы қатысатын параболалық типтегі теңдеулердің шешімдерінің күйреу және өшу әрекеттерін көрсетеміз.

Түйінді сөздер: Хёрмандер вектор өрістері,субэллиптикалық p-Лапласиан, спектралды есеп, Люстерник-Шнирельман теориясы, вариациялық меншікті мәндер.

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$*$	binary operation on the Heisenberg-Weyl group $\mathbb{H}^1 = (\mathbb{R}^3, *)$
$[\cdot,\cdot]$	Lie bracket
$\bullet$	binary operation on the Engel group $\mathcal{E} = (\mathbb{R}^4, \bullet)$
δ_λ	dilation mapping on a homogeneous Carnot group $\mathbb{G}$
Δ_p	p-Laplace operator
$\Delta_{\mathbb{G}}$	sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$
$\Delta_{\mathbb{H}^n}$	Kohn-Laplacian on the Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \diamond)$
Δ_X	sum of squares operator constructed from Hörmander vector fields
$\diamond$	binary operation on the Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \diamond)$
$\operatorname{ess\,sup}_\Omega u$	essential supremum of u in Ω
$\gamma(M)$	Krasnoselskii genus of a nonempty, compact and symmetric set M
$\Lambda(x, r)$	Nagel-Stein-Wainger polynomial at $x \in \Omega$
$\langle\cdot,\cdot\rangle$	duality pairing between a normed space and its dual
$\|\cdot\|_{\mathbb{G}}$	homogeneous norm on a homogeneous Carnot group $\mathbb{G}$
$\|\cdot\|_p$	norm on $L^p(\Omega)$
$\mathbb{G} = (\mathbb{R}^n, \star, \delta_\lambda)$	homogeneous Carnot group
$\mathbb{G} = (\mathbb{R}^n, \star)$	Lie group

$\mathbb{H}^1$	Heisenberg-Weyl group $\mathbb{H}^1 = (\mathbb{R}^3, *)$
$\mathbb{H}^n$	Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \diamond)$
$\mathbb{N}$	set of all natural numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}_+$	set of all nonnegative real numbers
$\mathbb{S}^{n-1}$	unit sphere in $\mathbb{R}^n$
$\mathcal{C}_X^{0,\alpha}(\Omega)$	α-Hölder class associated with vector fields $X_1, \dots, X_m$
$\mathcal{E}$	Engel group $\mathcal{E} = (\mathbb{R}^4, \bullet)$
$\mathcal{V}$	$\{M \subset \mathbb{B} \setminus \{0\} : \quad M \text{ is compact and symmetric}\}$
$\mathcal{W}_{X,0}^{1,p}(\Omega)$	horizontal Sobolev space with respect to Hörmander vector fields
$\mathfrak{g}$	Lie algebra of a Lie group $\mathbb{G}$
$\mathfrak{X}(U)$	collection of all real smooth vector fields in U
∇	standard gradient
$\nabla_{\mathbb{G}}$	horizontal gradient on a homogeneous Carnot group $\mathbb{G}$
$\Omega \Subset U$	Ω is compactly contained in U
Ω_T	$\Omega \times (0, T)$
div	divergence operator
$\mathrm{div}_{\mathbb{G}}$	horizontal divergence on a homogeneous Carnot group $\mathbb{G}$
$\mathrm{d}(Y_i)$	formal degree of a vector field Y_i
$\overline{\mathrm{conv}}\mathcal{A}$	closed convex hull of a set $\mathcal{A}$
Σ_T	$\partial\Omega \times (0, T)$
$\lvert\Omega\rvert$	n-dimensional Lebesgue measure of a set $\Omega \subset \mathbb{R}^n$
$B_X(x, r)$	open metric ball centred at x with the radius $r > 0$ with respect to the Carnot-Carathéodory metric

$C_0^\infty(\Omega)$	class of all smooth functions $u : \Omega \to \mathbb{R}$ with compact support in Ω
$d_X(x, y)$	Carnot-Carathéodory metric
J	multi-index $J = (j_1, \ldots, j_s) \in \{1, \ldots, m\}^s$
$L^p\left(0, T; \mathcal{W}_{X,0}^{1,p}(\Omega)\right)$	This space consists of all strongly measurable functions $u : [0, T] \to \mathcal{W}_{X,0}^{1,p}(\Omega)$ with $\displaystyle \| u \|_{L^p\left(0,T;\mathcal{W}_{X,0}^{1,p}(\Omega)\right)} := \left(\int_0^T \| u(t) \|^p dt \right)^{1/p} < \infty$
$L^1_{loc}(\Omega)$	space of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_K \|f\| dx < \infty$ for all compact subsets $K \subset \Omega$
$L^\infty(\Omega)$	space of all essentially bounded measurable functions $u : \Omega \to \mathbb{R}$
$L^\infty(\Omega_T)$	space of all essentially bounded measurable functions $u : \Omega \times (0, T) \to \mathbb{R}$
$L^p(\Omega)$	space of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega \|u\|^p dx < \infty$
$L^p(\Omega_T)$	space of all measurable functions $u : \Omega \times (0, T) \to \mathbb{R}$ such that $\int_0^T \int_\Omega \|u(x, t)\|^p dx dt < \infty$
$N_\delta(A)$	set of all points within a distance $\delta > 0$ from a set A
n_1	number of generators of a homogeneous Carnot group $\mathbb{G}$
Q	local homogeneous dimension relative to Ω
$Q(x)$	pointwise homogeneous dimension at $x \in \Omega$ relative to Ω
T^*	blow up time
u^+	$\max\{u, 0\}$
u^-	$\min\{u, 0\}$
$u_j \rightharpoonup u$	u_j converges weakly to u
$u_j \to u$	u_j converges strongly to u

$W_0^{1,p}(\Omega)$	standard trace zero Sobolev space
$X = (X_1, \ldots, X_m)$	horizontal gradient
$X^{(k)}$	collection of all commutators of $X_1, \ldots, X_m$ of length k
$X_1, \ldots, X_{n_1}$	Jacobian generators of a homogeneous Carnot group $\mathbb{G}$
X_i^*	formal adjoint of a vector field X_i
X_J	$\left[X_{j_1}, \left[X_{j_2}, \ldots \left[X_{j_{s-1}}, X_{j_s}\right] \ldots\right]\right]$

Chapter 1

Preliminaries on the p-Laplacian, Hörmander vector fields and critical point theory

> The p-Laplacian – Mascot of Nonlinear Analysis.
>
> *P. Drábek [Dra07]*

1.1 The origin of the p-Laplacian

Darcy law stemming from [Dar56] is commonly used in fluid mechanics, especially, in porous media flow. Generalized form of the Darcy law has the following differential form

$$\mathbf{v} = -K(\theta)\nabla\Phi(\theta), \tag{1.1.1}$$

where $\mathbf{v}$ stands for the velocity of the given fluid, the composite function $K = K(\theta)$ stands for the hydraulic conductivity depending on the moisture content denoted by $\theta = \theta(x,t)$, and the potential Φ is the combination of the hydrostatic and gravitational potentials $\psi(\theta)$ and z, respectively, that is,

$$\Phi(\theta) = \psi(\theta) + z.$$

Let us denote the dynamic viscosity and permeability of the medium by μ and K, respectively. In particular, when $\Phi = p$ and $K(\theta) = K/\mu$, the Darcy law (1.1.1) becomes

$$\nabla p = -\frac{\mu}{K}\mathbf{v}.$$

In this case, the Darcy law describes direct proportionality between the gradient pressure ∇p and velocity $\mathbf{v}$ through the porous medium.

When the fluid flow is turbulent, the Darcy law (1.1.1) does not yield an accurate representation of the relationship between the velocity $\mathbf{v}$ given in (1.1.1) and pressure slope or force expressed as

$$\mathbf{F} = -\nabla\varphi(\theta) + K(\theta)\mathbf{e},$$

where $\mathbf{e} \in \mathbb{R}^n$ is a unit vector. O. Smreker's work [Smr79] demonstrates limitations of the linear Darcy law (1.1.1) in practical applications, such as dug wells. To address these limitations, he proposed a nonlinear version of the Darcy law

$$\mathbf{F} = -K(\theta)\nabla\Phi(\theta) = -\nabla\varphi(\theta) + K(\theta)\mathbf{e} \tag{1.1.2}$$

with

$$\mathbf{F} = |\mathbf{v}|^{p'-2}\mathbf{v} \qquad \text{for some } p' > 2,$$

where $s = (p'-2)+1 = p'-1$. Since $1/p + 1/p' = 1$, we have

$$\mathbf{v} = |\mathbf{F}|^{p-2}\mathbf{F}. \tag{1.1.3}$$

The power law (1.1.3) with $p = 5/3$ is usually attributed to O. Smreker [Smr78] in the literature. Then the continuity equation

$$\frac{\partial\theta}{\partial t} + \operatorname{div}\mathbf{v} = 0 \tag{1.1.4}$$

with prescribed (1.1.2) and (1.1.3) is a parabolic type equation involving the p-Laplacian, given in the following form:

$$\frac{\partial\varphi^{-1}}{\partial t} - \operatorname{div}\left(|\nabla\varphi(\theta) - K(\theta)e|^{p-2}(\nabla\varphi(\theta) - K(\theta)e)\right) = 0, \tag{1.1.5}$$

where φ^{-1} denotes the inverse of φ. The general case of (1.1.5) was considered in [DT94]. So, it is rightful to say that the operator

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \qquad \text{for } 1 < p < \infty, \tag{1.1.6}$$

appears to be originated from the power law (1.1.3).

Next, we turn to the following question: Who was the first who combined (1.1.4) or its stationary case $\operatorname{div}\mathbf{v} = 0$ with (1.1.3) to get parabolic or elliptic type equations involving p-Laplacian? According to [Ben+18], two engineers, O. Smreker [Smr81] and

N.E. Zhukovskii [Zhu89] seem to be the first authors who dealt with the p-Laplacian. They showed that the function

$$u(r) = C_0 + C_1 \cdot r^{1-\mu} \quad \text{for every } r = |x| > 0, \tag{1.1.7}$$

is a radially symmetric solution to $\Delta_p u = 0$ for $1 < p < \infty$ with $p \neq n$, see [Smr14]. Here

$$\mu = \frac{n-1}{p-1} \geq 0 \quad \text{with } \mu \neq 1,$$

and

$$C_0 = \begin{cases} u(0) & \text{if } \mu < 1, \\ u(+\infty) = \lim_{r \to +\infty} u(r) & \text{if } \mu > 1, \end{cases}$$

and C_1 is a constant. While the formula (1.1.7) remains a radially symmetric solution to $\Delta_p u = 0$ in any dimension $n \geq 1$, they focused exclusively on the planar case $n = 2$ as described by the hydroengineering model. Interestingly, there is no explicit formula for the equation $\Delta_p u = 0$ in their papers [Smr81] and [Zhu89], instead, they both used Smreker's results [Smr78] for the power law.

The following type of a nonlinear parabolic equation

$$\frac{\partial u^m}{\partial t} = c\Delta_p u \quad \text{in } \mathbb{R}^3 \times (0, T) \tag{1.1.8}$$

first appeared in [Lei45b], where $m + 1 = p = 3/2$ and $c > 0$. Since $m = p - 1$, the equation (1.1.8) is $(p-1)$-homogeneous. The author applied the separation of variables in time and space

$$u(x_1, x_2, x_3, t) = v(t)w(x_1, x_2, x_3)$$

to derive the equation involving the 1-Laplacian

$$\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right) + A\sqrt{w} = 0,$$

where $A > 0$. To the best of our knowledge, the paper [Lei45b] is likely to be the pioneering work, which investigates quasilinear parabolic equations with the p-Laplacian in $\mathbb{R}^3 \times (0, T)$, although $p = 3/2$ only. Right after publishing [Lei45b], L. Leibenson increased the range of p from $p = 3/2$ to $3/2 \leq p \leq 2$ and considered

$$\frac{\partial}{\partial t}\left(u^{\frac{1}{m+1}}\right) = c\Delta_p u \quad \text{in } \mathbb{R}^3 \times (0, T), \tag{1.1.9}$$

in [Lei45a], where $m > 0$. Unlike (1.1.8), the equation (1.1.9) is not $(p-1)$-homogeneous.

By the end of this section, we review key properties of the p-Laplacian (1.1.6). It is well-known that the p-Laplacian is singular for $1 < p < 2$ and is degenerate for $p > 2$. When $p = 2$, it is linear and known as the Laplacian. Singularity and degeneracy characterize the structure of the p-Laplacian at critical points of u, where $|\nabla u| = 0$. If $1 < p < 2$, then $|\nabla u|^{p-2}\nabla u \to \infty$ as $\nabla u \to 0$ pointwise. If $p > 2$, then $|\nabla u|^{p-2}\nabla u \to 0$ as $\nabla u \to 0$ pointwise. From the decomposition

$$\begin{aligned} \Delta_p u &= \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \\ &= |\nabla u|^{p-4}\left(|\nabla u|^2\Delta u + (p-2)\sum_{i,j=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}\right), \end{aligned} \tag{1.1.10}$$

we see that $\Delta_p u = 0$ is a quasilinear elliptic equation. Let us point out some difficulties that one faces in the case $p \neq 2$:

1. $W_0^{1,p}(\Omega)$ is not a Hilbert space;
2. The Lyapunov-Schmidt reduction fails to decompose the following operator

   $$(J(u), v) = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, \mathrm{d}x$$

   to invariant subspaces when $p \neq 2$;
3. It is not known whether variational eigenvalues exhaust the whole spectrum or not;
4. Multiplicity of eigenvalues turns into a significantly different problem;
5. Any linear combination of two linearly independent eigenfunctions corresponding to a particular eigenvalue is not necessarily an eigenfunction associated with the same eigenvalue;
6. To the best of our knowledge, unique continuation property is an open problem.

1.2 Eigenvalue problem for the p-Laplacian

Let Ω be a bounded connected open set in $\mathbb{R}^n$ with $n \geq 2$ and $1 < p < \infty$. The Dirichlet eigenvalue problem for the p-Laplacian, given by

$$\begin{aligned} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) &= -\Lambda |u|^{p-2}u \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.2.1}$$

has been extensively studied since the 1980s and 1990s, we refer to [Thé86], [AL87], [Sak87], [GP87], [Bha89] and [Lin90] for pioneering contributions.

The first eigenvalue $\Lambda_{1,p}$ of (1.2.1) is simple and isolated. The corresponding eigenfunctions can be chosen positive in Ω. Moreover, if Λ is an eigenvalue distinct from the first eigenvalue $\Lambda_{1,p}$, then all eigenfunctions corresponding to Λ change their sign in Ω. We refer to [Lin90] for these results.

The first eigenvalue $\Lambda_{1,p}$ satisfies Cheeger's inequality [LW97], which is a generalization of the Dirichlet Cheeger inequality [Che70]. Cheeger's inequality states that

$$\Lambda_{1,p} \geq \left(\frac{h_1(\Omega)}{p}\right)^p,$$

where

$$h_1(\Omega) := \inf\left\{\frac{P(F)}{|F|} : F \subset \Omega, |F| > 0\right\}.$$

is known as the Cheeger constant and

$$P(F) = P\left(F; \mathbb{R}^n\right) = \sup\left\{\int_F \operatorname{div} g dx : g \in C_0^1\left(\mathbb{R}^n; \mathbb{R}^n\right), |g(x)| \leq 1\right\}.$$

is the distributional perimeter of F with respect to $\mathbb{R}^n$, see e.g. [Bae19].

The function $\Lambda_{1,p}$ is right continuous with respect to p, that is,

$$\lim_{s \to p+} \Lambda_{1,s} = \Lambda_{1,p}$$

for every $p > 1$, see [Lin93, Theorem 3.5]. If Ω is a Lipschitz domain, then the variational eigenvalue $\Lambda_{k,p}$ is continuous as a function of p for every $k \in \mathbb{N}$, see [Par11] and [Par09]. If $p \to 1+0$, then $\Lambda_{1,p}$ approaches the Cheeger constant [KF03]. Later on, E. Parini proved that the second eigenvalue $\Lambda_{2,p}$ goes to the second Cheeger constant as $p \to 1+0$ in [Par10]. Moreover, it is shown that the kth variational eigenvalue $\Lambda_{k,p}$ tends to $\Lambda_{k,1}$ as $p \to 1+0$ in [LS14] (see also [SS21] for the direct proof), where $\Lambda_{k,1}$ is the kth variational eigenvalue of the 1-Laplacian $-\Delta_1$. For the spectrum of the 1-Laplacian we refer to [Cha09]. Regarding the recent results on the monotonicity and non-monotonicity of variational eigenvalues with respect to p, we refer to [BT15], [BM21], [Mih22] and [KTT24]. We begin Chapter 3 with a review of the variational eigenvalues of (1.2.1)

According to [DiB83] and [Tol84], every eigenfunction of (1.2.1) possesses $C^{1,\alpha}$ regularity for some $0 < \alpha < 1$. When $p = 2$, all eigenfunctions of (1.2.1) are smooth, see e.g. [LMP23, Theorem 2.2.1].

1.3 Lie algebras

Definition 1.3.1. *Let* $\mathfrak{g}$ *be a vector space over* $\mathbb{R}$. *If the binary operaton*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfies

$$[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z] \tag{1.3.1}$$

$$[X, Y] = -[Y, X] \tag{1.3.2}$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{1.3.3}$$

for all $X, Y, Z \in \mathfrak{g}$ *and* $\alpha, \beta \in \mathbb{R}$, *then we call* $(\mathfrak{g}, +, \cdot, [\cdot,\cdot])$ *a Lie algebra over* $\mathbb{R}$.

The map $[\cdot,\cdot]$ is called the Lie bracket. The identity (1.3.3) is known as the Jacobi identity. We call (1.3.1) and (1.3.2) the bilinearity and anticommutativity properties, respectively. From (1.3.2) we see that $[X, X] = 0$ for all $X \in \mathfrak{g}$.

Definition 1.3.2. *Let* $\mathfrak{g}$ *be a Lie algebra. We define* $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^j = [\mathfrak{g}, \mathfrak{g}^{j-1}]$ *for all* $j \in \mathbb{N}$, *where*

$$[\mathfrak{g}, \mathfrak{g}^{j-1}] = \{[X, Y] : X \in \mathfrak{g}, Y \in \mathfrak{g}^{j-1}\}.$$

If $\mathfrak{g}^j = \{0\}$ *for some* $j \in \mathbb{N}$, *then we call* $\mathfrak{g}$ *a nilpotent Lie algebra.*

1.4 Smooth vector fields

Let U be a bounded connected open subset in $\mathbb{R}^n$ with $n \geq 2$. A real smooth vector field Z on U assigns to each point $x \in U$ a tangent vector of the form

$$Z(x) = \sum_{i=1}^{n} c_i(x)\partial_{x_i},$$

where the coefficients $c_i : U \to \mathbb{R}$ are smooth. Given two smooth vector fields Y and Z, we define their commutator as follows

$$[Y, Z] := YZ - ZY. \tag{1.4.1}$$

It is clear that the commutator (1.4.1) is also a smooth vector field.

We denote the set of all real smooth vector fields on U by $\mathfrak{X}(U)$.

Proposition 1.4.1. *[BB23, Propositon 1.11]*

The set $\mathfrak{X}(\Omega)$ *with the addition of vector fields and multiplication of a vector field by a real scalar, that is, with*

$$X + Y \in \mathfrak{X}(\Omega) \quad \text{for all } X, Y \in \mathfrak{X}(\Omega),$$

$$\alpha X \in \mathfrak{X}(\Omega) \quad \text{for all } X \in \mathfrak{X}(\Omega) \text{ and } \alpha \in \mathbb{R},$$

is a real vector space. Moreover, if we equip this vector space with the binary operation $[\cdot, \cdot]$ *defined by*

$$[X, Y] = XY - YX \in \mathfrak{X}(\Omega) \quad \text{for all } X, Y \in \mathfrak{X}(\Omega),$$

then $(\mathfrak{X}(\Omega), +, \cdot, [\cdot, \cdot])$ *is a Lie algebra over* $\mathbb{R}$.

1.5 Hörmander's finite rank condition

Let $U \subset \mathbb{R}^n$, $n \geq 2$ be a bounded connected open subset. Also, let $X_1, \ldots, X_m$ be a collection of smooth vector fields of the form

$$X_i = \sum_{k=1}^{m} b_{ik}(x)\partial_{x_k}, \quad i = 1, \ldots, m.$$

For a multi-index $J = (j_1, \ldots, j_s) \in \{1, \ldots, m\}^s$, we define a commutator X_J of length s as follows

$$X_J := \left[X_{j_1}, \left[X_{j_2}, \ldots \left[X_{j_{s-1}}, X_{j_s}\right] \ldots\right]\right].$$

We say that $X_1, \ldots, X_m$ satisfy Hörmander's finite rank condition at step s if at any point $x \in U$ there exist n linearly independent vector fields amongst $X_1, \ldots, X_m$ and all their commutators of length s. The history of the Hörmander finite rank condition dates back to 1960s, when Lars Hörmander, in his celebrated paper [Hör67], proved that if $X_1, \ldots, X_m$ satisfy Hörmander's finite rank condition, then the operator $\Delta_X = \sum_{j=1}^{m} X_j^2$ is hypoelliptic.

Hörmander vector fields satisfy the so-called *"connectivity property"*, that is, the Carnot-Carathéodory metric, which we will introduce it later on, is finite in the given domain. Connectivity property of Hörmander vector fields plays a crucial role in the geometric control theory, because this property ensures controllability of physical systems so that starting from any initial state, one can reach the final state by appropriately manipulating the controls. If the Hörmander rank condition holds when

$m < n$, then the number of controls is fewer than the number of degrees of freedom. This phenomenon is striking for the reason that there is a possibility to manipulate the behavior of a wide range of physical systems within a limited number of controls.

The connectivity property of Hörmander vector fields has also applications in a more recent interdisciplinary field called *"neurogeometry"* that links neuroscience with geometry. Neurophysiologists Torsten N. Wiesel and David H. Hubel were awarded the Nobel Prize in Physiology or Medicine in 1981 for their discoveries on a part of the brain, so-called visual cortex, that receives and processes visual data. In particular, they observed that some cells in the visual cortex respond specifically to edges or contours with certain orientations. Their breakthrough results led the question of how our brain is capable of reconstructing or assembling the whole visual contour of an object, when many neurons in the visual cortex are active and each of them is responsive to specific parts of the contour. This discovery became a reason for the development of numerous mathematical models of this process, we cite only the pioneering paper [Hof89]. This question has connections with the perceptual completion: the ability of a brain to fill in missing parts of images that are not visible. For instance, in the Kanizsa triangle shown in Figure 1.1, our brain perceives the area enclosed by six black shapes as a bright white triangle. However, the edges of the triangle are illusory. In reality, it is not a triangle, and the enclosed area is not actually brighter either.

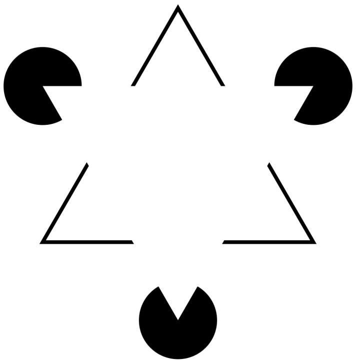

Figure 1.1: Kanizsa triangle [Kan76]

Another mathematical model of the perceptual completion was proposed by G. Citti and A. Sarti [CS06]. Their model is based on Hörmander vector fields. There is a book on neuromathematics, we refer to [CS14].

1.6 Carnot-Carathéodory spaces

Despite possibly having fewer vector fields $\{X_j\}$ than the dimension of the space, it remains possible to establish a connection between any two points within the space by following a sequence of arcs of integral lines of $\{X_j\}$, that is, any two points can be connected via the integral lines of $\{X_j\}$. This phenomenon is a consequence of Hörmander's finite rank condition, and it is known as the connectivity property in the literature. A fundamental result in this area was established independently by P. Rashevsky [Ras38] and W.-L. Chow [Cho40]. As a result, this finding is commonly referred to as the Chow-Rashevsky connectivity theorem. We refer to [BLU07] for the connectivity theorem for stratified Lie groups. By the connectivity property, we can introduce the notion so-called *"Carnot-Carathéodory metric"* induced by the vector fields $X_1, \ldots, X_m$. We refer to the [Gro96] for the basic theory of Carnot-Carathéodory spaces. Pioneering results were obtained mostly by M. Gromov and P. Pansu, see [Le 17] and references therein.

Although numerous techniques from $\mathbb{R}^n$ can be effectively extended to Carnot-Carathéodory spaces, several open problems remain, particularly in the context of Heisenberg and Carnot groups. For instance, consider a Carnot group $\mathbb{G}$. It is still an open question whether every sub-Laplacian $\Delta_{\mathbb{G}}$ on $\mathbb{G}$ possesses the unique continuation property. For a detailed discussion on such open problems, we refer to [BBS16].

Now, let $\Omega \Subset U$ be an open connected subset. We recall the notion of the Carnot-Carathéodory metric associated with $X_1, \ldots, X_m$.

Definition 1.6.1. *Given $\delta > 0$, let $C_1(\delta)$ denote the class of absolutely continuous mappings $\varphi : [0,1] \to \Omega$ satisfying*

$$\varphi'(t) = \sum_{i=1}^{m} a_i(t) X_i I\left(\varphi(t)\right) \quad a.e. \ in \ [0,1],$$

where a_i is a given measurable function such that

$$|a_i| \leq \delta \quad a.e. \ in \ [0,1]$$

for $i = 1, \ldots, m$. Given two points $x, y \in \Omega$, the Carnot-Carathéodory metric is defined by

$$d_X(x,y) := \inf\left\{\delta > 0 : \ \exists \varphi \in C_1(\delta) \ with \ \varphi(0) = x \ and \ \varphi(1) = y\right\}. \tag{1.6.1}$$

The following example illustrates that from the Carnot-Carathéodory metric we can recover the Euclidean metric.

Example 1.6.2. *Let* $X_1 = \partial_{x_1}$*,* $X_2 = \partial_{x_2}$*, and* $X_3 = \partial_{x_3}$ *in* $\mathbb{R}^3$*. Also, let* $a_j(t) = a_j$ *satisfy*

$$\sqrt{\sum_{j=1}^{3} a_j^2} \leq \delta \quad \textit{for } j = 1, 2, 3.$$

Since

$$X_1 I(x) = (1,0,0)^\top = e_1, \;\; X_2 I(x) = (0,1,0)^\top = e_2 \textit{ and } \;\; X_3 I(x) = (0,0,1)^\top = e_3,$$

we have

$$\begin{cases} \varphi'(t) = \sum_{j=1}^{3} a_j(t) X_j(I(\varphi(t))) = \sum_{j=1}^{3} a_j e_j & \textit{a.e. in } [0,1], \\ \varphi(0) = x, \;\; \varphi(1) = y. \end{cases}$$

This is a system of ordinary differential equations with initial and boundary conditions. In component form,

$$\frac{d\varphi_j(t)}{dt} = a_j, \quad \textit{with } \varphi_j(0) = x_j \textit{ and } \varphi_j(1) = y_j,$$

for $j = 1, 2, 3$*. The solution is* $\varphi_j(t) = (y_j - x_j)\,t + x_j$*, where* $a_j = y_j - x_j$ *for* $j = 1, 2, 3$*. Then we have*

$$\sqrt{\sum_{j=1}^{3} (y_j - x_j)^2} \leq \delta.$$

Hence,

$$d_X(x, y) := \inf \delta = \sqrt{\sum_{j=1}^{3} (y_j - x_j)^2}$$

is the Euclidean metric.

Example 1.6.3. *Let* $X_1 = \partial_{x_1}$ *be the single vector field in* $\mathbb{R}^2$*. Then the Carnot-Carathéodory distance* d_{X_1} *between* $x = (x_1, x_2) \in \mathbb{R}^2$ *and* $y = (y_1, y_2) \in \mathbb{R}^2$ *is given by*

$$d_{X_1}(x, y) = \begin{cases} |x - y|, & \textit{if } x \textit{ and } y \textit{ lie on a line parallel to the } x_1\textit{-axis}, \\ \infty, & \textit{otherwise.} \end{cases}$$

Remark 1.6.4. *For simplicity, we adopt the notation* d_X*, although it also depends on the domain* Ω*, see [BB23, Remark 1.30].*

Theorem 1.6.5. *[BB23, Theorem 1.45] Let $x \in \Omega$. Then for every neighborhood $V \subset \Omega$ of x there is a neighborhood $W \subset V$ of x such that any two points in W can be joined by a curve lying within V. This curve is formed by a finite number of arcs, which are integral curves of the vector fields $X_1, \ldots, X_m$.*

Theorem 1.6.5 is a localized version of connectivity. We can also state the global connectivity.

Theorem 1.6.6. *[BB23, Theorem 1.45] Let $x, y \in \Omega$. Then there is a curve connecting x to y that lies entirely within Ω. This curve is formed by a finite number of arcs, which are integral curves of the vector fields $X_1, \ldots, X_m$. In particular, the Carnot-Carathéodory metric associated with Hörmander vector fields $X_1, \ldots, X_m$ is finite in Ω. This property is called the connectivity property.*

Remark 1.6.7. *Hörmander's finite rank conditon is only a sufficient condition for the connectivity property, but not necessary, since there exist vector fields that do not satisfy Hörmander's finite rank conditon in Ω, but satisfy the connectivity property. For example, consider the Franchi-Lanconelli vector fields $X_1 = \partial_{x_1}$ and $X_2 = |x_1|^{\alpha} \partial_{x_2}$ in $\mathbb{R}^2$, where $\alpha > 0$ is not an integer. These vector fields are not smooth and do not satisfy Hörmander's finite rank conditon on $\{x \in \mathbb{R}^2 : x_1 = 0\}$, nevertheless, B. Franchi and E. Lanconelli showed that these vector fields satisfy the connectivity property in [FL83].*

Since d_X is finite in Ω, it follows that (Ω, d_X) is a metric space, known as the Carnot-Carathéodory space. Then an open ball centred at $x \in \Omega$ with the radius $r > 0$ in (Ω, d_X) is defined as follows

$$B_X(x, r) := \{y \in \Omega : \ d_X(x, y) < r\}.$$

1.6.1 Homogeneous Carnot groups

Homogeneous Carnot groups are a special case of Carnot-Carathéodory spaces. Since they also form a subclass of Lie groups, we first introduce Lie groups to establish the necessary foundation before discussing their more specialized structure.

Definition 1.6.8. *Let $\mathbb{G} = (\mathbb{R}^n, \star)$ be a group. If the mapping*

$$\mathbb{R}^n \times \mathbb{R}^n \ni (x, y) \mapsto y^{-1} \star x \in \mathbb{R}^N$$

is smooth, then we call $\mathbb{G}$ a Lie group on $\mathbb{R}^n$. Here y^{-1} is the inverse of y.

Next, we define left-invariant vector fields, which play a key role in the structure of homogeneous Carnot groups. To illustrate their significance, we also provide an example.

Definition 1.6.9. *Let X be a smooth vector field on a Lie group $\mathbb{G}$. We say that X is left invariant if*

$$X\left(\varphi(\alpha\star x)\right)=(X\varphi)(\alpha\star x)$$

holds for all $\alpha\in\mathbb{G}$ and $\varphi\in C^\infty(\mathbb{R}^n)$.

The set of all left invariant vector fields forms a Lie algebra.

Example 1.6.10. *Let us show that $X_1=\partial_{x_1}+2x_2\partial_{x_3}$ in $\mathbb{R}^3$ is left invariant with respect to*

$$x*\tilde{x}=(x_1,x_2,x_3)*(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)=(x_1+\tilde{x}_1,x_2+\tilde{x}_2,x_3+\tilde{x}_3+2(x_2\tilde{x}_1-x_1\tilde{x}_2)).$$

where $x,\tilde{x}\in\mathbb{R}^3$. Let $\alpha\in\mathbb{R}^3$ and $\varphi\in C^\infty(\mathbb{R}^3)$. Then

$$(X\varphi)(\alpha\star x)=\partial_1\varphi(\alpha\star x)+2(\alpha_2+x_2)\partial_3\varphi(\alpha\star x)$$

and

$$\begin{aligned}X\left(\varphi(\alpha\star x)\right)&=X\big(\varphi(\alpha_1+x_1,\alpha_2+x_2,\alpha_3+x_3+2(\alpha_2x_1-\alpha_1x_2))\big)\\&=X\left(\varphi(f_1(x_1),f_2(x_2),f_3(x_1,x_2,x_3))\right)=\partial_{x_1}\varphi(f_1,f_2,f_3)+2x_2\partial_{x_3}\varphi(f_1,f_2,f_3)\\&=\sum_{j=1}^{3}(\partial_{x_1}f_j)\partial_j\varphi(f_1,f_2,f_3)+2x_2\sum_{j=1}^{3}(\partial_{x_3}f_j)\partial_j\varphi(f_1,f_2,f_3)\\&=\partial_1\varphi(f_1,f_2,f_3)+2\alpha_2\partial_1\varphi(f_1,f_2,f_3)+2x_2\partial_3\varphi(f_1,f_2,f_3)\\&=\partial_1\varphi(\alpha\star x)+2(\alpha_2+x_2)\partial_3\varphi(\alpha\star x),\end{aligned}$$

where ∂_j denotes the partial derivative with respect ot jth coordinate and

$$f_1(x_1)=\alpha_1+x_1,\quad f_2(x_2)=\alpha_2+x_2,\quad f_3(x_1,x_2,x_3)=\alpha_3+x_3+2(\alpha_2x_1-\alpha_1x_2).$$

With these preliminaries in place, we are now ready to define homogeneous Carnot groups.

Definition 1.6.11. *[BLU07, Definition 1.4.1] Let $\mathbb{G}=(\mathbb{R}^n,\star)$ be a Lie group. If the following statements hold true:*

(i) the space $\mathbb{R}^n$ can be decomposed as $\mathbb{R}^n=\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_s}$ and for each $\lambda>0$ the dilation map $\delta_\lambda:\mathbb{R}^n\to\mathbb{R}^n$ given by

$$\delta_\lambda(x)=\delta_\lambda\left(x^{(1)},\ldots,x^{(s)}\right)=\left(\lambda x^{(1)},\lambda^2x^{(2)},\ldots,\lambda^sx^{(s)}\right),\quad x^{(i)}\in\mathbb{R}^{n_i},$$

is automorphism of $\mathbb{G}$;

(ii) Given n_1 as above, let $X_1, \ldots, X_{n_1}$ be the left invariant vector fields on $\mathbb{G}$ such that $X_i(0) = \partial_{x_i}|_0$ for $i = 1, \ldots, n_1$. Then

$$\operatorname{rank}\left(\operatorname{Lie}\left\{X_1, \ldots, X_{n_1}\right\}(x)\right) = n \qquad \text{for every } x \in \mathbb{R}^n,$$

then the triple $\mathbb{G} = (\mathbb{R}^n, \star, \delta_\lambda)$ is called a homogeneous Carnot group.

We say that $\mathbb{G}$ is a group of step s with n_1 generators. The vector fields $X_1, \ldots, X_{n_1}$ are known as the Jacobian generators of $\mathbb{G}$.

1.6.1.1 Homogeneous norms

Definition 1.6.12. *[BB23, Definition 3.8]*

Let $\mathbb{G} = (\mathbb{R}^n, \star, \delta_\lambda)$ be a homogeneous Carnot group. Any continuous function $\|\cdot\|_{\mathbb{G}} : \mathbb{R}^n \to [0, +\infty)$ satisfying

(i) $x \in \mathbb{G}$ satisfies $\|x\|_{\mathbb{G}} = 0$ if and only if $x = 0$;

(ii) for all $x \in \mathbb{G}$ and $\lambda > 0$ we have $\|\delta_\lambda(x)\|_{\mathbb{G}} = \lambda \|x\|_{\mathbb{G}}$;

(iii) for all $x \in \mathbb{G}$ there exists $C > 0$ such that $\|x^{-1}\|_{\mathbb{G}} \leq C\|x\|_{\mathbb{G}}$;

(iv) for all $x, y \in \mathbb{G}$ there exists $C > 0$ such that $\|x \star y\|_{\mathbb{G}} \leq C(\|x\|_{\mathbb{G}} + \|y\|_{\mathbb{G}})$.

is called a homogeneous norm on $\mathbb{G}$.

Moreover, if $\|\cdot\|_{\mathbb{G}}$ satisfies

$$\left\|x^{-1}\right\|_{\mathbb{G}} = \|x\|_{\mathbb{G}} \qquad \text{for every } x \in \mathbb{R}^n,$$

then $\|\cdot\|_{\mathbb{G}}$ is called a symmetric homogeneous norm.

If $\|\cdot\|_{\mathbb{G}} \in C^\infty(\mathbb{R}^n \setminus \{0\})$, then it is a smooth homogeneous norm.

In the following example, we demonstrate the connection between the Carnot-Carathéodory metric and homogeneous norms.

Example 1.6.13. *[BLU07, Theorem 5.2.8]*

Let $\mathbb{G} = (\mathbb{R}^n, \star, \delta_\lambda)$ be a homogeneous Carnot group with n_1 generators, say, $X_1, \ldots, X_{n_1}$. The Carnot-Carathéodory metric $d_X(\cdot, 0)$ associated with $X_1, \ldots, X_{n_1}$ is a symmetric homogeneous norm on $\mathbb{G}$. However, in general, it is not smooth, see [BLU07, Remark 5.2.9]

We can define a smooth homogeneous norms as follows

$$\|x\|_{\mathbb{G}} := \left(\sum_{j=1}^{r} \left|x^{(j)}\right|^{\frac{2r!}{j}}\right)^{\frac{1}{2r!}}, \quad x = \left(x^{(1)}, \ldots, x^{(r)}\right) \in \mathbb{G},$$

where $x^{(i)} \in \mathbb{R}^{n_i}$ with $n_1 + \ldots + n_r = n$.

Proposition 1.6.14. *[BLU07, Proposition 5.1.4] All homogeneous norms on a homogeneous Carnot group $\mathbb{G}$ are equivalent.*

1.6.2 Local homogeneous dimension

The notion of the local homogeneous dimension Q is important in the thesis, for instance, to prove Theorem 2.1.6, which appears in Chapter 2.

Let $X^{(k)}$ denote the set of all commutators of $X_1, \ldots, X_m$ of length k, that is,

$$X^{(k)} := \left\{ X_J : \quad J = (j_1, \ldots, j_k) \in \{1, \ldots, m\}^k, \ \ |J| = k \right\},$$

where $k \in \mathbb{N}$ with $k \leq s$. Here s is the step in Hörmander's finite rank condition. From the components of $X^{(1)}, \ldots, X^{(s)}$, we select the ones necessary to span $\mathbb{R}^n$ and label them as $Y_1, \ldots, Y_l$. If Y_i belongs to $X^{(k)}$, then the formal degree $\mathrm{d}(Y_i) = k$ is assigned, so we define

$$\mathrm{d}(I) := \sum_{k=1}^{n} \mathrm{d}\left(Y_{i_k}\right) \qquad \text{and} \qquad a_I(x) := \det\left(Y_{i_1}, \ldots, Y_{i_n}\right)(x)$$

for multi-indices $I = (i_1, \ldots, i_n) \in \{1, \ldots, l\}^n$. Lastly, we introduce the Nagel-Stein-Wainger polynomial at $x \in \Omega$ (see [NSW85])

$$\Lambda(x, r) := \sum_{I} |a_I(x)|\, r^{\mathrm{d}(I)} \quad \text{for } r > 0.$$

This allows us to define the local homogeneous dimension.

Definition 1.6.15. *[CDG93, p. 1771] Given $x \in \Omega$, let*

$$Q(x) := \lim_{r \to 0+} \frac{\log \Lambda(x, r)}{\log r}. \tag{1.6.2}$$

We call $Q(x)$ the pointwise homogeneous dimension at $x \in \Omega$ relative to Ω.

Definition 1.6.16. *[CDG93, p. 1771] Let*

$$Q := \sup_{x \in \Omega} Q(x).$$

We call Q the local homogeneous dimension relative to Ω.

Remark 1.6.17. *It is obvious that $n \leq Q(x) \leq Q$. We need to mention that the local homogeneous dimension Q is equal to the generalized Métivier index $\tilde{\nu}$, see [CC19, Proposition 2.2 and Definition 1.2].*

The following example illustrates that the pointwise homogeneous dimension Q may not be a constant across Ω.

Example 1.6.18. *Let* $X_1 = \partial_{x_1}$*,* $X_2 = x_1\partial_{x_2}$ *and* $X_3 = \partial_{x_3}$ *in* $\mathbb{R}^3$*. It is clear that* $[X_1, X_2] = \partial_{x_2}$ *and the other commutators are zero. Thus,*

$$|\det(X_1, X_2, X_3)(x)| = \left|\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = |x_1| \tag{1.6.3}$$

and

$$|\det(X_1, [X_1, X_2], X_3)(x)| = \left|\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = 1. \tag{1.6.4}$$

Now, we construct the Nagel-Stein-Wainger polynomial $\Lambda(x, r)$*. In the case of* (1.6.3)*,* $d(I) = 3$ *and* $|a_I(x)| = |x_1|$*. In the case of* (1.6.4)*,* $d(I) = 4$ *and* $|a_I(x)| = 1$*. So, we have*

$$\Lambda(x, r) = \begin{cases} |x_1|r^3 + r^4 & \text{if } x_1 \neq 0, \\ r^4 & \text{if } x_1 = 0. \end{cases}$$

We use the formula (1.6.2) *to find the pointwise homogeneous dimension. Let* $x \in \mathbb{R}^3$ *be such that* $x_1 = 0$*. Then*

$$Q(x) = \lim_{r\to 0+} \frac{\log \Lambda(x, r)}{\log r} = \lim_{r\to 0+} \frac{\log r^4}{\log r} = 4.$$

Let $x \in \mathbb{R}^3$ *be such that* $x_1 \neq 0$*. Then*

$$Q(x) = \lim_{r\to 0+} \frac{\log \Lambda(x, r)}{\log r} = \lim_{r\to 0+} \frac{\log (|x_1|r^3 + r^4)}{\log r} = \lim_{r\to 0+} \frac{\log r^3 + \log(|x_1| + r)}{\log r} = 3.$$

This concludes that the pointwise homogeneous dimension relative to $\mathbb{R}^3$ *is*

$$Q(x) = \begin{cases} 3 & \text{if } x_1 \neq 0, \\ 4 & \text{if } x_1 = 0. \end{cases}$$

Proposition 1.6.19. *[BLU07, Proposition 2.2.8] Let* $\mathbb{G}$ *be a homogeneous Carnot group with the stratification* $(\mathscr{W}_1, \ldots, \mathscr{W}_r)$*. Then the local homogeneous dimension is*

$$Q := \sum_{i=1}^{r} i \dim(\mathscr{W}_i). \tag{1.6.5}$$

Example 1.6.20. *Any homogeneous Carnot group which is not Euclidean has* $Q \geq 4$*.*

1.6.3 An interesting phenomenon regarding the volume of open balls in Carnot-Carathéodory spaces

Let $X_1 = \partial_{x_1}$ and $X_2 = x_1\partial_{x_2}$ in $\mathbb{R}^2$. Since X_1 and X_2 are Hörmander vector fields, it follows from Theorem 1.6.6 that $d_X(x,y) < \infty$ for all $x, y \in \mathbb{R}^2$. Let $x_1 > 0$ and $r < x_1$. Then

$$x_1 r^2 \le |B_X((x_1,0),r)| \le 6x_1 r^2$$

by [BB23, Proposition 1.85]. However,

$$\frac{r^3}{16} \le |B_X((0,0),r)| \le 2r^3$$

by [BB23, Proposition 1.84]. The volume of an open ball in the Carnot-Carathéodory space is proportional to either r^2 or r^3 depending on the location of a point. In general, for every $(x_1,x_2) \in \mathbb{R}^2$ and $r > 0$ we have

$$C_1\left(r^3 + r^2|x_1|\right) \le |B_X((x_1,x_2),r)| \le C_2\left(r^3 + r^2|x_1|\right), \tag{1.6.6}$$

where C_1 and C_2 depend on $\sqrt{x_1^2 + x_2^2}$ and an upper bound on r, see [BB23, Example 9.2]. Indeed, one can derive

$$|a_I(x)|\, r^{\mathrm{d}(I)} = \begin{cases} |x_1|r^2 + r^3 & \text{if } x_1 \neq 0, \\ r^3 & \text{if } x_1 = 0. \end{cases} \tag{1.6.7}$$

by following Example 1.6.18. Applying [BB23, Theorem 9.1], we get (1.6.6).

1.7 Examples of Hörmander vector fields

1.7.1 Left invariant vector fields generating the Lie algebra of Heisenberg groups

The history of Heisenberg groups and their algebras dates back to the first half of the twentieth century when quantum mechanics formalized mathematically, see e.g. [Wey50]. Researchers found applications of Heisenberg groups in complex analysis of several variables ([CCG07, pp. 5-6]), Cauchy–Riemann geometry, subelliptic partial differential equations as well as Fourier analysis.

Heisenberg groups can be viewed as a particular case of homogeneous Carnot groups as well as Carnot-Carathéodory spaces. If we equip $\mathbb{R}^3$ with the following binary operation

$$(x_1, x_2, t) * \left(\tilde{x}_1, \tilde{x}_2, \tilde{t}\right) = \left(x_1 + \tilde{x}_1, x_2 + \tilde{x}_2, t + \tilde{t} + 2(x_2\tilde{x}_1 - x_1\tilde{x}_2)\right),$$

it gives the Heisenberg-Weyl group $\mathbb{H}^1 = (\mathbb{R}^3, *)$ on $\mathbb{R}^3$. The unit element is $e = (0,0,0)$, the inverse of (x_1, x_2, t) is $(-x_1, -x_2, -t)$.

On the other hand, the following vector fields

$$X_1 = \frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial t} \quad \text{and} \quad X_2 = \frac{\partial}{\partial x_2} - 2x_1\frac{\partial}{\partial t},$$

which are left invariant with respect to $*$ (see Example 1.6.10 for X_1, similarly, one can show that X_2 is also left invariant), together with their commutator generate the Lie algebra of $\mathbb{H}^1$. Moreover,

$$X_1(0) = \left.\frac{\partial}{\partial x_1}\right|_0 \quad \text{and} \quad X_2(0) = \left.\frac{\partial}{\partial x_2}\right|_0.$$

Since $[X_1, X_2] = -4\partial_t$, it is immediate that

$$\operatorname{rank}\left(\operatorname{Lie}\{X_1, X_2\}(x_1, x_2, t)\right) = 3 \quad \text{for every } (x_1, x_2, t) \in \mathbb{R}^3.$$

Note that $\operatorname{Lie}\{X_1, X_2\}$ is nilpotent of step two, that is,

$$[[X_1, X_2], X_1] = [[X_1, X_2], X_2] = 0.$$

To be more precise, the Heisenberg-Weyl group $\mathbb{H}^1$ is a homogeneous Carnot group of step 2. It has 2 generators (X_1 and X_2 are the Jacobian basis) and the dilation is

$$\delta_\lambda(x_1, x_2, t) = \left(\lambda x_1, \lambda x_2, \lambda^2 t\right).$$

The following function

$$d(x) = |x_1| + |x_2| + |t|^{1/2}$$

defines a homogeneous norm on $\mathbb{H}^1$ [BLU07, p. 751]. Another homogeneous norm is

$$d(x,t) = \left(\left(x_1^2 + x_2^2\right)^2 + t^2\right)^{\frac{1}{4}},$$

see e.g. [BLU07, p. 569].

Analogously, we can define $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \diamond)$ endowed with

$$(x, y, t) \diamond (\tilde{x}, \tilde{y}, \tilde{t}) = \left(x + \tilde{x}, t + \tilde{t} + 2\sum_{i=1}^{n}(y_i\tilde{x}_i - x_i\tilde{y}_i)\right),$$

where $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}^n$ and $t, \tilde{t} \in \mathbb{R}$. In this case, the vector fields

$$X_i := \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial t} \quad \text{for } 1 \leq i \leq n$$

and

$$Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t} \quad \text{for } 1 \leq i \leq n,$$

which are left invariant with respect to $\diamond$, together with their commutators generate the Lie algebra of $\mathbb{H}^n$. Since $[X_i, Y_i] = -4\partial_t$ for all $i = 1, \ldots, n$, it follows that

$$\operatorname{rank}\left(\operatorname{Lie}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \partial_t\}(x, y, t)\right) = 2n + 1$$

for every $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. The sum of squares operator

$$\Delta_{\mathbb{H}^n} = \sum_{i=1}^{n} X_i^2 + Y_i^2$$

is called the Kohn-Laplacian, which has applications in the theory of functions of several complex variables, see [Bra14] and [CS01].

1.7.2 Left invariant vector fields generating the Lie algebra of the Engel group

If we equip $\mathbb{R}^4$ with the following binary operation

$$x \bullet \tilde{x} = (x_1 + \tilde{x}_1, x_2 + \tilde{x}_2, x_3 + \tilde{x}_3 + P_1, x_4 + \tilde{x}_4 + P_2),$$

it gives the Engel group $\mathcal{E} = (\mathbb{R}^4, \bullet)$ (see e.g. [CC09, Section 12.3]), where

$$\begin{aligned} P_1 &= \frac{1}{2}(x_1\tilde{x}_2 - x_2\tilde{x}_1), \\ P_2 &= \frac{1}{2}(x_1\tilde{x}_3 - x_3\tilde{x}_1) + \frac{1}{12}\left(x_1^2\tilde{x}_2 - x_1\tilde{x}_1(x_2 + \tilde{x}_2) + x_2\tilde{x}_1^2\right). \end{aligned}$$

The following vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \left(\frac{x_3}{2} - \frac{x_1x_2}{12}\right)\frac{\partial}{\partial x_4}, \qquad & X_3 &= \frac{\partial}{\partial x_3} + \frac{x_1}{2}\frac{\partial}{\partial x_4}, \\ X_2 &= \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2}{12}\frac{\partial}{\partial x_4}, & X_4 &= \frac{\partial}{\partial x_4}, \end{aligned}$$

which are left invariant with respect to $\bullet$, together with their commutators generate the Lie algebra of $\mathcal{E}$. The Lie algebra of the Engel group has step 3, since

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4 \quad \text{and} \quad [X_3, X_4] = 0.$$

1.7.3 Baouendi-Grushin vector fields

We consider the Baouendi-Grushin vector fields introduced in [Bao67],[Gru70] and [Gru71]. Let $X_1 = \partial_{x_1}$ and $X_2 = x_1\partial_{x_2}$ in $\mathbb{R}^2$. Since $[X_1, X_2] = \partial_{x_2}$, it follows that

$$\operatorname{rank}\left(\operatorname{Lie}\left\{X_1, X_2\right\}(x_1, x_2)\right) = 2 \quad \text{for every } (x_1, x_2) \in \mathbb{R}^2. \tag{1.7.1}$$

However, since $X_1 I(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $X_2 I(x) = \begin{pmatrix} 0 \\ x \end{pmatrix}$ are linearly dependent in $\mathbb{R}^2$, it is apparent to see that X_1 and X_2 cannot be left invariant by [BLU07, Proposition 1.2.13].

The sum of squares operator

$$\Delta_X = X_1^2 + X_2^2 = \partial^2_{x_1x_1} + x_1^2\partial^2_{x_2x_2}$$

is called the Baouendi-Grushin operator, it is elliptic on $\mathbb{R}^2\backslash\{x_1 = 0\}$ and subelliptic on $\mathbb{R}^2\backslash\{x_1 \neq 0\}$. The Baouendi-Grushin operator is not locally solvable on $\{x \in \mathbb{R}^2 : x_1 = 0\}$, see e.g. [CC09, Appendix A], so subelliptic operators serve as classic examples of partial differential operators that are not locally solvable.

1.7.4 Non-nilpotent vector fields

In the previous three examples, the vector fields generate nilpotent Lie algebras. However, as shown in the following example, Hörmander vector fields may generate Lie algebras that are not nilpotent.

Example 1.7.1. *Consider* $X_1 = \partial_{x_1} + x_2\partial_{x_2}$ *and* $X_2 = \partial_{x_2}$ *in* $\mathbb{R}^2$. *Since* $[X_2, X_1] = X_2$, *inductively we have*

$$[\cdots[[[X_2, \underbrace{X_1], X_1], X_1]\cdots X_1}_{k\ times}] = X_2 \quad \textit{for all } k \in \mathbb{N}.$$

The corresponding Lie algebra $\operatorname{Lie}\{X_1, X_2\}$ *is not nilpotent, although*

$$\operatorname{rank}\left(\operatorname{Lie}\left\{X_1, X_2\right\}(x_1, x_2)\right) = 2 \quad \textit{for every } (x_1, x_2) \in \mathbb{R}^2.$$

1.8 Sobolev spaces induced by Hörmander vector fields

Here we introduce the Sobolev space $\mathcal{W}^{1,p}_X(\Omega)$ induced by $X_1, \ldots, X_m$. We begin with the notion of weak derivatives with respect to $X_1, \ldots, X_m$. We say that a function $f \in L^1_{loc}(\Omega)$ is differentiable in the weak sense with respect to X_i if there exists a locally integrable function $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega g(x)\varphi(x)dx = \int_\Omega f(x)X_i^*\varphi(x)dx$$

holds for all $\varphi \in C_0^\infty(\Omega)$. In this case, we denote $X_if = g$. We call X_i^* the formal adjoint of X_i, which has the explicit representation

$$X_i^*\varphi(x) = -\sum_{k=1}^n \partial_{x_k}(b_{ik}\varphi(x)) = -\sum_{k=1}^n b_{ik}\partial_{x_k}\varphi - \varphi\sum_{k=1}^n \partial_{x_k}b_{ik} = -X_i\varphi + b\varphi,$$

where $b := -\sum_{k=1}^n \partial_{x_k}b_{ik}$. If $b \not\equiv 0$ in Ω, the formal adjoint X_i^* is not a vector field.

Now, we define the Sobolev space adapted to $X_1, \dots, X_m$ as follows

$$\mathcal{W}^{1,p}_X(\Omega) := \{f \in L^p(\Omega) : \ \ X_if \in L^p(\Omega) \quad \text{for } i = 1, \dots, m\}$$

with

$$\|u\|_{\mathcal{W}^{1,p}_X(\Omega)} := \left(\int_\Omega (|u|^p + |Xu|^p)\,dx\right)^{\frac{1}{p}}, \tag{1.8.1}$$

where the notation $X := (X_1, \dots, X_m)$ stands for the horizontal gradient and its length is given by

$$|Xf| = \left(\sum_{i=1}^m (X_if)^2\right)^{\frac{1}{2}}.$$

If $m = n$ and $X_i = \partial_{x_i}$ for every $i = 1, \dots, n$, we recover the classical Sobolev space $W^{1,p}(\Omega)$.

The completion of $C_0^\infty(\Omega)$ in $\mathcal{W}^{1,p}_X(\Omega)$ gives the trace zero Sobolev space denoted by $\mathcal{W}^{1,p}_{X,0}(\Omega)$. It is known that $\mathcal{W}^{1,p}_{X,0}(\Omega)$ is a reflexive Banach space (see e.g. [Xu90, Theorem 1]). Therefore, by the Eberlein-Šmulian theorem, every bounded sequence in $\mathcal{W}^{1,p}_{X,0}(\Omega)$ contains a weakly convergent subsequence. Consequently, it is necessary to characterize weak convergence in $\mathcal{W}^{1,p}_{X,0}(\Omega)$.

Proposition 1.8.1. *A sequence v_j converges weakly to v in $\mathcal{W}^{1,p}_{X,0}(\Omega)$ if and only if there exist $g_1, \dots, g_m \in L^p(\Omega)$ such that*

$$v_j \rightharpoonup v \quad \textit{weakly in } L^p(\Omega) \qquad \textit{and} \qquad X_iv_j \rightharpoonup g_i \quad \textit{weakly in } L^p(\Omega)$$

for $i = 1, \dots, m$. In this case, $g_i = X_iv$.

Proof. The proof follows a standard approach; please refer to [Le 18, Proposition 1.8] and [Leo17, Corollary 11.70], as well as [Cap+24, Proposition 2.2]. ☐

The following theorem is known as the L^p Poincaré-Friedrichs inequality for Hörmander vector fields and can be found in [Cap+24, Theorem 2.5], see also [CCL24, Proposition 2.6].

Theorem 1.8.2. *There exists a constant $C > 0$ such that*

$$\int_\Omega |u|^p dx \leq C \int_\Omega |Xu|^p dx \tag{1.8.2}$$

holds for all $u \in \mathcal{W}^{1,p}_{X,0}(\Omega)$. Here the constant C does not depend u.

We will establish the best constant in (1.8.2) in Chapter 2. The L^p Poincaré-Friedrichs inequality (1.8.2) allows us to equip $\mathcal{W}^{1,p}_{X,0}(\Omega)$ with the equivalent norm

$$\|u\| := \left(\int_\Omega |Xu|^p \, dx \right)^{\frac{1}{p}}. \tag{1.8.3}$$

We recall continuous and compact Sobolev embeddings, which serve as essential tools for deriving the main results in Chapters 2 and 3.

Theorem 1.8.3. *[Dan91, Corollary 3.3] Let $1 \leq p < \infty$. Then the embedding $\mathcal{W}^{1,p}_{X,0}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.*

Theorem 1.8.4. *Let $1 < p < Q$. Then the embedding $\mathcal{W}^{1,p}_{X,0}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $1 \leq q \leq \frac{Qp}{Q-p}$.*

Proof. The proof follows from [CDG93, Theorem 2.3] and partition of the unity. ☐

Definition 1.8.5. *Let $0 < \alpha < 1$. We say that $f : \Omega \to \mathbb{R}$ is α-Hölder continuous with respect to d_X, if*

$$\sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{d_X(x,y)^\alpha} < \infty.$$

The set of α-Hölder continuous functions with respect to d_X form the Hölder class $\mathcal{C}^{0,\alpha}_X(\Omega)$ induced by $X_1, \ldots, X_m$.

Theorem 1.8.6. *Let $p \geq Q$. Then the embedding $\mathcal{W}^{1,p}_{X,0}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $1 \leq q < \infty$.*

Proof. Let $p = Q$. Then the embedding $\mathcal{W}^{1,p}_{X,0}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $1 \leq q < \infty$ by [CCL24, Remark 1.1]. Now, let $p > Q$. Then the continuous embedding $\mathcal{W}^{1,p}_{X,0}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha}_X(\overline{\Omega})$ follows from [CCL24, Remark 1.1] provided $0 < \alpha < 1$. By [BB23, Proposition 2.14], we have the continuous embedding $\mathcal{C}^{0,\alpha}_X(\overline{\Omega}) \hookrightarrow C^{0,\alpha/s}(\overline{\Omega})$. Combining them, we complete the proof. ☐

For the sake of completeness, we provide a proof of the following theorem, which will be used to apply the Radon–Riesz property in Chapter 3.

Theorem 1.8.7. *$\mathcal{W}^{1,p}_{X,0}(\Omega)$ equipped with (1.8.3) is a uniformly convex space.*

Proof. We treat $1 < p < 2$ and $p \geq 2$ cases separately. First, let $p \geq 2$ and let $u, v \in \mathcal{W}^{1,p}_{X,0}(\Omega)$ satisfy

$$\|u\| = \|v\| = 1 \ \text{ and } \ \|u - v\| \geq \varepsilon \quad \text{for } \varepsilon \in (0, 2].$$

We apply the elementary inequality (see, [AF03, Lemma 2.37])

$$\left|\frac{\omega_1 + \omega_1}{2}\right|^p + \left|\frac{\omega_1 - \omega_1}{2}\right|^p \leq \frac{1}{2}\left(|\omega_1|^p + |\omega_1|^p\right) \quad \text{for all } \omega_1, \omega_2 \in \mathbb{R}^m,$$

when $\omega_1 = Xu(x)$ and $\omega_2 = Xv(x)$ for a fixed $x \in \Omega$, and integrate both sides over Ω to get

$$\begin{aligned}
\left\|\frac{u + v}{2}\right\|^p + \left\|\frac{u - v}{2}\right\|^p &= \int_\Omega \left(\left|\frac{Xu + Xv}{2}\right|^p + \left|\frac{Xu - Xv}{2}\right|^p\right) dx \\
&\leq \frac{1}{2}\int_\Omega \left(|Xu|^p + |Xv|^p\right) dx \\
&= \frac{1}{2}\left(\|u\|^p + \|v\|^p\right) = 1,
\end{aligned}$$

which gives

$$\left\|\frac{u + v}{2}\right\|^p \leq 1 - \left(\frac{\varepsilon}{2}\right)^p.$$

Then there exists $\delta(\varepsilon) > 0$ such that

$$\left\|\frac{u + v}{2}\right\|^p \leq 1 - \delta(\varepsilon).$$

Now, let $1 < p < 2$. Let $u, v \in \mathcal{W}^{1,p}_{X,0}(\Omega)$ satisfy

$$\|u\| = \|v\| = 1 \ \text{ and } \ \|u - v\| \geq \varepsilon \quad \text{for } \varepsilon \in (0, 2].$$

It is easy to check that $|Xu|^{p'}, |Xv|^{p'} \in L^{p-1}(\Omega)$,

$$\||Xu|^{p'}\|_{p-1} = \|u\|^{p'} \ \text{ and } \ \||Xv|^{p'}\|_{p-1} = \|v\|^{p'},$$

where $p' = \frac{p}{p-1}$. Then by using the reverse Minkowski inequality

$$\|\, |Xu|^{p'} + |Xv|^{p'} \,\|_{p-1} \geq \|\, |Xu|^{p'} \,\|_{p-1} + \|\, |Xv|^{p'} \,\|_{p-1}$$

and the following elementary vector inequality (see e.g. [AF03, Lemma 2.37])

$$\left|\frac{\omega_1+\omega_2}{2}\right|^{p'}+\left|\frac{\omega_1-\omega_2}{2}\right|^{p'}\le\left[\frac{1}{2}\left(|\omega_1|^p+|\omega_2|^p\right)\right]^{\frac{1}{p-1}}\quad\text{for all } \omega_1,\omega_2\in\mathbb{R}^m,$$

when $\omega_1 = Xu$ and $\omega_2 = Xv$, we have

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|^{p'}+\left\|\frac{u-v}{2}\right\|^{p'}&=\left\|\left|\frac{Xu+Xv}{2}\right|^{p'}\right\|_{p-1}+\left\|\left|\frac{Xu-Xv}{2}\right|^{p'}\right\|_{p-1}\\&\le\left\|\left|\frac{Xu+Xv}{2}\right|^{p'}+\left|\frac{Xu-Xv}{2}\right|^{p'}\right\|_{p-1}\\&=\left[\int_\Omega\left(\left|\frac{Xu+Xv}{2}\right|^{p'}+\left|\frac{Xu-Xv}{2}\right|^{p'}\right)^{p-1}\right]^{\frac{1}{p-1}}\\&\le\left[\frac{1}{2}\int\left(|Xu|^p+|Xv|^p\right)\right]^{\frac{1}{p-1}}\\&=\left[\frac{1}{2}\|u\|^p+\frac{1}{2}\|v\|^p\right]^{\frac{1}{p-1}}.\end{aligned}$$

Hence,

$$\left\|\frac{u+v}{2}\right\|^{p'}\le 1-\left(\frac{\varepsilon}{2}\right)^{p'}.$$

We have shown that there exists $\delta(\varepsilon)>0$ such that

$$\left\|\frac{u+v}{2}\right\|^{p}\le 1-\delta(\varepsilon).$$

□

1.9 Operators on Banach spaces

Let $\mathbb{B}$ be an infinite dimensional real reflexive Banach space with its dual $\mathbb{B}^*$. For all $\omega\in\mathbb{B}^*$ and $u\in\mathbb{B}$, we set a dual pair $\langle\omega,u\rangle=\omega(u)$. Strong convergence in $\mathbb{B}$ and $\mathbb{B}^*$ is denoted by $\to$. Weak convergence in $\mathbb{B}$ and $\mathbb{B}^*$ is denoted by $\rightharpoonup$.

Definition 1.9.1. *An operator $T:\mathbb{B}\to\mathbb{B}^*$ is odd if*

$$T(-u)=-T(u)\quad\textit{for all } u\in\mathbb{B}. \tag{1.9.1}$$

Definition 1.9.2. *[Zei90, p. 555] If an operator $T:\mathbb{B}\to\mathbb{B}^*$ satisfies*

$$\textit{whenever}\quad u_j\rightharpoonup u\quad\Longrightarrow\quad Tu_j\to Tu \tag{1.9.2}$$

then it is called strongly continuous. It is apparent to see that stronly continuous operators are continuous.

Definition 1.9.3. *If an operator $T : \mathbb{B} \to \mathbb{B}^*$ satisfies*

$$\lim_{\|u\|\to\infty} \frac{\langle Tu, u\rangle}{\|u\|} = +\infty,$$

then it is called coercive.

Definition 1.9.4. *If an operator $T : \mathbb{B} \to \mathbb{B}^*$ satisfies*

$$\langle Tu - Tv, u - v\rangle \geq \|u - v\|^p \quad \textit{for all } u, v \in \mathbb{B},$$

then it is called uniformly monotone. The definition is restricted to our case, which is sufficient, indeed, the general definition can be found in e.g. [Zei90, p. 501].

Definition 1.9.5. *An operator $\Phi : \mathbb{B} \to \mathbb{B}^*$ is said to satisfy condition $(S)_0$, whenever a sequence $\{u_n\}$ in $\mathbb{B}$ satisfies*

$$\begin{aligned} u_n &\rightharpoonup u && \textit{in } \mathbb{B}, \\ \Phi(u_n) &\rightharpoonup v && \textit{in } \mathbb{B}^*, \\ \langle \Phi(u_n), u_n\rangle &\to \langle v, u\rangle && \textit{in } \mathbb{R}, \end{aligned}$$

then $u_n \to u$ in $\mathbb{B}$.

1.10 Critical point theory

1.10.1 Palais-Smale condition

The notion of compactness plays an important role in analysis. Heine-Cantor theorem asserts that every continuous function on a compact subset of $\mathbb{R}^n$ is uniformly continuous. Extreme value theorem states that every continuous function on a nonempty compact subset of $\mathbb{R}^n$ attains the maximum and minimum values. According to the Bolzano–Weierstrass theorem, every bounded sequence in $\mathbb{R}^n$ admits a convergent subsequence. Therefore, thanks to the Bolzano–Weierstrass theorem, if a set E in $\mathbb{R}^n$ is compact, then it is sequentially compact. Recall that a set E in $\mathbb{R}^n$ is sequentially compact, if every sequence in E admits a subsequence converging to a point in E. Another application of the Bolzano–Weierstrass theorem is the following: *Let E be closed and nonempty in $\mathbb{R}^n$ and $f : E \to \mathbb{R}$ be lower semicontinuous. Then*

f attains a minimum in E if and only if there exists a bounded minimizing sequence in E. On top of the lower semicontinuity of f, if we assume that f is coercive, that is,

$$f(x) \to \infty \quad \text{as } |x| \to \infty,$$

then f admits a minimizing sequence in E, since every minimizing sequence in E is bounded for coercive functions.

Now we try to extend these results to infinite dimensional normed spaces. Note that, compactness and sequentially compactness coincide in normed spaces. However, infinite dimensional normed spaces do not possess the Heine-Borel property, that is, if a set is closed and bounded, then the set is not necessarily compact. Compactness is a highly restrictive assumption in infinite-dimensional normed spaces. In order to have a large class of compact subsets, we can weaken their topologies, however, this process shrinks the class of lower semicontinuous real functions. To avoid swinging from one extreme to the other, we can replace the class of weakly lower semicontinuous real functions with the class of weakly sequentially lower semicontinuous real functions in infinite dimensional Hilbert spaces. So we have the following extension: *Let E be nonempty and weakly sequentially closed in a infinite dimensional Hilbert space $\mathscr{H}$. Also, let $f : E \to \mathbb{R}$ be weakly sequentially lower semicontinuous. Then f admits a minimum in E if and only if there exists a bounded minimizing sequence in E.* Here the Bolzano–Weierstrass theorem is replaced with the following statement: *Let E be a nonempty subset of $\mathscr{H}$. Then E is weak sequentially compact if and only if it is weak sequentially closed and bounded.* When E is a convex set and f is a convex function, then it suffices to assume that E is closed and nonempty and f is lower semicontinuous. We can derive the same results if we consider reflexive infinite dimensional Banach spaces.

Now let us consider infinite dimensional Banach spaces which also covers non-reflexive spaces. Can we ensure that there exists a minimum for a function f in this case under some assumptions? The positive answer is given by R.S. Palais and S. Smale [PS64], who introduced a compactness condition known as Palais-Smale condition or in short PS condition on f.

Definition 1.10.1. *[Jab03, p. 16] A functional $\Phi \in C^1(\mathbb{B}, \mathbb{R})$ is said to satisfy Palais-Smale condition, in short* PS*-condition: if $\{u_j\}$ is a sequence in $\mathbb{B}$ such that*

(i) $\{\Phi(u_j)\}$ *is bounded;*

(ii) $\Phi'(u_j) \to 0$ *in* $\mathbb{B}^*$,

then it admits a convergent subsequence.

Also, we present the localized version of the Palais-Smale condition.

Definition 1.10.2. *[Jab03, p. 16] We say that a functional* $\Phi \in C^1(\mathbb{B}, \mathbb{R})$ *satisfies the local Palais-Smale condition at level* $c \in \mathbb{R}$*, or shortly* $(PS)_c$*, whenever a sequence* $\{u_n\}$ *in* $\mathbb{B}$ *satisfies*

(i) $\Phi(u_n) \to c$*;*

(ii) $\Phi'(u_n) \to 0$ *in* $\mathbb{B}^*$*,*

then there exists a convergent subsequence of $\{u_n\}$*.*

We present an example of a function satisfying the Palais–Smale condition at some $c \in \mathbb{R}$ to gain a better understanding of this condition.

Example 1.10.3. *Let* $\mathbb{B} = \mathbb{R}$*. The function* $\Phi(t) = \sin(t)$ *on* $\mathbb{R}$ *satisfies* $(\mathrm{PS})_{\mathrm{c}}$ *condition for all* $c \in \mathbb{R} \setminus \{-1, 1\}$*, but not for* $c = \pm 1$*.*

Indeed, for $|c| > 1$ *there is no sequence* $\{t_j\}$ *such that*

(i) $\Phi(t_j) \to c$*;*

(ii) $\Phi'(t_j) \to 0$ *in* $\mathbb{R}$*,*

because the range of Φ *lies between* -1 *and* 1*.*

Now let $|c| < 1$*. In this case, if* $\sin(t_j) \to c$ *for some* $\{t_j\}$*, then we cannot have* $\cos(t_j) \to 0$*, It becomes easier to verify when we visualize it by drawing two graphs:* $\Phi(t) = \sin(t)$ *and* $\Phi'(t) = \cos(t)$ *in* $\mathbb{R}^2$*.*

Lastly, let $c \in \{-1, 1\}$*. Let* $t_j = \pi/2 + 2\pi j$ *and* $\tau_j = -\pi/2 + 2\pi j$ *for all* $j \in \mathbb{N}$*. Then*

(i) $\Phi(t_j) \to 1$ *and* $\Phi(\tau_j) \to -1$*;*

(ii) $\Phi'(t_j) \to 0$ *and* $\Phi'(\tau_j) \to 0$ *in* $\mathbb{R}$*,*

however, $\{t_j\}$ *and* $\{\tau_j\}$ *do not possess convergent subsequences.*

We refer to [MW10] for the history, development of the Palais-Smale condition and we refer to [Jab03] for other types of compactness conditions such as $(\mathrm{WPS})_c$ and (WPS) conditons.

1.10.2 Krasnoselskii genus

Definition 1.10.4. *Let* $M \subset \mathbb{B}$ *be a symmetric compact subset such that* $0 \notin M$*. The set* M *is said to have genus* n*, denoted by* $\gamma(M) = n$*, if there is an odd continuous mapping* $h : M \to \mathbb{R}^n \setminus \{0\}$*, where* n *is the smallest natural number holding this property. If there does not exist such a natural number* n*, we set* $\gamma(A) = \infty$*. In the literature,* γ *is called the Krasnoselskii genus.*

Let us review properties of the genus γ. We define a class of sets

$$\mathcal{V} = \{M \subset \mathbb{B} \setminus \{0\} : \quad M \text{ is compact and symmetric}\}. \tag{1.10.1}$$

Proposition 1.10.5. *[Rab73, Lemma 3.5] Let* $A, B \in \mathcal{V}$*. Then*

(i) If there exists an odd continuous mapping from A *to* B*, then* $\gamma(A) \leq \gamma(B)$*;*

(ii) If $A \subseteq B$*, then* $\gamma(A) \leq \gamma(B)$*;*

(iii) If $f : A \to B$ *is an odd homeomorphism, then* $\gamma(A) = \gamma(B)$*;*

(iv) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$*;*

(v) If $\gamma(B) < \infty$*, then* $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$*;*

(vi) If A *is a compact set , then* $\gamma(A) < \infty$*;*

(vii) If A *is a compact set, then there is a uniform neighborhood* $N_\delta(A)$ *(that is, the set of all points within a distance* δ *from* A *) such that* $\gamma(N_\delta(A)) = \gamma(A)$*.*

1.10.3 Lusternik-Schnirelmann theory

We begin with the Courant-Fischer theorem, which can be viewed as a particular case of the Lusternik-Schnirelman theory.

Theorem 1.10.6. *[Jab03, p. 121] Let* $\mathbf{A}$ *be a real symmetric* $n \times n$ *matrix. Then* k*th eigenvalue is given by*

$$\eta_k = \min_{\substack{\dim \mathbb{V}=k \\ \mathbb{V} \subset \mathbb{R}^n}} \max_{\substack{\vartheta \in \mathbb{V} \\ \vartheta \neq 0}} \frac{\langle \mathbf{A}\vartheta, \vartheta \rangle}{\langle \vartheta, \vartheta \rangle} = \max_{\substack{\dim \mathbb{W}=n-k+1 \\ \mathbb{W} \subset \mathbb{R}^n}} \min_{\substack{\vartheta \in \mathbb{W} \\ \vartheta \neq 0}} \frac{\langle \mathbf{A}\vartheta, \vartheta \rangle}{\langle \vartheta, \vartheta \rangle},$$

where $k = 1, \dots, n$*.*

It is possible to extend this result to real separable infinite-dimensional Hilbert spaces; see [Zei85, Proposition 44.1]. Moreover, even more general cases can be considered in the setting of infinite-dimensional real reflexive Banach spaces, which is the case we focus on here.

Let F and G be functionals defined in $\mathbb{B}$ and let $\mu \in \mathbb{R}$. Define the level set

$$\mathcal{G} := \{u \in X : G(u) = 1\}. \tag{1.10.2}$$

Then the following eigenvalue problem

$$F'(u) = \mu G'(u) \quad \text{on } \mathcal{G} \tag{1.10.3}$$

can be addressed by using the Lusternik-Schnirelman theory (see e.g. [Zei85, Section 44.5]).

We assume that

(H1) F and G are continuously differentiable even functionals such that $F(0) = G(0) = 0$;

(H2) F' is a strongly continuous operator. Additionally,

$$\langle F'(u), u\rangle = 0, \ \ u \in \overline{\text{conv}}\mathcal{G} \quad \Longrightarrow \quad F(u) = 0, \tag{1.10.4}$$

where $\overline{\text{conv}}\mathcal{G}$ is the closed convex hull of the level set $\mathcal{G}$;

(H3) G' is bounded, continuous and satisfies condition $(S)_0$;

(H4) $\mathcal{G}$ is a bounded set and

$$u \neq 0 \quad \Longrightarrow \quad \begin{aligned} &\langle G'(u), u\rangle > 0; \\ &\lim_{t\to+\infty} G(tu) = +\infty; \\ &\inf_{u\in\mathcal{G}} \langle G'(u), u\rangle > 0. \end{aligned}$$

Proposition 1.10.7. *([Zei85, Proposition 43.21]). A function $u \neq 0$ is an eigenfunction of (1.10.3) if and only if u is a critical point of the functional F with respect to $\mathcal{G}$.*

We define

$$\mathcal{G}_n := \{\mathcal{A} \subset \mathcal{G} : \mathcal{A} \ \text{ is a compact and symmetric set}, \ \gamma(\mathcal{A}) \geq n\}.$$

Also, let

$$\beta_n := \begin{cases} \sup_{\mathcal{A}\in\mathcal{G}_n} \inf_{u\in\mathcal{A}} F(u) & \text{if } \mathcal{G}_n \neq \varnothing, \\ 0 & \text{if } \mathcal{G}_n = \varnothing, \end{cases}$$

and

$$\chi := \begin{cases} \sup\{n \in \mathbb{N} : \beta_n > 0\} & \text{if } \beta_1 > 0, \\ 0 & \text{if } \beta_1 = 0. \end{cases}$$

The following theorem can be found in [Zei85, Theorem 44.A] and is known as the Lusternik-Schnirelman principle.

Theorem 1.10.8. *Assume that* ***(H1)-(H4)*** *are valid. Then*

(i) If $\beta_n > 0$, then eigenvalue problem (1.10.3) admits eigenpairs (μ_n, u_n) such that $\mu_n \neq 0$ and $F(u_n) = \beta_n$;

(ii) If $\chi = \infty$, then eigenvalue problem (1.10.3) admits infinitely many eigenfunctions u_n associated with nonzero eigenvalues μ_n;

(iii) The sequence $\{\beta_n\}$ is nonincreasing, nonnegative and converges to 0;

(iv) If $\chi = \infty$ and

$$F(u) = 0, \;\; u \in \overline{\operatorname{conv}}\mathcal{G} \quad\Longrightarrow\quad \langle F'(u), u\rangle = 0, \tag{1.10.5}$$

then there exists a sequence of distinct eigenvalues $\{\mu_n\}$ of (1.10.3) converging to 0;

(v) Assume that

$$F(u) = 0 \;\text{ and }\; u \in \overline{\operatorname{conv}}\mathcal{G} \quad\Longrightarrow\quad u = 0.$$

Then $\chi = \infty$ and there exists a sequence of eigenpairs $\{(u_n, \mu_n)\}$ of (1.10.3) such that

$$u_n \rightharpoonup 0 \quad \text{in} \quad \mathbb{B} \quad \text{and} \quad \mu_n \to 0,$$

where $\mu_n \neq 0$ for all $n \in \mathbb{N}$.

Remark 1.10.9. *We refer to [Zei85, Remark 44.23] for the assumptions* ***(H2)-(H3)****.*

At the end of this subsection, we highlight some important historical developments. The Lusternik-Schnirelmann theory dates back to 1934 [LS34]. It was first established on C^2 Finsler manifolds in [Pal66] and later extended to C^1 Finsler manifolds by A. Szulkin in [Szu88].

Chapter 2

Subelliptic p-Laplacian spectral problem for Hörmander vector fields

> The Latin word *"spectrum"* was adopted by I. Newton in 1671 to describe the range of colors observed when white light passes through a prism.
>
> From *Spectral theory – basic concepts and applications [Bor20, Introduction]*

Let Ω be a bounded domain in $\mathbb{R}^n$ and let $1 < p < \infty$. The following eigenvalue problem

$$\begin{aligned} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) &= -\lambda |u|^{p-2}u \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.0.1}$$

serves as the prominent prototype of our objective. There is extensive literature on the problem (2.0.1), see, for example, [FPS15], [Lin08] and the references cited therein.

Now, let $\mathbb{G} = (\mathbb{R}^n, \star, \delta_\lambda)$ be a homogeneous Carnot group. Then (2.0.1) has the following generalization

$$\begin{aligned} \operatorname{div}_{\mathbb{G}}\left(|\nabla_{\mathbb{G}} u|^{p-2}\,\nabla_{\mathbb{G}} u\right) &= -\lambda |u|^{p-2}u \quad && \text{in } \Omega \subset \mathbb{G}, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.0.2}$$

where $\nabla_{\mathbb{G}}$ is the horizontal gradient and $\operatorname{div}_{\mathbb{G}} v := \nabla_{\mathbb{G}} \cdot v$ is the horizontal divergence on $\mathbb{G}$. To the best of our knowledge, there is relatively limited research conducted on (2.0.2). For instance, N. Wei et al. established several estimates of the first and second eigenvalues of (2.0.2) in [WNL09].

When $p = 2$, then the problem (2.0.1) reduces to

$$\begin{aligned} \Delta_{\mathbb{G}} u &= -\lambda u \quad \text{in } \Omega \subset \mathbb{G}, \\ u &= 0 \quad\quad\ \ \text{on } \partial\Omega, \end{aligned} \tag{2.0.3}$$

where the operator $\Delta_{\mathbb{G}}$ is known as the sub-Laplacian. Recently, M. Carfagnini and M. Gordina proved that the first eigenvalue is simple and all eigenfunctions are continuous in [CG24].

The sub-Laplacian $\Delta_{\mathbb{G}}$ is a sum of squares operator derived from left-invariant vector fields that generate the Lie algebra of the homogeneous Carnot group $\mathbb{G}$. If we replace these left invariant vector fields with smooth vector fiels that satisfy Hörmander's rank condition, then the eigenvalue problem (2.0.3) takes the following form

$$\begin{aligned} \Delta_X u &= -\lambda u \quad \text{in } \Omega, \\ u &= 0 \quad\quad\ \ \text{on } \partial\Omega. \end{aligned} \tag{2.0.4}$$

H. Chen and H.-G. Chen studied the eigenvalue problem (2.0.4) in Carnot-Carathéodory spaces by assuming only the Hörmander finite rank condition for smooth vector fields in Ω, see [CC21]. It is worthwhile to mention that they found the best constant of the L^2 Poincaré-Friedrichs inequality for Hörmander vector fields in [CC21]. Since we deal with a generalization of the eigenvalue problem (2.0.4), our findings allow us to find the best constant of the L^p Poincaré-Friedrichs inequality for Hörmander vector fields within the entire range $1 < p < \infty$ (see Corollary 2.1.8).

Now, we can turn to our objective. Let $X_1, \ldots, X_m$ be smooth Hörmander vector fields with $m \leq n$. We aim to study eigenvalues and eigenfunctions of

$$\begin{aligned} \sum_{j=1}^{m} X_j^* \left(|Xu|^{p-2} X_j u \right) = \lambda |u|^{p-2} u \quad &\text{in } \Omega, \\ u = 0 \quad\quad\quad\quad\quad &\text{on } \partial\Omega, \end{aligned} \tag{2.0.5}$$

where X_j^* is the formal adjoint of X_j. The operator on the left hand side of (2.0.5) is called the subelliptic p-Laplacian. It is worth mentioning that M. Ruzhansky et al. proved the simplicity of the first eigenvalue of (2.0.5) by assuming the existence of a positive eigenfunction u via the Picone identity in [RSS21], whereas we provide an alternative proof involving a suitable choice of test functions without assuming the existence of a positive eigenfunction u.

Let us review some established results related to our topic. A.M. Hansson [Han08] obtained the Friedlander-Filonov inequality (the inequality between $(k+1)$th

Neumann and kth Dirichlet eigenvalues of the Heisenberg Laplacian) for the Heisenberg Laplacian in domains with some geometric properties in $\mathbb{H}^1$, later this result was extended to any domain in $\mathbb{H}^1$ by R.L. Frank and A. Laptev [FL10].

A.M. Hansson and A. Laptev [HL08] proved the sharp Berezin-Li-Yau type inequality for the Heisenberg Laplacian, later H. Kovařík, et al. [KRW18] improved it by adding a lower order term.

S. Eswarathasan and C. Letrouit obtained the upper bound for the number of nodal domains of eigenfunctions of sub-Laplacians on compact manifolds under various assumptions in [EL23], for instance, when the boundary of the domain is smooth and non-characteristic. The upper bound depends either on the index of the corresponding eigenvalue or on the sum of the index and multiplicity of the corresponding eigenvalue. R.L. Frank and B. Helffer [FH24] extended Pleijel's theorem in the setting of sub-Riemannian manifolds.

H. Chen and P. Luo [CL15] derived lower bounds for a finite sum of eigenvalues of (2.0.4) provided that $\partial\Omega$ is non-characteristic for $X_1, \ldots, X_m$. When the sum of square operator is the Baouendi-Grushin operator, they showed the lower bound explicitly.

Y.C. de Verdiere et al. derived small-time asymptotics of the microlocal and local Weyl law for sub-Riemannian Laplacians in [VHT22].

We refer to [GKR24] for the fractional counterpart of (2.0.2) on stratified Lie groups: they established simplicity and isolatedness of the first eigenvalue and positivity of the first eigenfunction.

The main results of this chapter are contained in [KS25].

2.1 Some properties of eigenvalues and eigenfunctions

We study the following eigenvalue problem

$$\begin{aligned} \sum_{i=1}^{m} X_i^* \left(|Xu|^{p-2} X_i u \right) = \lambda |u|^{p-2} u \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{2.1.1}$$

in the weak sense.

Definition 2.1.1. *We say that $u \in \mathcal{W}^{1,p}_{X,0}(\Omega) \setminus \{0\}$ is an eigenfunction of (2.1.1) if*

$$\int_\Omega |Xu|^{p-2} Xu \cdot X\varphi dx = \lambda \int_\Omega |u|^{p-2} u \varphi dx \tag{2.1.2}$$

holds for all $\varphi \in \mathcal{W}^{1,p}_{X,0}(\Omega)$*. In this case, we call* λ *an eigenvalue.*

First, we show that all eigenvalues of (2.1.1) can be bounded below.

Proposition 2.1.2. *Let* λ *be an eigenvalue of* (2.1.1) *with a corresponding eigenfunction* u*. Then there exists* $C > 0$ *such that*

$$\lambda \geq \frac{1}{C|\Omega|^{1-\frac{p}{p^*}}},$$

where $|\Omega|$ *is* n*-dimensional Lebesgue measure of* Ω *and*

$$p^* = \begin{cases} \frac{Qp}{Q-p} & \text{if } 1 < p < Q, \\ 2p & \text{if } p \geq Q, \end{cases} \tag{2.1.3}$$

The constant C *is independent of* u *and* λ*.*

Proof. By Theorem 1.8.4 and Theorem 1.8.6, we have the following continuous embedding

$$\mathcal{W}^{1,p}_{X,0}(\Omega) \hookrightarrow \begin{cases} L^{\frac{Qp}{Q-p}}(\Omega) & \text{if } 1 < p < Q, \\ L^{2p}(\Omega) & \text{if } p \geq Q. \end{cases} \tag{2.1.4}$$

Then applying the Hölder inequality and (2.1.4), we get

$$\int_\Omega |Xu|^p dx = \lambda \int_\Omega |u|^p dx \leq \lambda |\Omega|^{1-\frac{p}{p^*}} \left(\int_\Omega |u|^{p^*} dx \right)^{\frac{p}{p^*}} \leq C\lambda |\Omega|^{1-\frac{p}{p^*}} \int_\Omega |Xu|^p dx,$$

where p^* is expressed as (2.1.3) and the norm (1.8.3) has been employed. □

Corollary 2.1.3. *Every eigenvalue of* (2.1.1) *is positive.*

Although the following result will not be used throughout the rest of the thesis, it is of interest on its own by demonstrating that the spectrum is closed in $\mathbb{R}$.

Theorem 2.1.4. *The spectrum of* (2.1.1) *is a closed set.*

Proof. Let $\{\tilde{\lambda}_j\}$ be a convergent sequence of eigenvalues. We denote its limit by $\tilde{\lambda} \in \mathbb{R}$. If $\tilde{\lambda}$ is also an eigenvalue of (2.1.1), the proof is complete. Let $\tilde{u}_j$ be an eigenfunction associated with $\tilde{\lambda}_j$ for every $j \in \mathbb{N}$. If u is an eigenfunction, then Cu is also an eigenfunction provided $C \neq 0$. Therefore, without loss of generality, it suffices to prove the theorem for $\|\tilde{u}_j\|_p = 1$. Then

$$\tilde{\lambda}_j = \int_\Omega |X\tilde{u}_j|^p \, dx$$

implies that $\{\tilde{u}_j\}$ is a bounded sequence in $\mathcal{W}^{1,p}_{X,0}(\Omega)$. Applying the Eberlein-Šmulian theorem, Proposition 1.8.1 and Theorem 1.8.3, we can find a subsequence $\{\tilde{u}_{j_k}\}$ and a function $\tilde{u} \in \mathcal{W}^{1,p}_{X,0}(\Omega)$ such that

$$\tilde{u}_{j_k} \to \tilde{u} \quad \text{strongly in } L^p(\Omega) \quad \text{and} \quad X\tilde{u}_{j_k} \rightharpoonup X\tilde{u} \quad \text{weakly in } L^p(\Omega). \tag{2.1.5}$$

The function $\tilde{u}_{j_k} - \tilde{u}$ belongs to $\mathcal{W}^{1,p}_{X,0}(\Omega)$, so we can insert $u = \tilde{u}_{j_k}$ and $\varphi = \tilde{u}_{j_k} - \tilde{u}$ into (2.1.2) to have

$$\begin{aligned} &\int_\Omega \left(|X\tilde{u}_{j_k}|^{p-2} X\tilde{u}_{j_k} - |X\tilde{u}|^{p-2} X\tilde{u}\right) \cdot (X\tilde{u}_{j_k} - X\tilde{u})\, dx \\ &= \tilde{\lambda}_{j_k} \int_\Omega |\tilde{u}_{j_k}|^{p-2}\, \tilde{u}_{j_k} (\tilde{u}_{j_k} - \tilde{u})\, dx - \int_\Omega |X\tilde{u}|^{p-2} X\tilde{u} \cdot (X\tilde{u}_{j_k} - X\tilde{u})\, dx. \end{aligned} \tag{2.1.6}$$

Keeping into account (2.1.5) and letting $k \to \infty$, the equality (2.1.6) leads to

$$\lim_{k\to\infty} \int_\Omega \left(|X\tilde{u}_{j_k}|^{p-2} X\tilde{u}_{j_k} - |X\tilde{u}|^{p-2} X\tilde{u}\right) \cdot (X\tilde{u}_{j_k} - X\tilde{u})\, dx = 0. \tag{2.1.7}$$

If

$$X\tilde{u}_{j_k} \to X\tilde{u} \quad \text{strongly in } L^p(\Omega), \tag{2.1.8}$$

then $\tilde{\lambda}$ is an eigenvalue and $\tilde{u}$ is the corresponding eigenfunction.

Let $p \geq 2$. The vector inequality (6.0.2), in particular, becomes

$$C\,|\omega_2 - \omega_1|^p \leq p\left(|\omega_2|^{p-2}\,\omega_2 - |\omega_1|^{p-2}\,\omega_1\right) \cdot (\omega_2 - \omega_1) \qquad \text{for all } \omega_1, \omega_2 \in \mathbb{R}^m. \tag{2.1.9}$$

Substituting $\omega_1 = X\tilde{u}$ and $\omega_2 = X\tilde{u}_{j_k}$ into the inequality (2.1.9) and integrating over Ω, we observe that the limit (2.1.7) implies

$$\lim_{k\to\infty} \int_\Omega |X\tilde{u}_{j_k} - X\tilde{u}|^p\, dx = 0.$$

Let $1 < p < 2$. The vector inequality (6.0.1), in particular, becomes

$$C\frac{|\omega_2 - \omega_1|^2}{(|\omega_2| + |\omega_1|)^{2-p}} \leq p\left(|\omega_2|^{p-2}\,\omega_2 - |\omega_1|^{p-2}\,\omega_1\right) \cdot (\omega_2 - \omega_1) \qquad \text{for all } \omega_1, \omega_2 \in \mathbb{R}^m. \tag{2.1.10}$$

Substituting $\omega_1 = X\tilde{u}$ and $\omega_2 = X\tilde{u}_{j_k}$ into the inequality (2.1.10) and integrating over Ω, we get

$$\lim_{k\to\infty} \int_\Omega \frac{|X\tilde{u}_{j_k} - X\tilde{u}|^2}{(|X\tilde{u}_{j_k}| + |X\tilde{u}|)^{2-p}} dx = 0.$$

To obtain $\|\tilde{u}_{j_k}-\tilde{u}\|\to 0$ for $1<p<2$, we employ the Hölder inequality

$$\int_\Omega |X\tilde{u}_{j_k}-X\tilde{u}|^p dx = \int_\Omega (|X\tilde{u}_{j_k}|+|X\tilde{u}|)^{\frac{p(2-p)}{2}} \frac{|X\tilde{u}_{j_k}-X\tilde{u}|^p}{(|X\tilde{u}_{j_k}|+|X\tilde{u}|)^{\frac{p(2-p)}{2}}} dx$$

$$\leq \left(\int_\Omega \left(|X\tilde{u}_{j_k}|^p+|X\tilde{u}|^p\right)dx\right)^{\frac{2-p}{2}} \left(\int_\Omega \frac{|X\tilde{u}_{j_k}-X\tilde{u}|^2}{(|X\tilde{u}_{j_k}|+|X\tilde{u}|)^{2-p}}dx\right)^{\frac{p}{2}}$$

$$\leq C\left(\int_\Omega \frac{|X\tilde{u}_{j_k}-X\tilde{u}|^2}{(|X\tilde{u}_{j_k}|+|X\tilde{u}|)^{2-p}}dx\right)^{\frac{p}{2}}.$$

Letting $k\to\infty$, we arrive at (2.1.8). The proof is complete. □

The following lemma will be crucial in proving Theorem 2.1.6 and Theorem 2.2.1.

Lemma 2.1.5. *Let (u,λ) be an eigenpair. Then u is essentially bounded in Ω.*

Proof. The case $p>Q$ is trivial, because we have the embedding $\mathcal{W}^{1,p}_{X,0}(\Omega)\hookrightarrow \mathcal{C}^{0,\alpha}_X(\overline{\Omega})$ provided $0<\alpha<1$, see e.g. [CCL24, Remark 1.1].

Now, let $1<p\leq Q$ and $k>0$. We will show that $\operatorname{ess\,sup}_\Omega u<\infty$. Assume that the set $\{x\in\Omega:\ u(x)>0\}$ has positive n-dimensional Lebesgue measure. Otherwise, it is sufficient to prove only $\operatorname{ess\,inf}_\Omega u>-\infty$. We can verify that

$$\varphi(x):=\max\{u(x)-k,0\},\quad x\in\Omega,$$

is admissible in $\mathcal{W}^{1,p}_{X,0}(\Omega)$. From

$$\int_\Omega \varphi^p dx=\int_{\Omega_k}(u-k)^p dx\leq \int_{\Omega_k} u^p dx\leq \int_\Omega |u|^p dx<\infty,$$

it follows that $\varphi\in L^p(\Omega)$, where

$$\Omega_k:=\{x\in\Omega:\ u(x)>k\}.$$

Moreover,

$$\int_\Omega |X\varphi|^p dx=\int_{\Omega_k}|Xu|^p dx\leq \int_\Omega |Xu|^p dx<\infty.$$

Lastly, $\varphi|_{\partial\Omega}=\max\{u|_{\partial\Omega}-k,0\}=0$, so φ can be viewed as a test function. Hence,

$$\int_{\Omega_k}|Xu|^p dx=\lambda\int_{\Omega_k} u^{p-1}(u-k)dx.$$

The embedding $\mathcal{W}^{1,p}_{X,0}(\Omega)\hookrightarrow L^1(\Omega)$ is continuous for all $1<p\leq Q$, that is, from Theorem 1.8.4 we infer the case $1<p<Q$ and from Theorem 1.8.6 we infer the case $p=Q$. Thus, $u\in L^1(\Omega)$. Moreover, from

$$k\,|\Omega_k|=\int_{\Omega_k} k dx\leq \int_{\Omega_k} u dx\leq \int_\Omega |u| dx=\|u\|_1, \tag{2.1.11}$$

it follows that

$$\lim_{k\to\infty} |\Omega_k| = 0. \tag{2.1.12}$$

We multiply the following inequality

$$u^{p-1} \le 2^{p-1}(u-k)^{p-1} + 2^{p-1}k^{p-1} \ \text{ in } \Omega_k$$

by $u-k$ and integrate over Ω_k to get

$$\int_{\Omega_k} u^{p-1}(u-k)dx \le 2^{p-1}\int_{\Omega_k}(u-k)^p dx + 2^{p-1}k^{p-1}\int_{\Omega_k}(u-k)dx. \tag{2.1.13}$$

Then we apply the Hölder inequality and Sobolev embedding (2.1.4) to get

$$\begin{aligned}\int_{\Omega_k}(u-k)^p dx &\le |\Omega_k|^{1-\frac{p}{p^*}}\left(\int_{\Omega_k}(u-k)^{p^*}dx\right)^{\frac{p}{p^*}} = |\Omega_k|^{1-\frac{p}{p^*}}\left(\int_{\Omega}\varphi^{p^*}dx\right)^{\frac{p}{p^*}}\\ &\le C\,|\Omega_k|^{1-\frac{p}{p^*}}\int_{\Omega}|X\varphi|^p dx = C\,|\Omega_k|^{1-\frac{p}{p^*}}\int_{\Omega_k}|Xu|^p dx \\ &= \lambda C\,|\Omega_k|^{1-\frac{p}{p^*}}\int_{\Omega_k}|u|^{p-2}u(u-k)dx,\end{aligned} \tag{2.1.14}$$

where p^* is expressed as (2.1.3). Multiplying both sides of (2.1.13) by $\lambda C\,|\Omega_k|^{1-\frac{p}{p^*}}$ (note that λ is positive) and using the inequality (2.1.14), we derive

$$\left(1-\lambda C\,|\Omega_k|^{1-\frac{p}{p^*}}\,2^{p-1}\right)\int_{\Omega_k}(u-k)^p dx \le \lambda C\,|\Omega_k|^{1-\frac{p}{p^*}}\,2^{p-1}k^{p-1}\int_{\Omega_k}(u-k)dx.$$

Then taking into account (2.1.11), we have

$$\lambda C\,|\Omega_k|^{1-\frac{p}{p^*}}\,2^{p-1} \le \lambda C\frac{\|u\|_1^{1-\frac{p}{p^*}}}{k^{1-\frac{p}{p^*}}}2^{p-1}.$$

Since $\|u\|_1/k$ goes to zero as $k\to\infty$, for any $\varepsilon>0$ there exists a natural number k_ε such that for all $k\ge k_\varepsilon$

$$\lambda C\frac{\|u\|_1^{1-\frac{p}{p^*}}}{k^{1-\frac{p}{p^*}}}2^{p-1} < \varepsilon \tag{2.1.15}$$

holds true. Here

$$\lambda^{\frac{p^*}{p^*-p}}C^{\frac{p^*}{p^*-p}}2^{\frac{(p-1)p^*}{p^*-p}}\|u\|_1\varepsilon^{-\frac{p^*}{p^*-p}} < k_\varepsilon.$$

We can choose

$$k_\varepsilon = \lambda^{\frac{p^*}{p^*-p}}C^{\frac{p^*}{p^*-p}}2^{\frac{(p-1)p^*}{p^*-p}}\|u\|_1\varepsilon^{-\frac{p^*}{p^*-p}} + 1$$

for (2.1.15). Now, let $\varepsilon = \frac{1}{2}$. Then

$$\lambda C\,|\Omega_k|^{1-\frac{p}{p^*}}\,2^{p-1} < \frac{1}{2}$$

is valid for every $k \geq k_\varepsilon$. As a result,

$$\int_{\Omega_k} (u-k)^p dx < \lambda C \left|\Omega_k\right|^{1-\frac{p}{p^*}} 2^p k^{p-1} \int_{\Omega_k} (u-k) dx \quad \text{for all } k \geq k_\varepsilon. \tag{2.1.16}$$

By using the Hölder inequality, we have

$$\left(\int_{\Omega_k} (u-k) dx\right)^p \leq |\Omega_k|^{p-1} \int_{\Omega_k} (u-k)^p dx. \tag{2.1.17}$$

Lastly, combining (2.1.16) and (2.1.17), we obtain

$$\int_{\Omega_k} (u-k) dx < (2^p \lambda C)^{\frac{1}{p-1}} k \left|\Omega_k\right|^{1+\frac{p^*-p}{p^*(p-1)}} \quad \text{for all } k \geq k_\varepsilon. \tag{2.1.18}$$

This is the required inequality to bound $\operatorname{ess\,sup}_\Omega u$ as given in [LU68, Lemma 5.1, p. 71], see also [Lin92]. To prove this, we introduce

$$f(k) := \int_{\Omega_k} (u-k) dx = \int_0^\infty \left|\Omega_{\tau+k}\right| d\tau = \int_k^\infty \left|\Omega_t\right| dt, \tag{2.1.19}$$

where the second equality in (2.1.19) follows from [LL01, Theorem 1.13]. Taking into account (2.1.12), we have $f'(k) = -\left|\Omega_k\right|$, that is, f is a decrasing nonnegative function. Hence, (2.1.18) can be expressed in terms of f

$$f(k) < \gamma k \left(-f'(k)\right)^{1+\delta} \quad \text{for all } k \geq k_\varepsilon,$$

where $\delta = \frac{p^*-p}{p^*(p-1)}$ and $\gamma = (2^p \lambda C)^{\frac{1}{p-1}}$. Then

$$k^{-\frac{1}{1+\delta}} < -\gamma^{\frac{1}{1+\delta}} f'(k) f(k)^{-\frac{1}{1+\delta}} \quad \text{for all } k \geq k_\varepsilon. \tag{2.1.20}$$

Let $k_{max} := \operatorname{ess\,sup}_\Omega u$. If $k_{max} \leq k_\varepsilon$, then $\operatorname{ess\,sup}_\Omega u < \infty$. So, assume that $k_{max} > k_\varepsilon$. Integrating (2.1.20) from k_ε to k_{max} we obtain

$$k_{max}^{\frac{\delta}{1+\delta}} - k_\varepsilon^{\frac{\delta}{1+\delta}} < \gamma^{\frac{1}{1+\delta}} \left(f\left(k_\varepsilon\right)^{\frac{\delta}{1+\delta}} - f(k_{max})^{\frac{\delta}{1+\delta}}\right) = \gamma^{\frac{1}{1+\delta}} f\left(k_\varepsilon\right)^{\frac{\delta}{1+\delta}},$$

where $f(k_{max}) = 0$. Hence,

$$\ln k_{max} < \ln k_\varepsilon + \ln \gamma^{\frac{1}{\delta}} f\left(k_\varepsilon\right).$$

Since

$$\lim_{k \to 0} \Omega_k = \Omega_0 := \{x \in \Omega : \ u(x) > 0\},$$

it follows that

$$f\left(k_\varepsilon\right) \leq f(0) = \int_{\Omega_0} u dx \leq \int_\Omega |u| dx = \|u\|_1.$$

Therefore, $k_{max} = \operatorname{ess\,sup}_\Omega u < \infty$. Similarly, one can prove that $\operatorname{ess\,inf}_\Omega u > -\infty$ by replacing u with $-u$. $\square$

We can derive the Hölder regularity of eigenfunctions.

Theorem 2.1.6. *Every eigenfunction is Hölder continuous with respect to the Carnot-Carathéodory metric in* Ω.

Proof. Let u be an eigenfunction. We treat $1 < p \leq Q$ and $p > Q$ cases separately. Let $1 < p \leq Q$. Since $u \in L^\infty(\Omega)$ by Lemma 2.1.5, there exists $0 < \alpha < 1$ such that

$$\operatorname*{ess\,sup}_{x,y \in \Omega} \frac{|u(x) - u(y)|}{d_X(x,y)^\alpha} < \infty$$

by [CDG93, Theorem 3.35]. We obtain the Hölder continuity of u with respect to d_X in Ω after a redefinition in a set of Lebesgue measure zero.

Let $p > Q$. Then by [CCL24, Remark 1.1], we have the Hölder continuity of u with respect to d_X in Ω. $\square$

Let $u \in \mathcal{W}^{1,p}_{X,0}(\Omega) \setminus \{0\}$. We call the following expression

$$\frac{\int_\Omega |Xu|^p dx}{\int_\Omega |u|^p dx} \tag{2.1.21}$$

the Rayleigh quotient. We will demonstrate that the Rayleigh quotient can be minimized in $\mathcal{W}^{1,p}_{X,0}(\Omega) \setminus \{0\}$. Following this, we will prove that this minimization process yields the first eigenvalue of (2.1.1).

Theorem 2.1.7. *There exists* $u_1 \in \mathcal{W}^{1,p}_{X,0}(\Omega) \setminus \{0\}$ *such that*

$$\frac{\int_\Omega |Xu_1|^p dx}{\int_\Omega |u_1|^p dx} = \inf_{u \in \mathcal{W}^{1,p}_{X,0}(\Omega) \setminus \{0\}} \frac{\int_\Omega |Xu|^p dx}{\int_\Omega |u|^p dx} = \lambda_1.$$

Moreover, (λ_1, u_1) *satisfies* (2.1.2) *for all* $\varphi \in \mathcal{W}^{1,p}_{X,0}(\Omega)$ *and* λ_1 *is the smallest eigenvalue. We call* (λ_1, u_1) *the first eigenpair.*

Proof. If u is an eigenfunction, then Cu is also an eigenfunction provided $C \neq 0$. Therefore, without loss of generality, it is sufficient to minimize $\int_\Omega |Xu|^p dx$ over

$$\{u \in \mathcal{W}^{1,p}_{X,0}(\Omega) : \|u\|_p = 1\}.$$

We begin with proving the existence of a minimizer. Let

$$E_0 := \inf_{\substack{u \in \mathcal{W}^{1,p}_{X,0}(\Omega) \\ \|u\|_p = 1}} \int_\Omega |Xu|^p dx.$$

It is possible to select a minimizing sequence $\{\tilde{u}_j\}$ such that $\|u_j\|_p = 1$ and

$$\int_\Omega |X\tilde{u}_j|^p \, dx < E_0 + \frac{1}{j}.$$

Thus, $\{\tilde{u}_j\}$ is a bounded sequence in $\mathcal{W}^{1,p}_{X,0}(\Omega)$. Then there exist a subsequence $\{\tilde{u}_{j_k}\}$ and a function $u_1 \in \mathcal{W}^{1,p}_{X,0}(\Omega)$ such that

$$\tilde{u}_{j_k} \rightharpoonup u_1 \quad \text{weakly in } \mathcal{W}^{1,p}_{X,0}(\Omega)$$

by the Eberlein-Šmulian theorem. Let us prove that u_1 is a minimizer. The weak lower semicontinuity of $\|\cdot\|$ yields

$$\int_\Omega |Xu_1|^p dx \leq \liminf_{j_k \to \infty} \int_\Omega |X\tilde{u}_{j_k}|^p dx = E_0.$$

It only remains to demonstrate that u_1 is an eigenfunction associated with

$$\lambda_1 := \int_\Omega |Xu_1|^p dx.$$

Given $\varphi \in \mathcal{W}^{1,p}_{X,0}(\Omega)$, we define the functional $f : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f(\varepsilon) := \frac{\int_\Omega |X(u_1 + \varepsilon\varphi)|^p dx}{\int_\Omega |u_1 + \varepsilon\varphi|^p dx}.$$

Since u_1 minimizes the Rayleigh quotient, it follows that $f'(0) = 0$ or more explicitly

$$\int_\Omega |u_1|^p dx \int_\Omega |Xu_1|^{p-2} Xu_1 \cdot X\varphi dx = \int_\Omega |u_1|^{p-2} u_1 \varphi dx \int_\Omega |Xu_1|^p dx.$$

Thus, keeping into account $\|u_1\|_p = 1$, we simplify to

$$\begin{aligned} \int_\Omega |Xu_1|^{p-2} Xu_1 \cdot X\varphi dx &= \int_\Omega |Xu_1|^p dx \int_\Omega |u_1|^{p-2} u_1 \varphi dx \\ &= \lambda_1 \int_\Omega |u_1|^{p-2} u_1 \varphi dx. \end{aligned} \tag{2.1.22}$$

Note that (2.1.22) is true for all $\varphi \in \mathcal{W}^{1,p}_{X,0}(\Omega)$, so u_1 is an eigenfunction associated with λ_1. We can show that λ_1 is the smallest eigenvalue. Let $\tilde{\lambda} \neq \lambda_1$ be an eigenvalue with a corresponding eigenfunction $\tilde{u}$ such that $\|\tilde{u}\|_p = 1$. Then

$$\lambda_1 = \int_\Omega |Xu_1|^p dx \leq \int_\Omega |X\tilde{u}|^p dx = \tilde{\lambda} \int_\Omega |\tilde{u}|^p dx = \tilde{\lambda},$$

so, λ_1 is the smallest eigenvalue of (2.1.1). □

As a consequence of Theorem 2.1.7, the best constant in (1.8.2) is established.

Corollary 2.1.8. *The L^p Poincaré-Friedrichs inequality for Hörmander vector fields*

$$\int_\Omega |u|^p dx \leq \lambda_1^{-1} \int_\Omega |Xu|^p dx$$

is valid for all $u \in \mathcal{W}^{1,p}_{X,0}(\Omega)$. The constant λ_1^{-1} is the best constant.

Furthermore, we can prove the nonnegativity of u_1.

Corollary 2.1.9. *If u_1 minimizes the Rayleigh quotient, then $|u_1|$ also minimizes the Rayleigh quotient.*

Proof. It follows from Proposition 6.0.1 that $|u_1| \in \mathcal{W}^{1,p}_{X,0}(\Omega)$ and $|Xu_1| = |X|u_1||$ a.e. in Ω. Therefore,

$$\frac{\int_\Omega |Xu_1|^p dx}{\int_\Omega |u_1|^p dx} = \inf_{u \in \mathcal{W}^{1,p}_{X,0}(\Omega)\setminus\{0\}} \frac{\int_\Omega |Xu|^p dx}{\int_\Omega |u|^p dx} = \frac{\int_\Omega |X|u_1||^p dx}{\int_\Omega |u_1|^p dx}.$$

□

Let us recall the Harnack inequality from [Lu96] that is adapted to the eigenvalue problem (2.1.1). The positivity of the first eigenfunction u_1 will be proved by using the Harnack inequality.

Theorem 2.1.10. *Let $x_0 \in \Omega$ such that $B_X(x_0, 3r) \subset \Omega$ for some $r > 0$. Suppose u is a nonnegative eigenfunction of (2.1.1) in $B_X(x_0, 3r)$. Then*

$$\operatorname*{ess\,sup}_{B_X(x_0,r)} u(x) \leq C \operatorname*{ess\,inf}_{B_X(x_0,r)} u(x), \tag{2.1.23}$$

where $C > 0$ is a constant.

Proof. The proof is provided in [Lu96, Corollary 3.11]. Note that, since $b_0 = 0$ in [Lu96, (3.1) formula, p. 314], we can relax the boundedness assumption in [Lu96, Corollary 3.11] in our case. For further details, we refer to [Lu96, p. 318] (see also [Tru67, p. 724] for comparison). □

Remark 2.1.11. *The Harnack inequality (2.1.23) for $1 < p \leq Q$ is also presented in [CDG93, Theorem 3.1].*

Corollary 2.1.12. *The first eigenfunction u_1 can be chosen to be positive in Ω.*

Proof. Besides the first eigenfunction u_1, it is known that $|u_1|$ also minimizes the Rayleigh quotient (2.1.21) by Corollary 2.1.9. Consequently, we can assume that $u_1 \geq 0$ in Ω. Theorem 2.1.10 yields

$$u_1 > 0 \quad \text{in } \Omega$$

or

$$u_1 = 0 \quad \text{in } \Omega.$$

The latter cannot be an eigenfunction, so $u_1 > 0$ in Ω. □

Proposition 2.1.13. *Let* $\Omega_1 \subset \Omega_2 \subset \Omega$. *Then* $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$.

Proof. There are more functions in Ω_2 for the Rayleigh quotient or in other words, $\mathcal{W}^{1,p}_{X,0}(\Omega_1) \subset \mathcal{W}^{1,p}_{X,0}(\Omega_2)$ (cf [Lin08, p. 187] and [Med18, Theorem 7.45.3]). The proof follows from the property of infima. □

Proposition 2.1.14. *Let*

$$\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots \subset \Omega$$

be an exhaustion of Ω, *that is,* $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. *Then*

$$\lim_{j \to \infty} \lambda_1(\Omega_j) = \lambda_1(\Omega).$$

Proof. By Proposition 2.1.13 we have

$$\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2) \geq \ldots \geq \lambda_1(\Omega),$$

therefore, $\lim_{j\to\infty} \lambda_1(\Omega_j)$ exists. Let $\varepsilon > 0$. Thanks to Theorem 2.1.7, there exists a $\varphi \in C_0^\infty(\Omega)$ such that

$$\frac{\int_\Omega |X\varphi|^p dx}{\int_\Omega |\varphi|^p dx} < \lambda_1(\Omega) + \varepsilon.$$

Moreover, we can assume that $\operatorname{supp}(\varphi) \subset \Omega_j$ for sufficiently large j. Hence,

$$\lambda_1(\Omega_j) \leq \frac{\int_{\Omega_j} |X\varphi|^p dx}{\int_{\Omega_j} |\varphi|^p dx} = \frac{\int_\Omega |X\varphi|^p dx}{\int_\Omega |\varphi|^p dx}.$$

We conclude that $\lambda_1(\Omega) \leq \lambda_1(\Omega_j) \leq \lambda_1(\Omega) + \varepsilon$ for j large enough.

□

As the first eigenvalue λ_1 depends on the parameter p, we can consider $\lambda_{1,p}$ as a function of p.

Lemma 2.1.15. *If* $1 < p < q < \infty$*, then*

$$p\,(\lambda_{1,p})^{\frac{1}{p}} < q\,(\lambda_{1,q})^{\frac{1}{q}}\,.$$

Proof. Let $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$. Then $\psi = \phi^{\frac{q}{p}} \in \mathcal{W}^{1,p}_{X,0}(\Omega)$. By the Hölder inequality, we obtain

$$\begin{aligned}(\lambda_{1,p})^{\frac{1}{p}} &\leq \frac{\left(\int_\Omega |X\psi|^p dx\right)^{\frac{1}{p}}}{\left(\int_\Omega \psi^p dx\right)^{\frac{1}{p}}} = \frac{q}{p}\frac{\left(\int_\Omega \phi^{q-p}|X\phi|^p dx\right)^{\frac{1}{p}}}{\left(\int_\Omega \phi^q dx\right)^{\frac{1}{p}}} \\ &\leq \frac{q}{p}\frac{\left[\left(\int_\Omega \phi^q dx\right)^{1-\frac{p}{q}}\left(\int_\Omega |X\phi|^q dx\right)^{\frac{p}{q}}\right]^{\frac{1}{p}}}{\left(\int_\Omega \phi^q dx\right)^{\frac{1}{p}}} \\ &= \frac{q}{p}\frac{\left(\int_\Omega |X\phi|^q dx\right)^{\frac{1}{q}}}{\left(\int_\Omega \phi^q dx\right)^{\frac{1}{q}}}.\end{aligned}$$

Therefore,

$$(\lambda_{1,p})^{\frac{1}{p}} \leq \frac{q}{p}\inf_{\substack{\phi\in C_0^\infty(\Omega)\setminus\{0\}\\ \phi\geq 0}} \frac{\left(\int_\Omega |X\phi|^q dx\right)^{\frac{1}{q}}}{\left(\int_\Omega \phi^q dx\right)^{\frac{1}{q}}} = (\lambda_{1,q})^{\frac{1}{q}}\,. \tag{2.1.24}$$

The inequality (2.1.24) is actually strict. By contradiction, suppose that,

$$(\lambda_{1,p})^{\frac{1}{p}} = (\lambda_{1,q})^{\frac{1}{q}}\,.$$

Therefore, $\psi = u_{1,q}^{q/p}$ is an eigenfunction which is equal to $Cu_{1,p}$ provided $C \neq 0$. The following inequality

$$\left(\int_\Omega \phi^{q-p}|X\phi|^p dx\right)^{\frac{1}{p}} \leq \left(\int_\Omega \phi^q dx\right)^{1-\frac{p}{q}}\left(\int_\Omega |X\phi|^q dx\right)^{\frac{p}{q}}$$

becomes equality if and only if ϕ and $|X\phi|$ are proportional in Ω. Hence, $u_{1,q}$ and $|Xu_{1,q}|$ are proportional, which is impossible. □

Corollary 2.1.16.

$$\lim_{q\to p-}\lambda_{1,q} \leq \lambda_{1,p} \leq \lim_{q\to p+}\lambda_{1,q}.$$

Proof. Let $f(q) := q\,(\lambda_{1,q})^{\frac{1}{q}}$ for $q > 1$. Then f is monotonically increasing by Lemma 2.1.15, thus f has one-sided limits. Limit laws ensure one-sided limits of a function $h(q) = q^q\lambda_{1,q}$. Hence, $g(q) := \lambda_{1,q}$ has one-sided limits. □

Theorem 2.1.17. *The function* $\lambda_{1,p}$ *is right continuous, that is,*

$$\lim_{q\to p+}\lambda_{1,q} = \lambda_{1,p} \tag{2.1.25}$$

for every $p > 1$.

Proof. Let $q > p > 1$. Then

$$\lambda_{1,q} \leq \frac{\int_\Omega |X\phi|^q dx}{\int_\Omega |\phi|^q dx}$$

holds for all $\phi \in C_0^\infty(\Omega) \setminus \{0\}$, which implies that

$$\lim_{q \to p+} \lambda_{1,q} \leq \frac{\int_\Omega |X\phi|^p dx}{\int_\Omega |\phi|^p dx}.$$

Hence,

$$\lim_{q \to p+} \lambda_{1,q} \leq \inf_{\phi \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |X\phi|^p dx}{\int_\Omega |\phi|^p dx} = \lambda_{1,p}. \tag{2.1.26}$$

Moreover,

$$\lambda_{1,p} \leq \lim_{q \to p+} \lambda_{1,q} \tag{2.1.27}$$

follows from Corollary 2.1.16. Combining (2.1.26) and (2.1.27), we get (2.1.25). ☐

2.2 Simplicity and isolatedness of the first eigenvalue

As is well-known, the first eigenvalue of the Dirichlet p-Laplacian is both simple and isolated, as noted in [Lin90]. Similarly, it is reasonable to expect such properties for the subelliptic p-Laplacian. However, unlike the $C^{1,\alpha}$ regularity of eigenfunctions for the Dirichlet p-Laplacian (see, for instance, [DiB83] and [Tol84]), we were only able to establish the Hölder regularity with respect to the Carnot-Carathéodory metric for the eigenfunctions of (2.1.1). Despite this, the results obtained still allow us to prove the simplicity and isolatedness of λ_1.

Theorem 2.2.1. *The first eigenvalue λ_1 of (2.1.1) is simple. In other words, if there exist two positive eigenfunctions u_1 and $\tilde{u}_1$ associated with λ_1, then they are proportional.*

Proof. Suppose we have two distinct eigenfunctions u_1 and $\tilde{u}_1$ associated with λ_1. Then $u_1 > 0$ and $\tilde{u}_1 > 0$ in Ω by Corollary 2.1.12. The central idea of the proof lies in the careful choice of test functions, which ultimately leads to the conclusion that the first eigenfunctions are unique modulo scaling.

Given $\varepsilon > 0$, let

$$\varphi := \frac{(u_1 + \varepsilon)^p - (\tilde{u}_1 + \varepsilon)^p}{(u_1 + \varepsilon)^{p-1}}$$

be a test function for u_1, and let

$$\psi := \frac{(\tilde{u}_1+\varepsilon)^p-(u_1+\varepsilon)^p}{(\tilde{u}_1+\varepsilon)^{p-1}}$$

be a test function for $\tilde{u}_1$. By Lemma 2.1.5 and Theorem 2.1.6, it follows that

$$\sup_{\Omega} u_1 < \infty \quad \text{and} \quad \sup_{\Omega} \tilde{u}_1 < \infty,$$

thus, φ and ψ are well defined. By Theorem 2.1.6, we have $u_1, \tilde{u}_1 \in \mathcal{C}^{0,\alpha}_X(\Omega)$. By computing the horizontal gradients

$$X\varphi = \left\{1+(p-1)\left(\frac{\tilde{u}_1+\varepsilon}{u_1+\varepsilon}\right)^p\right\}Xu_1 - p\left(\frac{\tilde{u}_1+\varepsilon}{u_1+\varepsilon}\right)^{p-1}X\tilde{u}_1$$

and

$$X\psi = \left\{1+(p-1)\left(\frac{u_1+\varepsilon}{\tilde{u}_1+\varepsilon}\right)^p\right\}X\tilde{u}_1 - p\left(\frac{u_1+\varepsilon}{\tilde{u}_1+\varepsilon}\right)^{p-1}Xu_1,$$

we see that $u_1, \tilde{u}_1 \in \mathcal{W}^{1,p}_{X,0}(\Omega)$. Inserting u_1, φ and $\tilde{u}_1, \psi$ into (2.1.2), respectively, and then combining the resulting expressions, we have

$$\begin{aligned}
&\lambda_1 \int_\Omega \left(\left(\frac{u_1}{u_1+\varepsilon}\right)^{p-1} - \left(\frac{\tilde{u}_1}{\tilde{u}_1+\varepsilon}\right)^{p-1}\right)\big((u_1+\varepsilon)^p-(\tilde{u}_1+\varepsilon)^p\big)dx \\
&= \int_\Omega \left(\left(1+(p-1)\left(\frac{\tilde{u}_1+\varepsilon}{u_1+\varepsilon}\right)^p\right)|Xu_1|^p + \left(1+(p-1)\left(\frac{u_1+\varepsilon}{\tilde{u}_1+\varepsilon}\right)^p\right)|X\tilde{u}_1|^p\right)dx \\
&- \int_\Omega \left(p\left(\frac{\tilde{u}_1+\varepsilon}{u_1+\varepsilon}\right)^{p-1}|Xu_1|^{p-2}Xu_1\cdot X\tilde{u}_1 + p\left(\frac{u_1+\varepsilon}{\tilde{u}_1+\varepsilon}\right)^{p-1}|X\tilde{u}_1|^{p-2}X\tilde{u}_1\cdot Xu_1\right)dx \\
&= \int_\Omega \big((u_1+\varepsilon)^p-(\tilde{u}_1+\varepsilon)^p\big)\big(\,|X\log(u_1+\varepsilon)|^p - |X\log(\tilde{u}_1+\varepsilon)|^p\,\big)dx \\
&- \int_\Omega p(\tilde{u}_1+\varepsilon)^p\,|X\log(u_1+\varepsilon)|^{p-2}X\log(u_1+\varepsilon)\cdot\big(X\log(\tilde{u}_1+\varepsilon)-X\log(u_1+\varepsilon)\big)dx \\
&- \int_\Omega p(u_1+\varepsilon)^p\,|X\log(\tilde{u}_1+\varepsilon)|^{p-2}X\log(\tilde{u}_1+\varepsilon)\cdot\big(X\log(u_1+\varepsilon)-X\log(\tilde{u}_1+\varepsilon)\big)dx.
\end{aligned} \tag{2.2.1}$$

Let $p \geq 2$. We multiply the inequality (6.0.2) with $\omega_1 = X\log(\tilde{u}_1+\varepsilon)$ and $\omega_2 = X\log(u_1+\varepsilon)$ by $(u_1+\varepsilon)^p$ and then integrate the resulting inequality over Ω to get

$$\begin{aligned}
0 &\leq C\int_\Omega \frac{1}{(\tilde{u}_1+\varepsilon)^p}\big|(\tilde{u}_1+\varepsilon)Xu_1-(u_1+\varepsilon)X\tilde{u}_1\big|^p dx \\
&\leq \int_\Omega (u_1+\varepsilon)^p\big(\,|X\log(u_1+\varepsilon)|^p - |X\log(\tilde{u}_1+\varepsilon)|^p\,\big)dx \\
&- \int_\Omega p(u_1+\varepsilon)^p\,|X\log(\tilde{u}_1+\varepsilon)|^{p-2}X\log(\tilde{u}_1+\varepsilon)\cdot\big(X\log(u_1+\varepsilon)-X\log(\tilde{u}_1+\varepsilon)\big)dx.
\end{aligned} \tag{2.2.2}$$

Also, we multiply the inequality (6.0.2) with $\omega_1 = X\log(u_1+\varepsilon)$ and $\omega_2 = X\log(\tilde{u}_1+\varepsilon)$ by $(\tilde{u}_1+\varepsilon)^p$ and then integrate the resulting inequality over Ω to get

$$\begin{aligned}
&0 \le C\int_\Omega \frac{1}{(u_1+\varepsilon)^p}\left|(\tilde{u}_1+\varepsilon)Xu_1-(u_1+\varepsilon)X\tilde{u}_1\right|^p dx\\
&\le -\int_\Omega (\tilde{u}_1+\varepsilon)^p\big(\,|X\log(u_1+\varepsilon)|^p-|X\log(\tilde{u}_1+\varepsilon)|^p\,\big)dx\\
&-\int_\Omega p(\tilde{u}_1+\varepsilon)^p\,|X\log(u_1+\varepsilon)|^{p-2}\,X\log(u_1+\varepsilon)\cdot\big(X\log(\tilde{u}_1+\varepsilon)-X\log(u_1+\varepsilon)\big)dx.
\end{aligned} \tag{2.2.3}$$

We combine (2.2.2) and (2.2.3), then use (2.2.1) to have

$$\begin{aligned}
0 &\le 2C\int_\Omega \left(\frac{1}{(\tilde{u}_1+\varepsilon)^p}+\frac{1}{(u_1+\varepsilon)^p}\right)\left|(\tilde{u}_1+\varepsilon)Xu_1-(u_1+\varepsilon)X\tilde{u}_1\right|^p dx\\
&\le \lambda_1\int_\Omega \left(\left(\frac{u_1}{u_1+\varepsilon}\right)^{p-1}-\left(\frac{\tilde{u}_1}{\tilde{u}_1+\varepsilon}\right)^{p-1}\right)\big((u_1+\varepsilon)^p-(\tilde{u}_1+\varepsilon)^p\big)dx.
\end{aligned}$$

Observe that

$$\lim_{\varepsilon\to 0+}\int_\Omega \left(\left(\frac{u_1}{u_1+\varepsilon}\right)^{p-1}-\left(\frac{\tilde{u}_1}{\tilde{u}_1+\varepsilon}\right)^{p-1}\right)\big((u_1+\varepsilon)^p-(\tilde{u}_1+\varepsilon)^p\big)dx = 0. \tag{2.2.4}$$

Taking into account the limit (2.2.4), we use the Fatou lemma to get

$$\int_\Omega \left(\frac{1}{\tilde{u}_1^p}+\frac{1}{u_1^p}\right)\left|\tilde{u}_1Xu_1-u_1X\tilde{u}_1\right|^p dx = 0.$$

Therefore, $X\left(u_1/\tilde{u}_1\right) = 0$ a.e. in Ω, implying $u_1 = c\tilde{u}_1$ a.e. in Ω for $c \neq 0$. Then $u_1 = c\tilde{u}_1$ in Ω by Theorem 2.1.6.

Let $1 < p < 2$. We can prove this case similarly. We multiply the inequality (6.0.1) with $\omega_1 = X\log(\tilde{u}_1+\varepsilon)$ and $\omega_2 = X\log(u_1+\varepsilon)$ by $(u_1+\varepsilon)^p$ and then integrate the resulting inequality over Ω to get

$$\begin{aligned}
&0 \le C\int_\Omega \frac{1}{(\tilde{u}_1+\varepsilon)^p}\frac{\left|(\tilde{u}_1+\varepsilon)Xu_1-(u_1+\varepsilon)X\tilde{u}_1\right|^2}{\big((\tilde{u}_1+\varepsilon)\,|Xu_1|+(u_1+\varepsilon)\,|X\tilde{u}_1|\,\big)^{2-p}}dx\\
&\le \int_\Omega (u_1+\varepsilon)^p\big(\,|X\log(u_1+\varepsilon)|^p-|X\log(\tilde{u}_1+\varepsilon)|^p\,\big)dx\\
&-\int_\Omega p(u_1+\varepsilon)^p\,|X\log(\tilde{u}_1+\varepsilon)|^{p-2}\,X\log(\tilde{u}_1+\varepsilon)\cdot\big(X\log(u_1+\varepsilon)-X\log(\tilde{u}_1+\varepsilon)\big)dx.
\end{aligned} \tag{2.2.5}$$

We multiply the inequality (6.0.1) with $\omega_1 = X\log(u_1+\varepsilon)$ and $\omega_2 = X\log(\tilde{u}_1+\varepsilon)$

by $(\tilde{u}_1+\varepsilon)^p$ and then integrate the resulting inequality over Ω to get

$$\begin{aligned}0 \le C\int_\Omega \frac{1}{(u_1+\varepsilon)^p}&\frac{\left|(\tilde{u}_1+\varepsilon)Xu_1-(u_1+\varepsilon)X\tilde{u}_1\right|^2}{\left((\tilde{u}_1+\varepsilon)\left|Xu_1\right|+(u_1+\varepsilon)\left|X\tilde{u}_1\right|\right)^{2-p}}dx\\ \le -\int_\Omega &(\tilde{u}_1+\varepsilon)^p\big(\left|X\log(u_1+\varepsilon)\right|^p-\left|X\log(\tilde{u}_1+\varepsilon)\right|^p\big)dx\\ -\int_\Omega &p(\tilde{u}_1+\varepsilon)^p\left|X\log(u_1+\varepsilon)\right|^{p-2}X\log(u_1+\varepsilon)\cdot\big(X\log(\tilde{u}_1+\varepsilon)-X\log(u_1+\varepsilon)\big)dx.\end{aligned} \tag{2.2.6}$$

We combine (2.2.5) and (2.2.6), then use (2.2.1) to have

$$\begin{aligned}0 \le 2C\int_\Omega\left(\frac{1}{(\tilde{u}_1+\varepsilon)^p}+\frac{1}{(u_1+\varepsilon)^p}\right)&\frac{\left|(\tilde{u}_1+\varepsilon)Xu_1-(u_1+\varepsilon)X\tilde{u}_1\right|^2}{\left((\tilde{u}_1+\varepsilon)\left|Xu_1\right|+(u_1+\varepsilon)\left|X\tilde{u}_1\right|\right)^{2-p}}dx\\ \le \lambda_1\int_\Omega\left(\left(\frac{u_1}{u_1+\varepsilon}\right)^{p-1}-\left(\frac{\tilde{u}_1}{\tilde{u}_1+\varepsilon}\right)^{p-1}\right)&\big((u_1+\varepsilon)^p-(\tilde{u}_1+\varepsilon)^p\big)dx.\end{aligned} \tag{2.2.7}$$

The right-hand side of (2.2.7) goes to 0 as $\varepsilon \to 0$. So, $X\left(u_1/\tilde{u}_1\right)=0$ a.e. in Ω by the Fatou lemma. Then $u_1 = c\tilde{u}_1$ a.e. in Ω for $c \neq 0$. By Theorem 2.1.6, we have $u_1 = c\tilde{u}_1$ in Ω . ☐

The methods employed in Theorem 2.2.1 enable us to establish Theorem 2.2.2 and Theorem 2.2.3 (cf [AL87]).

Theorem 2.2.2. *All eigenfunctions associated with an eigenvalue $\lambda \neq \lambda_1$ of* (2.1.1) *change their sign in Ω.*

Proof. Let $\lambda \neq \lambda_1$ be an eigenvalue of (2.1.1). The corresponding eigenfunction is denoted by u. Since λ_1 is the smallest eigenvalue of (2.1.1), it follows that $\lambda > \lambda_1$. Suppose, by contradiction, u does not change sign in Ω, so, without loss of generality, assume that $u \ge 0$ in Ω. By following the same steps as in the proof of Theorem 2.2.1, we get

$$\int_\Omega\left(\lambda_1\left(\frac{u_1}{u_1+\varepsilon}\right)^{p-1}-\lambda\left(\frac{u}{u+\varepsilon}\right)^{p-1}\right)\big((u_1+\varepsilon)^p-(u+\varepsilon)^p\big)dx \ge 0.$$

Then

$$(\lambda_1-\lambda)\int_\Omega\big(u_1^p-u^p\big)dx \ge 0$$

as $\varepsilon \to 0$. Using Theorem 2.2.1, u_1 can be multiplied by a sufficiently large positive constant C to obtain $u_1^p - u^p > 0$ in Ω. However, this results in a contradiction, which implies that u must change sign in Ω. ☐

Theorem 2.2.3. *The first eigenvalue* λ_1 *of* (2.1.1) *is isolated in the spectrum.*

Proof. Assume, by contradiction, λ_1 is not isolated, so we can find a sequence of eigenvalues $\{\tilde{\lambda}_j\}$ converging to λ_1. Let $\tilde{u}_j$ be an eigenfunction corresponding to $\tilde{\lambda}_j$ such that $\|\tilde{u}_j\|_p = 1$. Therefore,

$$\tilde{\lambda}_j = \int_\Omega |X\tilde{u}_j|^p dx.$$

Since $\{\tilde{\lambda}_j\}$ is bounded, $\{\tilde{u}_j\}$ is bounded in $\mathcal{W}^{1,p}_{X,0}(\Omega)$. So, we can extract a subsequence $\{\tilde{u}_{j_k}\}$ such that

$$\tilde{u}_{j_k} \to \tilde{u} \quad \text{strongly in } L^p(\Omega) \quad \text{and} \quad X\tilde{u}_{j_k} \rightharpoonup X\tilde{u} \quad \text{weakly in } L^p(\Omega)$$

for some $\tilde{u} \in \mathcal{W}^{1,p}_{X,0}(\Omega)$ by combining the Eberlein-Šmulian theorem, Proposition 1.8.1 and Theorem 1.8.3.

The norm $\|\cdot\|$ is weakly lower semicontinuous, thus

$$\int_\Omega |X\tilde{u}|^p dx \leq \lim_{j_k \to \infty} \tilde{\lambda}_{j_k} = \lambda_1.$$

We see that $\tilde{u}$ is a minimizer of the Rayleigh quotient. Subsequently by Theorem 2.1.7, $\tilde{u}$ is the first eigenfunction of (2.1.1). Hence, $\tilde{u}$ is positive in Ω by Corollary 2.1.12.

If $\tilde{\lambda}_{j_k} \neq \lambda_1$, the function $\tilde{u}_{j_k}$ changes its sign in Ω by Theorem 2.2.2. Therefore,

$$\Omega^+_{j_k} := \{x \in \Omega : \ \tilde{u}_{j_k}(x) > 0\} \quad \text{and} \quad \Omega^-_{j_k} := \{x \in \Omega : \ \tilde{u}_{j_k}(x) < 0\}$$

are nonempty sets. We will prove that

$$\left|\Omega^+_{j_k}\right| \geq \left(C\tilde{\lambda}_{j_k}\right)^{\frac{p^*}{p^*-p}}, \tag{2.2.8}$$

where p^* is given by (2.1.3). Indeed, since $\tilde{u}^+_{j_k} = \max\{\tilde{u}_{j_k}, 0\}$ is admissible in $\mathcal{W}^{1,p}_{X,0}(\Omega)$ by Proposition 6.0.1, taking it as a test function we have

$$\begin{aligned}\int_{\Omega^+_{j_k}} |X\tilde{u}_{j_k}|^p dx &= \int_\Omega |X\tilde{u}^+_{j_k}|^p dx = \int_\Omega |X\tilde{u}_{j_k}|^{p-2} X\tilde{u}_{j_k} \cdot X\tilde{u}^+_{j_k} dx \\ &= \tilde{\lambda}_{j_k} \int_\Omega |\tilde{u}_{j_k}|^{p-2}\tilde{u}_{j_k}\tilde{u}^+_{j_k} dx = \tilde{\lambda}_{j_k} \int_\Omega |\tilde{u}^+_{j_k}|^p dx = \tilde{\lambda}_{j_k} \int_{\Omega^+_{j_k}} |\tilde{u}_{j_k}|^p dx.\end{aligned}$$

We use the Hölder inequality and Sobolev embedding (2.1.4) to obtain

$$\begin{aligned}\int_{\Omega^+_{j_k}} |X\tilde{u}_{j_k}|^p dx = \tilde{\lambda}_{j_k} \int_{\Omega^+_{j_k}} |\tilde{u}_{j_k}|^p dx &\leq \tilde{\lambda}_{j_k} |\Omega^+_{j_k}|^{1-\frac{p}{p^*}} \left(\int_\Omega |\tilde{u}^+_{j_k}|^{p^*} dx\right)^{\frac{p}{p^*}} \\ &\leq C\tilde{\lambda}_{j_k} |\Omega^+_{j_k}|^{1-\frac{p}{p^*}} \int_{\Omega^+_{j_k}} |X\tilde{u}_{j_k}|^p dx,\end{aligned}$$

which proves (2.2.8). Similarly, one can show that

$$\left|\Omega_{j_k}^-\right| \geq \left(C\tilde{\lambda}_{j_k}\right)^{\frac{p^*}{p^*-p}}$$

with $\tilde{u}_{j_k}^- = \min\{\tilde{u}_{j_k}, 0\}$. Hence,

$$\left|\Omega^\pm\right| = \left|\limsup \Omega_{j_k}^\pm\right| > 0.$$

By [Bre11, Theorem 4.9], we can extract $\{\tilde{u}_{j_{k_l}}\} \subset \{\tilde{u}_{j_k}\}$ such that $\tilde{u}_{j_{k_l}} \to \tilde{u}$ a.e. in Ω. It follows that $\tilde{u} > 0$ a.e. in Ω^+ and $\tilde{u} < 0$ a.e. in Ω^-, that is, the first eigenfunction $\tilde{u}$ changes its sign in Ω. This leads to a contradiction, proving that λ_1 is isolated. $\square$

Chapter 3

Variational eigenvalues

> The earliest known problem in the calculus of variations was formulated and solved by I. Newton in late 1685.
>
> ---
>
> From *A History of the Calculus of Variations from the 17th through the 19th Century [Gol80]*

Let Ω be a bounded domain in $\mathbb{R}^n$ and let $1 < p < \infty$. J.P. García Azorero and I. Peral Alonso [GP87, Theorem 5.3] established the existence of variational eigenvalues of

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) &= \lambda |u|^{p-2}u && \text{in } \Omega \subset \mathbb{R}^n, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.0.1}$$

by the Lusternik-Schnirelman theory based on the Krasnoselskii genus of symmetric closed sets. A. Anane and N. Tsouli [AT96] showed that the second eigenvalue coincides with the second variational eigenvalue obtained by the Lusternik-Schnirelmann theory [GP87]. P. Drabek and S.B. Robinson [DR99, Section 3] found variational eigenvalues of (3.0.1) via a narrower class of sets, where symmetric closed sets are obtained by continuous odd surjective mappings from the unit sphere in $\mathbb{R}^n$ to a subset of the given manifold. K. Perera [Per03] found variational eigenvalues of (3.0.1) by the minimax principle involving the Yang index.

Nonlinear eigenvalue problems for the p-Laplacian subject to the Neumann, Robin, Steklov and no-flux conditions were summarized by A. Lê [Lê06] in the case of a bounded domain Ω with regular boundary $\partial\Omega$.

When $n = 1$, the spectrum of (3.0.1) is discrete (see [Nec71] and [Fuč+73, Appendix 5]) and coincides with the variational spectrum of (3.0.1) (see [FP03, Theorem 1.1] and [DM99, Theorem 4.1]), however, it is not still known whether the variational spectrum coincides with the spectrum for $n \geq 2$. When $n = 1$, the exact representation of eigenpairs of (3.0.1) was found in [GV88, Theorem 1.1], see also [Dra80, Theorem 4.4], [PEM89, Section 3] and [Ôta84, Remark 8].

The content of this chapter is contained in [Kar24].

3.1 Lusternik-Schnirelmann type variational eigenvalues

In order to establish the existence of a sequence of variational eigenvalues of (2.1.1), we follow the Lusternik-Schnirelmann theory, see e.g. [Zei85, Section 44.5]. Let us consider $G, F : \mathcal{W}^{1,p}_{X,0}(\Omega) \to \mathbb{R}$ defined by

$$G(u) := \int_\Omega |Xu|^p dx \quad \text{and} \quad F(u) := \int_\Omega |u|^p dx.$$

Then we see that the level set $\mathcal{G}$ given by

$$\mathcal{G} := \{u \in \mathcal{W}^{1,p}_{X,0}(\Omega) : G(u) = 1\} \tag{3.1.1}$$

becomes the unit sphere in $\mathcal{W}^{1,p}_{X,0}(\Omega)$ with respect to the norm (1.8.3). Their Gâteaux derivatives are

$$\langle F'(u), v\rangle = p \int_\Omega |u|^{p-2} uv dx \quad \text{and} \quad \langle G'(u), v\rangle = p \int_\Omega |Xu|^{p-2} Xu \cdot Xv dx \tag{3.1.2}$$

for all $v \in \mathcal{W}^{1,p}_{X,0}(\Omega)$.

We have $\langle G'(u), u\rangle = pG(u)$ and $G(u) \neq 0$ for all $u \in \mathcal{W}^{1,p}_{X,0}(\Omega) \setminus \{0\}$. Thus, $G'(u) = 0$ implies $u = 0$. Conversely, as $G(0) = 0$, the only one critical value of G is $u = 0$. Hence, by the implicit function theorem, the level set $\mathcal{G}$ is a C^1-Finsler manifold.

In view of (3.1.2), Definition 2.1.1 converts to the following form

$$\langle F'(u), \varphi\rangle = \mu\langle G'(u), \varphi\rangle \quad \text{on } \mathcal{G},$$

for all $\varphi \in \mathcal{W}^{1,p}_{X,0}(\Omega)$ or in the operator form

$$F'(u) = \mu G'(u) \qquad \text{on } \mathcal{G}, \tag{3.1.3}$$

where $\mu = \lambda^{-1} > 0$.

Our aim is to apply Theorem 1.10.8, so we begin by verifying assumptions **(H1)-(H4)**. Since continuous differentiability of G and F requires the continuity of G' and F' (see [Wil96, Proposition 1.3]), the verification of **(H2)** and **(H3)** ensures **(H1)**. We begin with checking **(H4)**. Recalling that $\mathcal{G}$ represents the unit sphere in $\mathcal{W}^{1,p}_{X,0}(\Omega)$, it is clear that $\mathcal{G}$ is bounded. For every $u \in \mathcal{W}^{1,p}_{X,0}(\Omega) \setminus \{0\}$, we have

$$\langle G'(u), u\rangle = p\int_{\Omega} |Xu|^p dx > 0$$

and

$$\lim_{t\to+\infty} G(tu) = t^p \int_{\Omega} |Xu|^p dx = +\infty.$$

By Theorem 2.1.7, it follows that

$$\inf_{u\in\mathcal{G}} \langle G'(u), u\rangle = \lambda_1 > 0.$$

Proposition 3.1.1. *The operator F' satisfies **(H2)**.*

Proof. It follows from $\langle F'(u), u\rangle = pF(u)$ that

$$\langle F'(u), u\rangle = 0, \ \ u \in \overline{\operatorname{conv}}\mathcal{G} \quad \Longrightarrow \quad F(u) = 0.$$

Therefore, it remains to show that F' is strongly continuous. Let $u_j \rightharpoonup u$ in $\mathcal{W}^{1,p}_{X,0}(\Omega)$. Then by the Hölder inequality

$$|\langle F'(u_j) - F'(u), v\rangle| \leq p \left\| |u_j|^{p-2}u_j - |u|^{p-2}u \right\|_{p'} \|v\|_p$$

holds for all $v \in \mathcal{W}^{1,p}_{X,0}(\Omega)$, where $p' = \frac{p}{p-1}$. If

$$|u_j|^{p-2}u_j \to |u|^{p-2}u \quad \text{in } \ L^{p'}(\Omega), \tag{3.1.4}$$

then

$$\|F'(u_j) - F'(u)\|_{\mathcal{W}^{1,p}_{X,0}(\Omega)^*} = \sup_{\substack{\|v\|\leq 1 \\ v\in\mathcal{W}^{1,p}_{X,0}(\Omega)}} |\langle F'(u_j) - F'(u), v\rangle| \to 0.$$

Combining Proposition 1.8.1 and Theorem 1.8.3, we derive

$$Xu_j \rightharpoonup Xu \quad \text{in } L^p(\Omega) \quad \text{and} \quad u_j \to u \quad \text{in } L^p(\Omega).$$

Now, let $1 < p < 2$. We use (6.0.5) for $\omega_1 = |u_j|^{p-2}u_j$ and $\omega_2 = |u|^{p-2}u$ and then integrate over Ω to get

$$\begin{aligned}\int_{\Omega} \left||u_j|^{p-2}u_j - |u|^{p-2}u\right|^{p'} dx &\leq C\int_{\Omega} |u_j - u|^{p'(p-1)} dx \\ &= C\|u_j - u\|_p^p .\end{aligned}$$

Therefore, $u_j \to u$ in $L^p(\Omega)$ ensures (3.1.4).

Let $p \geq 2$. We use (6.0.6) for $\omega_1 = |u_j|^{p-2} u_j$ and $\omega_2 = |u|^{p-2}u$, integrate over Ω and then apply the Hölder and triangle inequalities to get

$$\begin{aligned} \int_\Omega \left| |u_j|^{p-2} u_j - |u|^{p-2}u \right|^{p'} dx &\leq C \int_\Omega |u_j - u|^{p'} \left(|u_j| + |u|\right)^{p'(p-2)} dx \\ &\leq C \, \|u_j - u\|_p^{p'} \left(\|u_j\|_p + \|u\|_p \right)^{p'(p-2)} . \end{aligned}$$

Therefore, $u_j \to u$ in $L^p(\Omega)$ ensures (3.1.4). We have proved that

$$u_j \rightharpoonup u \quad \text{in } \mathcal{W}^{1,p}_{X,0}(\Omega) \implies F'(u_j) \to F'(u) \quad \text{in } \mathcal{W}^{1,p}_{X,0}(\Omega)^*.$$

□

The following lemma will be applied to verify **(H3)**.

Lemma 3.1.2. *For all $u, v \in \mathcal{W}^{1,p}_{X,0}(\Omega)$ we have*

$$\langle G'(u) - G'(v), u - v \rangle \geq p \left(\|u\|^{p-1} - \|v\|^{p-1} \right) \left(\|u\| - \|v\| \right).$$

Furthermore, $\langle G'(u) - G'(v), u - v \rangle = 0$ if and only if $u = v$ a.e. in Ω.

Proof. Let $u, v \in \mathcal{W}^{1,p}_{X,0}(\Omega)$. Then

$$\langle G'(u) - G'(v), u - v \rangle = p\|u\|^p + p\|v\|^p - \langle G'(u), v \rangle - \langle G'(v), u \rangle.$$

Using the Hölder inequality for the last two terms we get

$$\langle G'(u), v \rangle \leq p\|u\|^{p-1}\|v\| \quad \text{and} \quad \langle G'(v), u \rangle \leq p\|v\|^{p-1}\|u\|.$$

Hence,

$$\langle G'(u) - G'(v), u - v \rangle \geq p \left(\|u\|^{p-1} - \|v\|^{p-1} \right) \left(\|u\| - \|v\| \right).$$

Now, let $u, v \in \mathcal{W}^{1,p}_{X,0}(\Omega)$ satisfy

$$\langle G'(u) - G'(v), u - v \rangle = 0. \tag{3.1.5}$$

If

$$\langle G'(u) - G'(v), u - v \rangle \geq C\|u - v\|^p \tag{3.1.6}$$

holds for all $u, v \in \mathcal{W}^{1,p}_{X,0}(\Omega)$, then combining (3.1.5) and (3.1.6), we get $u = v$ a.e. in Ω.

Let $1 < p < 2$. Then by the Hölder inequality and vector inequality (6.0.3), we have

$$\begin{aligned}
\int_\Omega |Xu - Xv|^p dx &= \int_\Omega \frac{|Xu - Xv|^p}{\big(|Xu| + |Xv|\big)^{\frac{p(2-p)}{2}}} \big(|Xu| + |Xv|\big)^{\frac{p(2-p)}{2}} dx \\
&\leq \left(\int_\Omega \frac{|Xu - Xv|^2}{\big(|Xu| + |Xv|\big)^{2-p}} dx\right)^{\frac{p}{2}} \left(\int_\Omega \big(|Xu| + |Xv|\big)^p dx\right)^{\frac{2-p}{2}} \\
&\leq \left(\int_\Omega \frac{|Xu - Xv|^2}{\big(|Xu| + |Xv|\big)^{2-p}} dx\right)^{\frac{p}{2}} \big(\|u\| + \|v\|\big)^{\frac{p(2-p)}{2}} \\
&\leq C \int_\Omega \big(|Xu|^{p-2}Xu - |Xv|^{p-2}Xv\big) \cdot (Xu - Xv) dx.
\end{aligned}$$

Let $p \geq 2$. The inequality (3.1.6) can be derived from the inequality (6.0.4). We use the vector inequality (6.0.4) for $\omega_1 = Xu$ and $\omega_2 = Xv$. Then we integrate it over Ω to get (3.1.6). $\square$

Proposition 3.1.3. *The operator G' satisfies **(H3)**.*

Proof. Let $u \in \mathcal{W}^{1,p}_{X,0}(\Omega)$. Then the Hölder inequality yields

$$\langle G'(u), v\rangle \leq p\|u\|^{p-1}\|v\| \quad \text{for all } v \in \mathcal{W}^{1,p}_{X,0}(\Omega).$$

Thus, G' is bounded.

Let $u_j \to u$ in $\mathcal{W}^{1,p}_{X,0}(\Omega)$. Then by the Hölder inequality, it follows that

$$|\langle G'(u_j) - G'(u), v\rangle| \leq \left(\int_\Omega \Big||Xu_j|^{p-2}Xu_j - |Xu|^{p-2}Xu\Big|^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \|v\|. \tag{3.1.7}$$

If

$$|Xu_j|^{p-2}\, Xu_j \to |Xu|^{p-2}Xu \quad \text{in} \quad L^{p'}(\Omega), \tag{3.1.8}$$

then the continuity of G' follows from (3.1.7).

Let $1 < p < 2$. We use the vector inequality (6.0.5) for $\omega_1 = |Xu_j|^{p-2}\, Xu_j$ and $\omega_2 = |Xu|^{p-2}Xu$, then integrate over Ω to get

$$\begin{aligned}
\int_\Omega \left||Xu_j|^{p-2}\, Xu_j - |Xu|^{p-2}Xu\right|^{p'} dx &\leq C \int_\Omega |Xu_j - Xu|^{p'(p-1)}\, dx \\
&= C\, \|u_j - u\|^p\, .
\end{aligned}$$

Therefore, $u_j \to u$ in $\mathcal{W}^{1,p}_{X,0}(\Omega)$ ensures (3.1.8).

Let $p \geq 2$. We use the vector inequality (6.0.6) for $\omega_1 = |Xu_j|^{p-2} Xu_j$ and $\omega_2 = |Xu|^{p-2}Xu$, integrate over Ω and then apply the Hölder and triangle inequalities to get

$$\begin{aligned}\int_\Omega \left||Xu_j|^{p-2} Xu_j - |Xu|^{p-2}Xu\right|^{p'} dx &\leq C \int_\Omega |Xu_j - Xu|^{p'} \left(|Xu_j| + |Xu|\right)^{p'(p-2)} dx \\ &\leq C \left\|u_j - u\right\|^{p'} \left(\|u_j\| + \|u\|\right)^{p'(p-2)}.\end{aligned}$$

Therefore, $u_j \to u$ in $\mathcal{W}^{1,p}_{X,0}(\Omega)$ ensures (3.1.8). We have proved that

$$u_j \to u \quad \text{in } \mathcal{W}^{1,p}_{X,0}(\Omega) \implies G'(u_j) \to G'(u) \quad \text{in } \mathcal{W}^{1,p}_{X,0}(\Omega)^*.$$

Now, let $\{u_j\} \subset \mathcal{W}^{1,p}_{X,0}(\Omega)$ be such that

$$\begin{aligned} u_j &\rightharpoonup u && \text{in } \mathcal{W}^{1,p}_{X,0}(\Omega), \\ G'(u_j) &\rightharpoonup v && \text{in } \mathcal{W}^{1,p}_{X,0}(\Omega)^*, \\ \langle G'(u_j), u_j\rangle &\to \langle v, u\rangle && \text{in } \mathbb{R}, \end{aligned}$$

where $v \in \mathcal{W}^{1,p}_{X,0}(\Omega)^*$ and $u \in \mathcal{W}^{1,p}_{X,0}(\Omega)$. It suffices to show $\|u_j\| \to \|u\|$, since $\mathcal{W}^{1,p}_{X,0}(\Omega)$ enjoys the Radon-Riesz property

$$\|u_j\| \to \|u\| \quad \text{and} \quad u_j \rightharpoonup u \quad \text{in } \mathcal{W}^{1,p}_{X,0}(\Omega) \implies u_j \to u \quad \text{in } \mathcal{W}^{1,p}_{X,0}(\Omega).$$

On the one hand, Lemma 3.1.2 gives

$$\langle G'(u_j) - G'(u), u_j - u\rangle \geq p \left(\|u_j\|^{p-1} - \|u\|^{p-1}\right) \left(\|u_j\| - \|u\|\right).$$

On the other hand, we have

$$\lim_{j\to\infty} \langle G'(u_j) - G'(u), u_j - u\rangle = 0.$$

Therefore, $\|u_j\| \to \|u\|$. $\square$

It only remains to show **(H1)**. We see that G and F are even functionals with $G(0) = F(0) = 0$. Since G' and F' are continuous, they are continuously differentiable by [Wil96, Proposition 1.3]. As we have verified **(H1)-(H4)**, let us state the main theorem of the chapter. First, we define

$$\mu_j := \sup_{\mathcal{A}\in\mathcal{G}_j} \inf_{u\in\mathcal{A}} F(u). \tag{3.1.9}$$

Theorem 3.1.4. *There exists a nonincreasing sequence of variational eigenvalues $\{\mu_j\}$ of the eigenvalue problem (3.1.3). Furthermore, $\mu_j \to 0^+$ as $j \to \infty$.*

Proof. Since $\mathcal{G}$ is the unit sphere in $\mathcal{W}^{1,p}_{X,0}(\Omega)$, it follows from [Zei85, Proposition 44.10] that $\gamma(\mathcal{G}) = \dim \mathcal{W}^{1,p}_{X,0}(\Omega) = \infty$. The continuity of G implies that the level set $\mathcal{G} = G^{-1}(\{1\})$ is closed in $\mathcal{W}^{1,p}_{X,0}(\Omega)$. Recalling that $\dim \mathcal{W}^{1,p}_{X,0}(\Omega) = \infty$, it follows that for every $j \in \mathbb{N}$, there exists a j-dimensional subspace of $\mathcal{W}^{1,p}_{X,0}(\Omega)$, denoted by V_j. Thus, $\mathcal{G} \cap V_j$ is closed and bounded, and hence compact. Furthermore, $\mathcal{G} \cap V_j$ represents the unit sphere in V_j, so $\gamma(\mathcal{G} \cap V_j) = j$ by [Zei85, Proposition 44.10]. Since V_j is a subspace and $\mathcal{G}$ is symmetric, their intersection $\mathcal{G} \cap V_j$ is also symmetric. We have demonstrated that $\mathcal{G} \cap V_j$ belongs to the class $\mathcal{G}_j$, which in turn gives $\mathcal{G}_j \neq \varnothing$ for all $j \in \mathbb{N}$.

Now, let $\mathcal{A} \in \mathcal{G}_j$. Using the fact that continuity preserves compactness, we see that $F(\mathcal{A})$ is compact. Moreover, $F(u) > 0$ for all $u \in \mathcal{G}$, therefore, $\inf_{u \in \mathcal{A}} F(u) > 0$ ensuring $\beta_j > 0$ for all $j \in \mathbb{N}$. Then there exist eigenpairs (μ_j, u_j) of (3.1.3) with $\mu_j \neq 0$ and $F(u_j) = \beta_j$ by Theorem 1.10.8.

Since $\beta_j > 0$ for all $j \in \mathbb{N}$, it follows that $\chi = \infty$. Moreover, F satisfies (1.10.5). Thus, eigenvalues μ_j are distinct. We can derive (3.1.9) from

$$\mu_j = \mu_j G(u_j) = \frac{\mu_j}{p} \langle G'(u_j), u_j \rangle = \frac{1}{p} \langle F'(u_j), u_j \rangle = F(u_j) = \beta_j.$$

□

Corollary 3.1.5. *$\lambda_j := \frac{1}{\mu_j}$ is an eigenvalue of (2.1.1). Furthermore, $\lambda_j \to \infty$.*

It is possible to find these eigenvalues on another manifold $\mathcal{F}$ defined by

$$\mathcal{F} := \{u \in \mathcal{W}^{1,p}_{X,0}(\Omega) : \ F(u) = 1\}.$$

Let us show that $\mathcal{F}$ is a symmetric closed C^1-Finsler manifold. Indeed, since F is an even functional, it is clear that $\mathcal{F}$ is a symmetric set. Continuity of F ensures the closedness of $\mathcal{F} = F^{-1}(\{1\})$. Additionally, for all $\mathcal{W}^{1,p}_{X,0}(\Omega)$ we have $\langle F'(u), u\rangle = pF(u)$. Therefore, 0 is the only one critical value of F. We conclude that $\mathcal{F}$ is a C^1-Finsler manifold by the implicit function theorem.

We define

$$E(u) := \frac{G(u)}{F(u)} \quad \text{for all } u \in \mathcal{W}^{1,p}_{X,0}(\Omega) \setminus \{0\}.$$

Lemma 3.1.6. *The operator $E|_{\mathcal{F}}$ satisfies $(PS)_c$ for all $c \in \mathbb{R}$.*

To prove Lemma 3.1.6, we need the following proposition.

Proposition 3.1.7. *The operator G' is continuously invertible, that is, $G'^{-1} : \mathcal{W}^{1,p}_{X,0}(\Omega)^* \to \mathcal{W}^{1,p}_{X,0}(\Omega)$ is continuous.*

Proof. The inequality (3.1.6) gives the uniform monotonicity of G'. By Proposition 3.1.3, the operator G' is continuous. Since every uniformly monotone operator is coercive, it follows from [Zei90, Theorem 26.A] that G'^{-1} is continuous. $\square$

Now, we can prove Lemma 3.1.6.

Proof of Lemma 3.1.6. Let $\{u_j\} \subset \mathcal{F}$ be such that

(i) $E|_{\mathcal{F}}(u_j) = G(u_j) \to c$;

(ii) $E|'_{\mathcal{F}}(u_j) = G'(u_j) - E|_{\mathcal{F}}(u_j)F'(u_j) \to 0 \quad \text{in } \mathcal{W}^{1,p}_{X,0}(\Omega)^*$.

Our purpose is to find a convergent subsequence of $\{u_j\}$. It is straightforward to observe that $\{u_j\}$ is bounded in $\mathcal{W}^{1,p}_{X,0}(\Omega)$. This implies that there exist a subsequence $\{u_{j_k}\}$ and $u \in \mathcal{W}^{1,p}_{X,0}(\Omega)$ such that $u_{j_k} \rightharpoonup u$ in $\mathcal{W}^{1,p}_{X,0}(\Omega)$ by the Eberlein-Šmulian theorem. Proposition 1.8.1 and Theorem 1.8.3 ensure that $u_{j_k} \to u$ in $L^p(\Omega)$. Since $\{E|_{\mathcal{F}}(u_{j_k})\}$ is bounded in $\mathbb{R}$, there exist a subsequence $\{E|_{\mathcal{F}}(u_{j_{k_l}})\}$ and $\bar{E} \in \mathbb{R}$ such that $E|_{\mathcal{F}}(u_{j_{k_l}}) \to \bar{E}$. Strong continuity of F' gives $F'(u_{j_{k_l}}) \to F'(u)$ in $\mathcal{W}^{1,p}_{X,0}(\Omega)^*$. From Proposition 3.1.7 it follows that $u_{j_{k_l}} \to G'^{-1}(\bar{E}F'(u))$ in $\mathcal{W}^{1,p}_{X,0}(\Omega)$. $\square$

Theorem 3.1.8. *[Szu88, Corollary 4.1]*

Let $\mathcal{M}$ a closed symmetric C^1-submanifold of a real infinite dimensional Banach space $\mathbb{B}$ such that $0 \notin \mathcal{M}$. Let $f \in C^1(\mathcal{M}, \mathbb{R})$ be an even and bounded below functional. We define

$$c_j := \inf_{\mathcal{A} \in \Gamma_j} \sup_{u \in \mathcal{A}} f(u),$$

where

$$\Gamma_j := \{\mathcal{A} \subset \mathcal{M} : \mathcal{A} \text{ is a compact and symmetric set, } \gamma(\mathcal{A}) \geq j\}.$$

If Γ_j is nonempty for some $j \in \mathbb{N}$ and if f satisfies $(PS)_c$ condition at $c = c_k$ for each $k = 1, \dots, j$, then there are at least j distinct pairs of critical points of f.

We define

$$\mathcal{F}_j := \{\mathcal{A} \subset \mathcal{F} : \mathcal{A} \ \text{ is a compact and symmetric set, } \ \gamma(\mathcal{A}) \geq j\}$$

and

$$\nu_j := \inf_{\mathcal{A} \in \mathcal{F}_j} \sup_{u \in \mathcal{A}} E(u).$$

We first show that $\mathcal{F}_j$ is nonempty for all $j \in \mathbb{N}$. Since $\dim L^p(\Omega) = \infty$, for every $j \in \mathbb{N}$ there exists a j-dimensional subspace of $L^p(\Omega)$, denoted by W_j. Then W_j is closed in $L^p(\Omega)$. Let S be the unit sphere in $L^p(\Omega)$, that is,

$$S = \{u \in L^p(\Omega) : \|u\|_p = 1\}.$$

Boundedness of S in $L^p(\Omega)$ implies the boundedness of $W_j \cap S$ in $L^p(\Omega)$. Moreover, S is closed, so $W_j \cap S$ is closed in $L^p(\Omega)$. By the Heine-Borel theorem, $W_j \cap S$ is compact. Note that, $W_j \cap S$ represents the unit sphere in W_j, so $\gamma(W_j \cap S) = j$ by [Zei85, Proposition 44.10]. We define $P : L^p(\Omega) \to L^p(\Omega)$ as follows

$$P(u) := \begin{cases} u & \text{if } u \in \mathcal{W}^{1,p}_{X,0}(\Omega), \\ 0 & \text{if } u \notin \mathcal{W}^{1,p}_{X,0}(\Omega). \end{cases}$$

We see that P is odd and continuous. Therefore, $P(W_j \cap S) = W_j \cap \mathcal{F}$ is compact and $\gamma(W_j \cap \mathcal{F}) \geq j$ by Proposition 1.10.5. Since $\mathcal{F}$ is symmetric and W_j is a subspace, it follows that $W_j \cap \mathcal{F}$ is a symmetric set. We have demonstrated that $W_j \cap \mathcal{F} \in \mathcal{F}_j$ for every $j \in \mathbb{N}$. Hence, $\mathcal{F}_j \neq \varnothing$ for every $j \in \mathbb{N}$.

Since the operator $E|_{\mathcal{F}} = G \in C^1(\mathcal{W}^{1,p}_{X,0}(\Omega), \mathbb{R})$ is bounded below and $E|_{\mathcal{F}}$ satisfies $(PS)_c$ for every $c = \nu_j$ with $j \in \mathbb{N}$, there exists a countable number of critical points of E by Theorem 3.1.8. Furthermore,

$$\inf_{\mathcal{A} \in \mathcal{F}_j} \sup_{u \in \mathcal{A}} E(u) = \frac{1}{\sup_{\mathcal{A} \in \mathcal{G}_j} \inf_{u \in \mathcal{A}} F(u)},$$

so $\nu_j = \lambda_j$ for each $j \in \mathbb{N}$.

3.2 Special collection of symmetric, compact subsets on $\mathcal{F}$

For all $j \in \mathbb{N}$, we define

$$\mathbb{F}_j := \left\{\mathcal{A} \subset \mathcal{F} : \ \exists \text{ a continuous odd surjective map } \ h : \mathbb{S}^{j-1} \to \mathcal{A}\right\}, \tag{3.2.1}$$

where $\mathbb{S}^{j-1}$ is the unit sphere in $\mathbb{R}^j$. Combining $\gamma(\mathbb{S}^{j-1}) = j$ and Proposition 1.10.5, we have

$$\gamma(h(\mathbb{S}^{j-1})) \geq j.$$

Since $\mathbb{S}^{j-1}$ is symmetric and h is odd, it follows that $h(-x) = -h(x) \in \mathcal{A}$ for all $x, -x \in \mathbb{S}^{j-1}$. Therefore, $\mathcal{A}$ is also a symmetric set. Furthermore, $h(\mathbb{S}^{j-1}) = \mathcal{A}$ is a compact set.

We first aim to show that $\mathbb{F}_j \neq \varnothing$ for every $j \in \mathbb{N}$. We can find a continuous odd map $\widetilde{h} : \mathbb{S}^{j-1} \to \mathbb{R}^j \setminus \{0\}$, because $\gamma(\mathbb{S}^{j-1}) = j$. Since $\dim \mathbb{R}^j = \dim W_j = j$, there exists an isomorphism $g : \mathbb{R}^j \to W_j$. It is clear that g is a bounded linear mapping, hence, $g : \mathbb{R}^j \setminus \{0\} \to W_j \setminus \{0\}$ is odd and continuous. We define a new function

$f(u) = \|u\|_p^{-1}u$ for all $W_j \setminus \{0\}$. Then $f(W_j \setminus \{0\}) = W_j \cap S$. Observe that, f is odd and continuous. Then

$$h := P \circ f \circ g \circ \widetilde{h} : \mathbb{S}^{j-1} \to W_j \cap \mathcal{F}$$

is an odd and continuous mapping.

Let

$$\gamma_j := \inf_{\mathcal{A} \in \mathbb{F}_j} \sup_{u \in \mathcal{A}} E|_{\mathcal{F}}(u). \tag{3.2.2}$$

We recall a version of the deformation lemma given in [DR99].

Lemma 3.2.1. *Let $\alpha \in \mathbb{R}$ be a regular value of $E|_{\mathcal{F}}$ and let $\bar{\varepsilon} > 0$. Then there exists $\varepsilon \in (0, \bar{\varepsilon})$ and a continuous deformation $\psi : \mathcal{F} \times [0,1] \to \mathcal{F}$ such that the following conditions hold:*

(i) $\psi(\cdot, t)$ is a homeomorphism for every $t \in [0,1]$;

(ii) $\psi(u,t) = u$, if $|E|_{\mathcal{F}}(u) - \alpha| \geq \bar{\varepsilon}$ or if $t = 0$;

(iii) $E|_{\mathcal{F}}(\psi(u,t))$ is nonincreasing in t for every $u \in \mathcal{F}$;

(iv) If $E|_{\mathcal{F}}(u) \leq \alpha + \varepsilon$, then $E|_{\mathcal{F}}(\psi(u,1)) \leq \alpha - \varepsilon$;

(v) ψ is odd with respect to u for any $t \in [0,1]$.

Theorem 3.2.2. *Every γ_j is a critical value of $E|_{\mathcal{F}}$.*

Proof. Suppose that the number γ_j is not a critical value of $E|_{\mathcal{F}}$, that is, a regular value of $E|_{\mathcal{F}}$. Fixing $\bar{\varepsilon} = 1$ and $\alpha = \gamma_j$, let $\varepsilon \in (0, \bar{\varepsilon})$. By Lemma 3.2.1, there is a continuous deformation $\psi : \mathcal{F} \times [0,1] \to \mathcal{F}$. According to the definition of γ_j, there exists $\mathcal{A} \in \mathbb{F}_j$ such that

$$\sup_{u \in \mathcal{A}} E|_{\mathcal{F}}(u) \leq \gamma_j + \varepsilon.$$

According to the definition of $\mathbb{F}_j$, there exists a continuous odd surjective mapping $h : \mathbb{S}^{j-1} \to \mathcal{A}$. The deformation ψ is odd with respect to u and continuous, so $\psi(h(\cdot), 1) : \mathbb{S}^{j-1} \to \psi(\mathcal{A}, 1)$ is also a continuous odd surjective mapping. Hence, $\psi(\mathcal{A}, 1)$ belongs to $\mathbb{F}_j$ and

$$\sup_{u \in \psi(\mathcal{A},1)} E|_{\mathcal{F}}(u) \leq \gamma_j - \varepsilon.$$

This leads to a contradiction, hence, γ_j is a critical value of $E|_{\mathcal{F}}$. □

Theorem 3.2.3. *Every γ_j is an eigenvalue of (2.1.1).*

Proof. The number γ_j is a critical value of $E|_{\mathcal{F}}$ on $\mathcal{F}$ by Theorem 3.2.2. By definition, there exist $\mathcal{A} \in \mathbb{F}_j$ and $u_j \in \mathcal{A}$ such that $E|_{\mathcal{F}}(u_j) = \gamma_j$ and $E|'_{\mathcal{F}}(u_j) = 0$. Moreover, $E|'_{\mathcal{F}} = G' - E|_{\mathcal{F}} F'$. This concludes

$$E|'_{\mathcal{F}}(u_j) = G'(u_j) - E|_{\mathcal{F}}(u_j)F'(u_j) = G'(u_j) - \gamma_j F'(u_j) = 0.$$

□

Finally, $\mathbb{F}_j \subset \mathcal{F}_j$ implies $\lambda_j \leq \gamma_j$ for every $j \in \mathbb{N}$. By Corollary 3.1.5, it follows that $\gamma_j \to \infty$ as $j \to \infty$.

Chapter 4

Applications

> The blow-up effect happens in different situations, like when a sea wave crashes onto the shore, when a computer breaks down due to an electrical issue, and in other similar cases.
>
> From *Blow-up in nonlinear equations of mathematical physics* *[Kor+18, Introduction]*

The first eigenvalue $\Lambda_{1,p}$ of the Dirichlet p-Laplacian (1.2.1) has numerous applications in partial differential equations. For instance, T. Bartsch et al. studied nodal solutions of

$$\Delta_p u = f(x,u), \quad u \in W_0^{1,p}(\Omega),$$

in [BLW05], under the assumption that

$$\limsup_{t\to 0} \frac{|f(x,t)|}{|t|^{p-1}} < \Lambda_{1,p} \quad \text{uniformly in } x \in \Omega.$$

Y. Chen and M. Wang established the existence of at least one positive solution to

$$\begin{aligned} -\Delta_p u &= a(x)h(u) - b(x)f(u) && x \in \Omega, \\ u &= \infty && x \in \partial\Omega, \end{aligned}$$

in [CW12], under the assumption that

$$\sup_{x\in\Omega} a(x) < \gamma \Lambda_{1,p}(\Omega_0)$$

for some positive constant γ. Here $\Omega_0 = \{x \in \Omega : b(x) = 0\}$ satisfies $\overline{\Omega}_0 \subset \Omega$ and $\Lambda_{1,p}(\Omega_0)$ is the first eigenvalue of

$$\begin{aligned} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) &= -\Lambda |u|^{p-2}u && \text{in } \Omega_0, \\ u &= 0 && \text{on } \partial\Omega_0. \end{aligned}$$

Some other applications can be found in [YJ07], [LPW12], [ZL16] and [CC18].

4.1 Blow-up phenomenon

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let $p > 2$. We are interested in blow up of solutions to

$$\begin{aligned} u_t + \sum_{j=1}^{m} X_j^* \left(|Xu|^{p-2} X_j u\right) &= f(u) && \text{in } \Omega_T, \\ u(x,0) &= u_0(x) \geq 0 && \text{in } \Omega, \\ u(x,t) &= 0 && \text{on } \Sigma_T, \end{aligned} \tag{4.1.1}$$

where $\Omega_T := \Omega \times (0,T)$, $\Sigma_T := \partial\Omega \times (0,T)$ and f is a locally Lipschitz continuous function in $\mathbb{R}$ such that

$$f(0) = 0 \quad \text{and} \quad f(u) > 0 \text{ for } u > 0.$$

Motivated by [CC18] (see also [RST23]), we impose the following condition on f

$$\alpha \int_0^u f(\tau) d\tau \leq u f(u) + \beta u^p + \alpha\gamma, \quad u > 0, \tag{4.1.2}$$

for

$$\alpha, \beta, \gamma > 0 \text{ with } 0 < \beta \leq \frac{(\alpha - p)\lambda_1}{p},$$

where λ_1 is the first eigenvalue of (2.1.1).

Definition 4.1.1. *We say that a function*

$$u \in L^\infty(\Omega_T) \cap L^p\left(0, T, \mathcal{W}_{X,0}^{1,p}(\Omega)\right) \quad \text{with} \quad u_t \in L^2(\Omega_T)$$

is a weak solution of (4.1.1), *if it satisfies*

$$\begin{aligned} &\int_0^t \int_\Omega \left(u\varphi_\tau - |Xu|^{p-2} Xu \cdot X\varphi + f\varphi\right) dx d\tau \\ &= \int_\Omega u(x,t)\varphi(x,t) dx - \int_\Omega u_0(x)\varphi(x,0) dx \quad \text{for a.e. } t \in (0,T) \end{aligned} \tag{4.1.3}$$

for all test functions $\varphi \in L^2(\Omega_T)$ *with* $\varphi_t \in L^2(\Omega_T)$ *and* $X\varphi \in L^p(\Omega_T)$.

Theorem 4.1.2. *Let f satisfy the condition* (4.1.2). *If* $u_0 \in L^\infty(\Omega) \cap \mathcal{W}_{X,0}^{1,p}(\Omega)$ *satisfies*

$$-\frac{1}{p} \int_\Omega |Xu_0(x)|^p dx + \int_\Omega \left(F(u_0(x)) - \gamma\right) dx > 0, \tag{4.1.4}$$

then the nonnegative weak solution to the problem (4.1.1) *blows up in finite time* T^* *as follows*

$$\lim_{t\to T^*}\int_0^t\int_\Omega u^2(x,\tau)dxd\tau=+\infty,$$

where

$$F(u):=\int_0^u f(\tau)d\tau.$$

Proof. We introduce a function Ψ by

$$\Psi(t):=-\frac{1}{p}\int_\Omega |Xu(x,t)|^p dx+\int_\Omega [F(u(x,t))-\gamma]dx.$$

This together with (4.1.4) gives $\Psi(0)>0$. Since

$$-\int_0^t\int_\Omega u\varphi_\tau+\int_\Omega u(x,t)\varphi(x,t)dx-\int_\Omega u_0(x)\varphi(x,0)dx=\int_0^t\int_\Omega u_\tau\varphi,$$

setting $\varphi=u_\tau$ in (4.1.3) yields

$$\int_0^t\int_\Omega\left(-|Xu|^{p-2}Xu\cdot Xu_\tau+fu_\tau\right)dxd\tau=\int_0^t\int_\Omega u_\tau^2dxd\tau \quad \text{for} \quad \text{a.e. } t\in(0,T).$$

This can be rewritten as

$$\int_0^t\Psi'(\tau)d\tau=\Psi(t)-\Psi(0)=\int_0^t\int_\Omega u_\tau^2dxd\tau \quad \text{for} \quad \text{a.e. } t\in(0,T). \tag{4.1.5}$$

Additionally, we introduce another function $J:\mathbb{R}_+\to\mathbb{R}_+$ by

$$J(t):=\int_0^t\int_\Omega u^2(x,\tau)dxd\tau+M, \tag{4.1.6}$$

where $M>0$ will be specified later. Next, we compute J'

$$\begin{aligned}J'(t)&=\int_\Omega u^2(x,t)dx\\&=\int_0^t\int_\Omega 2u(x,\tau)u_\tau(x,\tau)dxd\tau+\int_\Omega u_0^2(x)dx.\end{aligned}$$

Since

$$\begin{aligned}\int_0^t\int_\Omega\frac{1}{2}\left(u^2(x,\tau)\right)_\tau dxds&=\frac{1}{2}\int_\Omega\left(u^2(x,t)-u_0^2(x)\right)dx\\&=\int_0^t\int_\Omega\left(-|Xu(x,\tau)|^p+u(x,\tau)f(u(x,\tau))\right)dxd\tau,\end{aligned}$$

it follows that

$$\begin{aligned}J''(t)&=\int_\Omega\left(u^2(x,t)\right)_t dx\\&=2\int_\Omega\left(-|Xu(x,t)|^p+u(x,t)f(u(x,t))\right)dx.\end{aligned}$$

By combining the condition (4.1.2), Corollary 2.1.8, and (4.1.5), we obtain

$$\begin{aligned} J''(t) &\geq -2\int_\Omega |Xu(x,t)|^p dx + 2\int_\Omega \left(\alpha F(u(x,t)) - \beta u^p(x,t) - \alpha\gamma\right) dx \\ &= 2\alpha\Psi(t) + \frac{2(\alpha-p)}{p}\int_\Omega |Xu(x,t)|^p dx - 2\beta\int_\Omega u^p(x,t)dx \\ &\geq 2\alpha\Psi(t) + 2\left(\frac{(\alpha-p)\lambda_1}{p} - \beta\right)\int_\Omega u^p(x,t)dx \\ &\geq 2\alpha\Psi(t) \\ &= 2\alpha\left(\Psi(0) + \int_0^t\int_\Omega u_\tau^2(x,\tau)dxd\tau\right). \end{aligned} \tag{4.1.7}$$

Then, we apply the following inequality

$$(a+b)^2 \leq (1+\varepsilon)a^2 + \left(1+\frac{1}{\varepsilon}\right)b^2 \quad \text{for all } a,b\in\mathbb{R} \text{ and } \varepsilon>0,$$

with

$$a = \int_0^t\int_\Omega 2u(x,\tau)u_\tau(x,\tau)dxd\tau \text{ and } b = \int_\Omega u_0^2(x)dx$$

to derive

$$J'(t)^2 \leq (1+\varepsilon)\left(\int_\Omega\int_0^t 2u(\tau,x)u_\tau(\tau,x)dxd\tau\right)^2 + \left(1+\frac{1}{\varepsilon}\right)\left(\int_\Omega u_0^2(x)dx\right)^2.$$

From the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned} J'(t)^2 \leq{}& 4(1+\varepsilon)\left(\int_0^t\int_\Omega u^2(x,\tau)dxd\tau\right)\left(\int_0^t\int_\Omega u_\tau^2(x,\tau)dxd\tau\right) \\ &+ \left(1+\frac{1}{\varepsilon}\right)\left(\int_\Omega u_0^2(x)dx\right)^2. \end{aligned} \tag{4.1.8}$$

Setting $\sigma = \varepsilon = \sqrt{\frac{\alpha}{2}} - 1 > 0$, we obtain the following estimate

$$\begin{aligned} &J''(t)J(t) - (1+\sigma)J'(t)^2 \\ &\geq 2\alpha\left(\Psi(0) + \int_0^t\int_\Omega u_\tau^2(x,\tau)dxd\tau\right)\left(\int_0^t\int_\Omega u^2(x,\tau)dxd\tau + M\right) \\ &- 4(1+\sigma)(1+\varepsilon)\left(\int_\Omega\int_0^t u^2(x,\tau)dxd\tau\right)\left(\int_\Omega\int_0^t u_\tau^2(x,\tau)dxd\tau\right) \\ &- (1+\sigma)\left(1+\frac{1}{\varepsilon}\right)\left(\int_\Omega u_0^2(x)dx\right)^2 \\ &\geq 2\alpha M\cdot\Psi(0) - (1+\sigma)\left(1+\frac{1}{\varepsilon}\right)\left(\int_\Omega u_0^2(x)dx\right)^2. \end{aligned}$$

Since $\Psi(0) > 0$, we can select

$$M := \frac{\frac{\alpha}{\alpha-2}\left(1+\sqrt{\frac{\alpha}{2}}\right)\left[\int_\Omega u_0^2(x)dx\right]^2}{2\alpha\left(-\frac{1}{p}\int_\Omega |\nabla u_0(x)|^p\,dx + \int_\Omega \left(F\left(u_0(x)\right)-\gamma\right)dx\right)}$$

so that

$$J''(t)J(t) - (1+\sigma)J'(t)^2 \geq 0.$$

Therefore,

$$\frac{d}{dt}\left(\frac{J'(t)}{J^{\sigma+1}(t)}\right) \geq 0 \quad \text{for } t \geq 0,$$

which implies that

$$\begin{cases} J'(t) \geq \left(\frac{J'(0)}{J^{\sigma+1}(0)}\right) J^{\sigma+1}(t), \\ J(0) = M, \end{cases}$$

for $t \geq 0$. Integrating both sides, we obtain

$$\int_0^t \frac{J'(\tau)}{J^{\sigma+1}(\tau)}d\tau = -\frac{1}{\sigma}\left(J^{-\sigma}(t) - J^{-\sigma}(0)\right) \geq \int_0^t \frac{J'(0)}{J^{\sigma+1}(0)}d\tau = \frac{J'(0)}{J^{\sigma+1}(0)}t.$$

Thus, we explicitly obtain the bound

$$J(t) \geq \left(\frac{1}{M^\sigma} - \frac{\sigma\int_\Omega u_0^2(x)dx}{M^{\sigma+1}}t\right)^{-\frac{1}{\sigma}}. \tag{4.1.9}$$

Since the right-hand side of (4.1.9) cannot remain finite for all $t > 0$, there exists $t_0 \in (0, T)$ such that

$$\frac{1}{M^\sigma} - \frac{\sigma\int_\Omega u_0^2(x)dx}{M^{\sigma+1}}t_0 = 0.$$

This implies that the solution u blows up in finite time T^*, where T^* satisfies

$$0 < T^* \leq \frac{\frac{\alpha}{\alpha-2}\left(1+\sqrt{\frac{\alpha}{2}}\right)\left(\int_\Omega u_0^2(x)dx\right)^2}{2\alpha\sigma\left(-\frac{1}{p}\int_\Omega |Xu_0(x)|^p\,dx + \int_\Omega \left(F\left(u_0(x)\right)-\gamma\right)dx\right)\int_\Omega u_0^2(x)dx}.$$

□

The following remark establishes the existence of a function $u_0 \in L^\infty(\Omega) \cap \mathcal{W}^{1,p}_{X,0}(\Omega)$ for some Hörmander vector fields $X_1, \ldots, X_m$ that satisfies (4.1.4).

Remark 4.1.3. *[CC18] When $X_1 = \partial_{x_1}, \ldots, X_n = \partial_{x_n}$, the problem (2.1.1) becomes*

$$\begin{aligned} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) &= -\Lambda|u|^{p-2}u \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.1.10}$$

where $p > 2$. The first eigenvalue of (4.1.10) is denoted by $\Lambda_{1,p}$.

Let Ω be a smooth domain such that $\Lambda_{1,p} > \frac{p}{p-1}$. Also, let f be a nonnegative locally Lipschitz continuous function satisfying (4.1.2) *with $\gamma = 1$ and*

$$f(s) \geq p\Lambda_{1,p} s^{p-1} \quad \text{for all } s > 0.$$

By choosing $u_0(x) := \phi_{1,p}(x)$, where $\phi_{1,p}$ is an eigenfunction corresponding to $\Lambda_{1,p}$ with

$$\int_\Omega |\phi_{1,p}(x)|^p \, dx = |\Omega|,$$

we have

$$\begin{aligned}
& -\frac{1}{p}\int_\Omega |\nabla u_0(x)|^p \, dx + \int_\Omega \left[F\left(u_0(x)\right) - \gamma\right] dx \\
&= -\frac{1}{p}\int_\Omega |\nabla \phi_{1,p}(x)|^p \, dx + \int_\Omega \int_0^{\phi_{1,p}(x)} f(s) ds - |\Omega| \\
&\geq -\frac{\Lambda_{1,p}}{p}\int_\Omega [\phi_{1,p}(x)]^p \, dx + \int_\Omega \int_0^{\phi_{1,p}(x)} p\Lambda_{1,p} s^{p-1} ds - |\Omega| \\
&= \Lambda_{1,p}\left(1 - \frac{1}{p}\right)\int_\Omega [\phi_{1,p}(x)]^p \, dx - |\Omega| \\
&= \left[\Lambda_{1,p}\left(1 - \frac{1}{p}\right) - 1\right]|\Omega| > 0.
\end{aligned}$$

Here we have used

$$\Lambda_{1,p}\int_\Omega |\phi_{1,p}|^p dx = \int_\Omega |\nabla \phi_{1,p}|^p dx.$$

Remark 4.1.4. *When $X = \nabla$, Z. Junning [Jun93] established the local existence of weak solutions to* (4.1.1)*. Additionally, he derived some blow-up results for the same problem.*

4.2 Critical extinction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n > 2$ with smooth boundary $\partial\Omega$ and let $1 < p < 2$. We investigate the extinction properties of solutions to

$$\begin{aligned}
u_t + \sum_{j=1}^m X_j^*\left(|Xu|^{p-2} X_j u\right) &= \kappa u^q && \text{in } \Omega_T, \\
u(x,0) = u_0(x) &\geq 0 && \text{in } \Omega, \\
u(x,t) &= 0 && \text{on } \Sigma_T,
\end{aligned} \tag{4.2.1}$$

where $\Omega_T = \Omega \times (0,T)$, $\Sigma_T = \partial\Omega \times (0,T)$. Here $u_0 \in L^\infty(\Omega) \cap \mathcal{W}^{1,p}_{X,0}(\Omega)$, $q > 0$ and $\kappa > 0$.

Definition 4.2.1. *We say that a function*

$$u \in L^{2q}(\Omega_T) \cap L^2(\Omega_T) \quad \text{with} \quad u_t \in L^2(\Omega_T) \ \text{ and } Xu \in L^p(\Omega_T)$$

is a weak solution of (4.2.1)*, if it satisfies*

$$\int_\Omega \left(u_t\varphi + |Xu|^{p-2}Xu \cdot X\varphi\right)dx = \int_\Omega \kappa u^q \varphi dx \quad \text{for} \quad \text{a.e. } t \in (0,T) \tag{4.2.2}$$

for all $\varphi \in L^2(\Omega_T)$ *such that* $\varphi_t \in L^2(\Omega_T)$ *and* $X\varphi \in L^p(\Omega_T)$.

Theorem 4.2.2. *Let* $u_0 \in L^\infty(\Omega) \cap \mathcal{W}^{1,p}_{X,0}(\Omega)$ *with* $u_0 \geq 0$. *If* $q = p-1$ *and* $\kappa < \lambda_1$, *then the nonnegative weak solution of* (4.2.1) *vanishes in the following sense*

$$\lim_{t\to\infty} \int_\Omega u^2(x,t)dx = 0, \tag{4.2.3}$$

where λ_1 *is the first eigenvalue of* (2.1.1).

Moreover, if $\frac{2Q}{Q+2} \leq p < 2$ *with* $\kappa < \lambda_1$, *then the nonnegative weak solution of* (4.2.1) *vanishes in finite time.*

Proof. Let $q = p-1$ and $\kappa < \lambda_1$. We need to show that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}. \tag{4.2.4}$$

For convenience, we set $l := \|u_0\|_{L^\infty(\Omega)}$. Taking $(u-l)_+$ as a test function, we have

$$\int_\Omega \left(u_t(u-l)_+ + |Xu|^{p-2}Xu \cdot X(u-l)_+\right)dx = \kappa \int_\Omega u^{p-1}(u-l)_+ dx$$

for a.e. $t \in (0,T)$. This equality can be given by

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega (u-l)_+^2 dx + \int_{\{u(\cdot,t)>l\}} |Xu|^p dx &= \kappa \int_\Omega u^{p-1}(u-l)_+ dx \\ &= \kappa \int_{\{u(\cdot,t)>l\}} u^p dx. \end{aligned} \tag{4.2.5}$$

Since Ω is bounded, the embedding $L^2(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for any $1 < p < 2$. Given that $u \in L^2(\Omega_T)$, the Fubini theorem yields $u(\cdot,t) \in L^2(\Omega)$, which in turn implies $u(\cdot,t) \in L^p(\Omega)$. Since $Xu \in L^p(\Omega_T)$, it follows from the Fubini theorem that $Xu(\cdot,t) \in L^p(\Omega)$. Therefore, $u(\cdot,t) \in \mathcal{W}^{1,p}_{X,0}(\Omega)$ and in view of Corollary 2.1.8, we have

$$\int_{\{u(\cdot,t)>l\}} |Xu|^p dx \geq \lambda_1 \int_{\{u(\cdot,t)>l\}} u^p dx. \tag{4.2.6}$$

Combining (4.2.5) and (4.2.6), we get

$$\frac{d}{dt}\int_\Omega (u-l)_+^2\,dx \le (\kappa-\lambda_1)\int_{\{u(\cdot,t)>l\}} u^p dx.$$

Therefore,

$$\frac{d}{dt}\int_\Omega (u-l)_+^2\,dx \le 0. \tag{4.2.7}$$

Setting

$$g(t) := \int_\Omega (u-l)_+^2\,dx,$$

we see that $g(0)=0$. From (4.2.7), it follows that g is non-increasing. On the other hand, g is nonnegative. Since

$$g(0) \ge g(t) \quad \text{for all } 0<t<T,$$

it follows that

$$g(t) = \int_\Omega (u-l)_+^2\,dx \equiv 0$$

holds for all $0<t<T$, which verifies (4.2.4). Taking u as a test function,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega |Xu|^p dx = \kappa\int_\Omega u^p dx. \tag{4.2.8}$$

Then by Corollary 2.1.8 it follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx \le -(\lambda_1-\kappa)\int_\Omega u^p dx.$$

Setting $w := u/l$, we have

$$\frac{d}{dt}\int_\Omega w^2 dx \le -2l^{p-2}(\lambda_1-\kappa)\int_\Omega w^p dx \le -2l^{p-2}(\lambda_1-\kappa)\int_\Omega w^2 dx. \tag{4.2.9}$$

Here we have used the fact that $\|w^{2-p}\|_{L^\infty(\Omega)} \le 1$, which follows from (4.2.4). We integrate (4.2.9) over $(0,t)$ to get

$$\int_\Omega w^2 dx \le e^{-2l^{p-2}(\lambda_1-\kappa)t}\int_\Omega w_0^2 dx,$$

that is,

$$\int_\Omega u^2 dx \le e^{-2\|u_0\|_{L^\infty(\Omega)}^{p-2}(\lambda_1-\kappa)t}\int_\Omega u_0^2\,dx.$$

Letting $t\to\infty$, we arrive at (4.2.3).

Next, we consider the case when $\frac{2Q}{Q+2} \le p < 2$. Applying Corollary 2.1.8 to (4.2.8) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega |Xu|^p dx \le \frac{\kappa}{\lambda_1}\int_\Omega |Xu|^p dx.$$

From $p < 2$ and $n > 2$, it follows that $p < Q$. Note that, $\frac{2Q}{Q+2} \leq p$ is the same as $2 \leq \frac{Qp}{Q-p}$. Thus, Theorem 1.8.4 yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx \leq -\left(1-\frac{\kappa}{\lambda_1}\right)\int_\Omega |Xu|^p dx \leq -C\left(1-\frac{\kappa}{\lambda_1}\right)\left(\int_\Omega u^2 dx\right)^{\frac{p}{2}}.$$

Lastly, we define

$$\Phi(t) := \int_\Omega u^2 dx \quad \text{for all } t > 0,$$

which is non-increasing. Moreover, $\Phi(0) \geq 0$. Hence, there exists a critical time $T^* \in (0, T)$ such that

$$\Phi(t) = \int_\Omega u^2(x,t) dx \equiv 0 \quad \text{for all } T^* \leq t \leq T,$$

which implies $u = 0$ a.e. in Ω for $T^* \leq t \leq T$. □

Remark 4.2.3. *When $X = \nabla$, Theorem 4.2.2 can be found in [YJ07, Theorem 4.1].*

Chapter 5

Conclusions

In summary, this thesis presents significant findings concerning the spectral properties of the eigenvalues and eigenfunctions of the subelliptic p-Laplacian subject to the Dirichlet condition in a given domain. We demonstrated that all eigenvalues form a closed subset of $\mathbb{R}$. Although this result is not directly applied elsewhere in the thesis, it holds independent significance for future research.

It was shown that every eigenfunction is essentially bounded, a property used to prove both the Hölder continuity of eigenfunctions with respect to the Carnot-Carathéodory metric and the simplicity of the first eigenvalue λ_1. The Hölder continuity of eigenfunctions with respect to the Carnot-Carathéodory metric, while sufficient for establishing the simplicity of the first eigenvalue, has stronger implications in certain cases, such as when $m = n$ and $X_i = \partial_{x_i}$ for $i = 1, \ldots, n$, where the eigenfunctions exhibit $C^{1,\alpha}$ regularity.

The reflexivity of $\mathcal{W}_{X,0}^{1,p}(\Omega)$ enabled the minimization of the Rayleigh quotient, leading to the determination of the first eigenvalue λ_1. This also led to the derivation of the best constant in the L^p Poincaré-Friedrichs inequality for Hörmander vector fields. The eigenfunctions corresponding to the first eigenvalue λ_1 were shown to be positive throughout the domain, a result based on the Harnack inequality, distinguishing the first eigenvalue λ_1 from others. Moreover, the first eigenvalue λ_1 was proven to be simple and isolated, with no other eigenvalues existing in its some neighborhood.

Additionally, we established a nondecreasing sequence of variational eigenvalues by leveraging the Lusternik-Schnirelman theory. This was achieved by converting the eigenvalue problem into the operator form, where the properties of the operators, along with the fact that $\mathcal{W}_{X,0}^{1,p}(\Omega)$ is an infinite-dimensional real reflexive uniformly convex Banach space, played a crucial role. These theoretical tools allowed us to gain deeper insights into the structure and progression of the eigenvalues.

By the end of the thesis, we also explored some applications of the first eigenvalue λ_1. Specifically, the first eigenvalue λ_1 can be applied to study extinction and blow-up phenomena in parabolic-type equations involving the subelliptic p-Laplacian. This provides valuable insights into the dynamic behavior of solutions to such equations.

These results not only contribute to the broader understanding of spectral theory but also lay a strong foundation for further developments in related fields.

Chapter 6

Appendix

The positive and negative parts of a function u are given by

$$u^{+} := \max\{u, 0\} \quad \text{and} \quad u^{-} := \min\{u, 0\}.$$

Proposition 6.0.1. *[GN96, Lemma 3.5]. Let Ω be a bounded open set in $\mathbb{R}^n$. If $u \in \mathcal{W}^{1,p}_{X,0}(\Omega)$, then $u^{\pm}, |u| \in \mathcal{W}^{1,p}_{X,0}(\Omega)$ and*

$$X|u| = \begin{cases} Xu, & \text{if } u > 0 \text{ a.e. in } \Omega, \\ 0, & \text{if } u = 0 \text{ a.e. in } \Omega, \\ -Xu & \text{if } u < 0 \text{ a.e. in } \Omega. \end{cases}$$

Proposition 6.0.2. *[Lin90, Lemma 4.2] Let $k \in \mathbb{N}$. Then the following inequalities hold:*

(i) If $1 < p < 2$, then there exists a constant $C > 0$ depending only on p such that

$$|\omega_2|^p \geq |\omega_1|^p + p|\omega_1|^{p-2}\omega_1 \cdot (\omega_2 - \omega_1) + \frac{C|\omega_2 - \omega_1|^2}{(|\omega_2| + |\omega_1|)^{2-p}} \tag{6.0.1}$$

for all $\omega_1, \omega_2 \in \mathbb{R}^k$.

(ii) If $p \geq 2$, then there exists a constant $C > 0$ depending only on p such that

$$|\omega_2|^p \geq |\omega_1|^p + p|\omega_1|^{p-2}\omega_1 \cdot (\omega_2 - \omega_1) + C|\omega_2 - \omega_1|^p \tag{6.0.2}$$

for all $\omega_1, \omega_2 \in \mathbb{R}^k$.

Lemma 6.0.3. *[Lin90, Appendix] Let $k \in \mathbb{N}$. Then the following inequalities hold:*

(i) If $1 < p \leq 2$, then there exists a constant $C > 0$ depending only on p such that

$$\left(|\omega_1|^{p-2}\omega_1 - |\omega_2|^{p-2}\omega_2\right) \cdot (\omega_1 - \omega_2) \geq C\frac{|\omega_1 - \omega_2|^2}{(|\omega_2| + |\omega_1|)^{2-p}} \tag{6.0.3}$$

for all $\omega_1, \omega_2 \in \mathbb{R}^k$.

(ii) If $p \geq 2$*, then there exists a constant* $C > 0$ *depending only on* p *such that*

$$\big(|\omega_1|^{p-2}\omega_1 - |\omega_2|^{p-2}\omega_2\big) \cdot (\omega_1 - \omega_2) \geq C|\omega_1 - \omega_2|^p \tag{6.0.4}$$

for all $\omega_1, \omega_2 \in \mathbb{R}^k$*.*

Lemma 6.0.4. *[DJM01] Let* $k \in \mathbb{N}$*. Then the following inequalities hold:*

(i) If $1 < p \leq 2$*, then there exists* $C > 0$ *depending only on* p *such that*

$$||\omega_1|^{p-2}\,\omega_1 - |\omega_2|^{p-2}\omega_2| \leq C|\omega_1 - \omega_2|^{p-1} \tag{6.0.5}$$

for all $\omega_1, \omega_2 \in \mathbb{R}^k$*.*

(ii) If $p \geq 2$*, then there exists* $C > 0$ *depending only on* p *such that*

$$||\omega_1|^{p-2}\,\omega_1 - |\omega_2|^{p-2}\omega_2| \leq C|\omega_1 - \omega_2\,|\,(|\omega_1| + |\omega_2|)^{p-2} \tag{6.0.6}$$

for all $\omega_1, \omega_2 \in \mathbb{R}^k$*.*

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