

Cross probability in the pushTASEP with initial condition 1...12

by

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Abstract

We study the 2 species PushTASEP, an stochastic model with finitely many particles each labeled by a positive integer acting on the $\mathbb{Z}$ lattice pushing one another. The main object of the research is the specific problem of the multi-species PushTASEP: the probability that particle of species 2 moving to the left-most position and initially left-most particle of species 1 going after particle of species 2 after some time t . We study the N -particle system in which particles are initially ordered as $11 \cdots 12$ — that is, $N - 1$ particles of species 1 followed by a single particle of species 2 — and derive an exact formula for the probability that the leftmost particle of species 1 is at position x at time t and particle of species 2 is to the left of the the leftmost species 1 particle. We show the exact formula using Bethe Ansatz method for $N=3$, $N=4$ cases and proof of the general formula using strong induction.

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Chapter 1

Introduction

In this paper, we analyze a two-species stochastic model with finitely many particles on the infinite lattice $\mathbb{Z}$, which has the exclusion rule that each site cannot be occupied by more than one particle. Each particle is labeled by a positive integer l , representing its species. A state of the system can be described by each particle's species and its location. In specific, if $X = (x_1, \dots, x_N) \in \mathbb{Z}^N$ with $x_1 < \dots < x_N$ represents the positions of N particles, and the species of the particle at x_i is denoted by π_i , then we denote the state of the system by $(X, \pi) = (x_1, \dots, x_N, \pi_1 \cdots \pi_N)$ with $\pi = \pi_1 \cdots \pi_N$.

The model we are interested in is PushTASEP, also well-known as the drop-push model [1, 5, 6]. In the PushTASEP, each particle waits an exponential time with rate 1 and then jumps to the right. Unlike TASEP model, a particle can jump even if the right site is already occupied.

In the multi-species PushTASEP, a particle of higher-numbered species has an advantage over a particle of lower-numbered species while jumping. To be more precise, if a particle of species l at position x chooses to jump to the position $x + 1$, which is already occupied by another particle of species l' and $l' < l$, then the particle l can jump to the site by pushing all other particles to the right. However, if $l' \geq l$, then the particle l chooses the nearest available position.

1.1 Definition of the Model

In this thesis, we study the two-species PushTASEP, that is, the light version of the multi-species asymmetric simple exclusion process introduced in [2]. The precise definition is derived from [2].

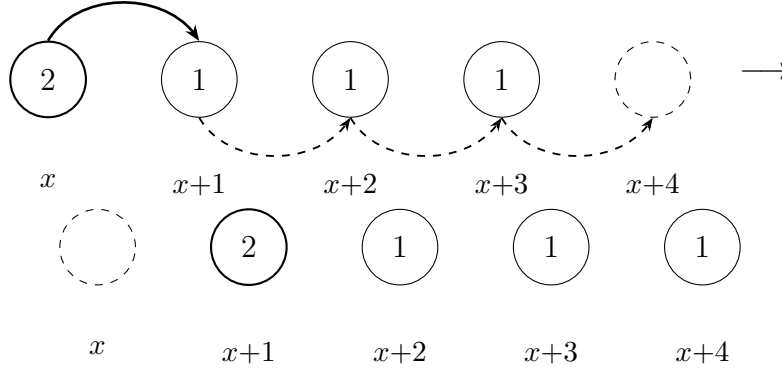


Figure 1: PushTASEP: species-2 particle at position x jumps to $x + 1$, which is occupied by a species-1 particle. The species-1 particle at $x + 1$ is pushed to $x + 2$, the one at $x + 2$ is pushed to $x + 3$, and so on. The configuration $(\dots, 2, 1, 1, 1, \dots)$ at positions $(x, x+1, x+2, x+3)$ becomes $(\dots, 0, 2, 1, 1, 1, \dots)$ at positions $(x, x+1, x+2, x+3, x+4)$.

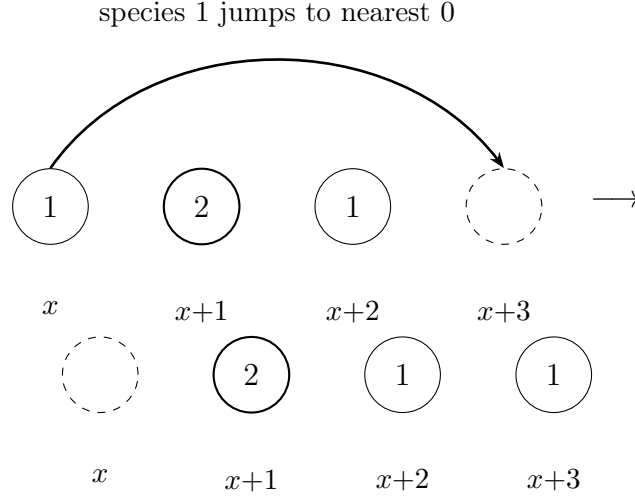


Figure 2: PushTASEP: species-1 particle at x jumps to the nearest site with $l' < 1$, i.e. the nearest empty site at $x + 3$. Since species 2 at $x + 1$ has higher priority ($2 > 1$), it is not pushed. The configuration $(1, 2, 1, 0)$ at positions $(x, x+1, x+2, x+3)$ becomes $(0, 2, 1, 1)$.

Definition 1.1. A particle of species l at position x waits an exponential time with rate 1 to jump to the right. It jumps to the nearest right site occupied by species of l'

and $l' < l$ (including $l' = 0$, which corresponds to an empty site), and pushing l' to the nearest right site occupied by species of l'' and $l'' < l'$. The two specific examples are shown in figures 1,2.

1.2 Notations

Suppose that there are N particles, of which $N - 1$ are of species 1 and one is of species 2, and let $\mathcal{M} = [1, 1, \dots, 1, 2]$ be the multi-set of those N integers. Let $\mathbf{P}_{(Y, 11\dots 12)}(X, 21\dots 11; t)$ be the probability that the system will be in state $(X, 21\dots 11)$ at time t , given that the system is in state $(Y, 11\dots 12)$ at time 0.

Throughout this thesis, $\mathbf{I}$ denotes the identity matrix, and $\mathbf{0}$ denotes the zero matrix. If needed, we write $\mathbf{I}_n$ and $\mathbf{0}_n$ to specify that their dimensions are n by n . We write $\mathbf{M}_{(\pi, \nu)}$ for the (π, ν) -element of a matrix $\mathbf{M}$. The two-body scattering coefficients are defined by:

$$S_{\beta\alpha} = -\frac{\xi_\beta(\xi_\alpha - 1)}{\xi_\alpha(\xi_\beta - 1)}, \quad T_{\beta\alpha} = \frac{\xi_\beta - \xi_\alpha}{\xi_\alpha(\xi_\beta - 1)}. \quad (1.1)$$

The transition probability matrix derived by Lee and Raimbekov in [4] is given by

$$\mathbf{P}_{(Y, 11\dots 12)}(X, 21\dots 11; t) = \oint_C \cdots \oint_C \sum_{\sigma \in S_N} [A_\sigma]_{21\dots 11, 11\dots 12} \prod_{i=1}^N \left(\xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\varepsilon(\xi_i)t} \right) d\xi_1 \cdots d\xi_N. \quad (1.2)$$

where A_σ is an $N^N \times N^N$ matrix, and $\oint_C$ implies $(1/2\pi i) \int_C$ where the contour C is a positively oriented circle centered at the origin with radius bigger than 1, so that all poles are included in C , and

$$\varepsilon(\xi_i) = \frac{1}{\xi_i} - 1. \quad (1.3)$$

The R -matrix for PushTASEP explicitly derived by Lee in [2]. The general form of

$R_{\beta\alpha}$ shown below:

$$(R_{\beta\alpha})_{(ij,lm)} = \begin{cases} S_{\beta\alpha} & \text{if } ij = lm \text{ and } i \geq j, \\ -1 & \text{if } ij = lm \text{ and } i < j, \\ T_{\beta\alpha} & \text{if } ij = ml \text{ and } i > j, \\ 0 & \text{otherwise.} \end{cases} \quad (1.4)$$

We also define $\mathbf{T}_i$ similar to [3] as the simple transposition in S_N that swaps the i -th and $(i+1)$ -th positions. Since any permutation $\sigma \in S_N$ can be decomposed into a product of simple transpositions, we write

$$\sigma = \mathbf{T}_{i_j}(\beta_j, \alpha_j) \mathbf{T}_{i_{j-1}}(\beta_{j-1}, \alpha_{j-1}) \cdots \mathbf{T}_{i_1}(\beta_1, \alpha_1), \quad (1.5)$$

where the notation $\mathbf{T}_{i_k}(\beta_k, \alpha_k)$ means the simple transposition $\mathbf{T}_{i_k}$ applied at step k , with β_k and α_k represent the two values being swapped at that step. To each such simple transposition $\mathbf{T}_i(\beta, \alpha)$ we associate an $N^N \times N^N$ matrix $\mathcal{T}_i(\beta, \alpha)$, defined by placing the $N^2 \times N^2$ matrix $R_{\beta\alpha}$ in the i -th slot and identity matrices I_N everywhere else:

$$\mathcal{T}_i(\beta, \alpha) = \underbrace{I_N \otimes \cdots \otimes I_N}_{(i-1) \text{ times}} \otimes R_{\beta\alpha} \otimes \underbrace{I_N \otimes \cdots \otimes I_N}_{(N-i-1) \text{ times}}, \quad (1.6)$$

where rows and columns of $R_{\beta\alpha}$ are labelled lexicographically by $11, 12, \dots, NN$. The matrix A_σ is then defined as the ordered product

$$A_\sigma = \mathcal{T}_{i_j}(\beta_j, \alpha_j) \mathcal{T}_{i_{j-1}}(\beta_{j-1}, \alpha_{j-1}) \cdots \mathcal{T}_{i_1}(\beta_1, \alpha_1). \quad (1.7)$$

The restricted form of $\mathcal{T}_i(\beta, \alpha)$ is taken to be an $N \times N$ matrix, which we again denote by $\mathcal{T}_i(\beta, \alpha)$, whose entries are given by

$$[\mathcal{T}_i(\beta, \alpha)]_{\pi^{(k)}, \pi^{(l)}} = \begin{cases} -1 & \text{if } k = l = i, \\ S_{\beta\alpha} & \text{if } k = l \text{ and } l \neq i, \\ T_{\beta\alpha} & \text{if } k = i + 1 \text{ and } l = i, \\ 0 & \text{otherwise.} \end{cases} \quad (1.8)$$

1.3 Organization of the Paper

In Chapter 2 and Chapter 3, we investigate the cases of $N = 3$ and $N = 4$ respectively. Both results then are used to derive the general formula for $\mathbf{P}_{(Y, 11 \dots 12)}(X, 21 \dots 11; t)$, that is the probability that the leftmost particle of species 1 coming right after the particle of species 2. This will help to obtain the conjecture of the formula of $[A_\sigma]_{21 \dots 11, 11 \dots 12}$. After deriving the general formula, we shall prove the result by strong induction.

Chapter 2

N=3 Particle System

The probability that leftmost particle of species 1 is at position x after time t for three particles system is given by

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbf{P}_{(y_1, y_2, y_3, 112)}(x-i, x, x+j, 211; t) &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \oint_C \oint_C \oint_C e^{t(\frac{1}{\xi_1} + \frac{1}{\xi_2} + \frac{1}{\xi_3} - 3)} \\ &\times [A_{\pi}]_{\sigma, \eta} \xi_{\pi(1)}^{x-j-y_{\pi(1)}-1} \xi_{\pi(2)}^{x-y_{\pi(2)}-1} \xi_{\pi(3)}^{x+j-y_{\pi(3)}-1} d\xi_1 d\xi_2 d\xi_3. \quad (2.1) \end{aligned}$$

We compute the transition from 112 to 211 with particles located at positions x_1, x_2, x_3 . the simple transposition matrices $\mathcal{T}_1, \mathcal{T}_2$ acting on the basis $\{112, 121, 211\}$ as follows

$$\mathcal{T}_1(2, 1) = \begin{pmatrix} S_{21} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & T_{21} & S_{21} \end{pmatrix}, \quad \mathcal{T}_2(3, 2) = \begin{pmatrix} -1 & 0 & 0 \\ T_{32} & S_{32} & 0 \\ 0 & 0 & S_{32} \end{pmatrix},$$

where $T_{\beta\alpha}$ and $S_{\beta\alpha}$ are defined in equation (1.1).

We then compute all matrix products needed to build A_{π} for $\pi \in S_3$

$$\mathcal{T}_1(3, 1)\mathcal{T}_2(3, 2) = \begin{pmatrix} -S_{31} & 0 & 0 \\ -T_{32} & -S_{32} & 0 \\ T_{31}T_{32} & T_{31}S_{32} & S_{31}S_{32} \end{pmatrix},$$

$$\mathcal{T}_2(3,1)\mathcal{T}_1(2,1) = \begin{pmatrix} -S_{21} & 0 & 0 \\ T_{31}S_{21} & -S_{31} & 0 \\ 0 & S_{31}T_{21} & S_{21}S_{31} \end{pmatrix},$$

$$\mathcal{T}_1(3,2)\mathcal{T}_2(3,1)\mathcal{T}_1(2,1) = \begin{pmatrix} -S_{32}S_{21} & 0 & 0 \\ -T_{31}S_{21} & S_{31} & 0 \\ T_{31}T_{32}S_{21} & -S_{31}T_{32} + S_{31}S_{32}T_{21} & S_{31}S_{32}S_{21} \end{pmatrix}.$$

Out of all $3! = 6$ permutations, only $(N-1)! = 2$ of them give nonzero coefficients

$$[A_{312}]_{211,112} = T_{31}T_{32}, \quad [A_{321}]_{211,112} = T_{31}T_{32}S_{21},$$

Giving the coefficients, we can now fully represent our equation we can represent our equation

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbf{P}_{(y_1, y_2, y_3, 112)}(x-i, x, x+j, 211; t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \oint_C \oint_C \oint_C e^{t\left(\frac{1}{\xi_1} + \frac{1}{\xi_2} + \frac{1}{\xi_3} - 3\right)} \\ \times \left[T_{31}T_{32} \xi_3^{x-i-y_3-1} \xi_1^{x-y_1-1} \xi_2^{x+j-y_2-1} + S_{21}T_{31}T_{32}\xi_3^{x-i-y_3-1} \xi_2^{x-y_2-1} \xi_1^{x+j-y_3-1} \right] d\xi_1 d\xi_2 d\xi_3.$$

Now we can factor out common terms from the equation, which yields

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbf{P}_{(y_1, y_2, y_3, 112)}(x-i, x, x+j, 211; t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \oint_C \oint_C \oint_C e^{t\left(\frac{1}{\xi_1} + \frac{1}{\xi_2} + \frac{1}{\xi_3} - 3\right)} \left(\xi_1^{x-y_1-1} \xi_2^{x-y_2-1} \xi_3^{x-y_3-1} \right) \\ \times \left[T_{31}T_{32} \xi_3^{-i} \xi_2^j + T_{31}T_{32}S_{21} \xi_3^{-i} \xi_1^j \right] d\xi_1 d\xi_2 d\xi_3.$$

For simplicity, from now on we set

$$B := e^{t\left(\frac{1}{\xi_1} + \frac{1}{\xi_2} + \frac{1}{\xi_3} - 3\right)} \left(\xi_1^{x-y_1-1} \xi_2^{x-y_2-1} \xi_3^{x-y_3-1} \right). \quad (2.2)$$

We can now push the sums inside the integrals to get

$$\oint_C \oint_C \oint_C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} B \times \left[T_{31}T_{32} \xi_3^{-i} \xi_2^j + T_{31}T_{32}S_{21} \xi_3^{-i} \xi_1^j \right] d\xi_1 d\xi_2 d\xi_3.$$

Using the formula for summation of geometric series, we can evaluate the summation for $-i$ without any restrictions, because the we are working on big contour C. We can split and obtain two integrals

$$\begin{aligned}\mathcal{I}_1 &= \oint_C \oint_C \oint_C \sum_{j=1}^{\infty} B \cdot T_{31} T_{32} \xi_2^j \frac{1}{(\xi_3 - 1)} d\xi_1 d\xi_2 d\xi_3, \\ \mathcal{I}_2 &= \oint_C \oint_C \oint_C \sum_{j=1}^{\infty} B \cdot T_{31} T_{32} S_{21} \xi_1^j \frac{1}{(\xi_3 - 1)} d\xi_1 d\xi_2 d\xi_3.\end{aligned}$$

To evaluate the summation for j , we need to shrink the contour to c , that is positively orientated circle centered at the origin with radius less than 1, where all the poles are located outside the c , except those located at the origin. We shrink both contours for respective ξ 's and after evaluating the summation, we obtain

$$\begin{aligned}\mathcal{I}_1 &= \oint_C \oint_c \oint_C B \left[T_{31} T_{32} \frac{\xi_2}{(1 - \xi_2)(\xi_3 - 1)} \right] d\xi_1 d\xi_2 d\xi_3, \\ \mathcal{I}_2 &= \oint_C \oint_C \oint_c B \left[T_{31} T_{32} S_{21} \frac{\xi_1}{(1 - \xi_1)(\xi_3 - 1)} \right] d\xi_1 d\xi_2 d\xi_3.\end{aligned}$$

To increase the contour back to big one, we further need to consider the singularities at $\xi_2 = 1$ and $\xi_1 = 0$. However, we can neglect the singularities at 0, since they are included in both contours. The only singularity we consider is at $\xi_2 = 1$ and after calculating its residue, we get

$$\text{Res}_{\xi_2=1} = \frac{(\xi_3 - \xi_1)(\xi_3 - 1)}{\xi_1(\xi_3 - 1)^3}.$$

Respective two integrals with big contours now become

$$\begin{aligned}\mathcal{I}_1 &= \oint_C \oint_C \oint_C B \left[T_{31} T_{32} \frac{\xi_2}{(1 - \xi_2)(\xi_3 - 1)} \right] d\xi_1 d\xi_2 d\xi_3 + \oint_C \oint_C B T_{31} \frac{1}{\xi_3 - 1} d\xi_1 d\xi_3, \\ \mathcal{I}_2 &= \oint_C \oint_C \oint_C B \left[T_{31} T_{32} S_{21} \frac{\xi_1}{(1 - \xi_1)(\xi_3 - 1)} \right] d\xi_1 d\xi_2 d\xi_3.\end{aligned}$$

Combining both $\mathcal{I}_1$ and $\mathcal{I}_2$ we get

$$\oint_C \oint_C \oint_C B \times T_{31} T_{32} \frac{1}{\xi_3 - 1} \left[\frac{\xi_2}{1 - \xi_2} + S_{21} \frac{\xi_1}{1 - \xi_1} \right] d\xi_1 d\xi_2 d\xi_3 + \oint_C \oint_C B \times T_{31} \frac{1}{\xi_3 - 1} d\xi_1 d\xi_3.$$

Putting the values for S_{21} into the equation, we get

$$\oint_C \oint_C \oint_C B \times T_{31} T_{32} \frac{1}{\xi_3 - 1} \left[\frac{\xi_2}{1 - \xi_2} + \frac{\xi_2(1 - \xi_1)}{\xi_1(\xi_2 - 1)} \frac{\xi_1}{1 - \xi_1} \right] d\xi_1 d\xi_2 d\xi_3 + \oint_C \oint_C B \times T_{31} \frac{1}{\xi_3 - 1} d\xi_1 d\xi_3.$$

Evaluating the full sum, we finally get the final result for probability, that starting from configuration $(Y, 112)$ the system reaches configuration $(X, 221)$ at time t , meaning the particle of species 2 has moved to the position with all species 1 to its right, and the initially leftmost particle of species 1 is at position x . To represent final result, we also need also to consider the value B from (2.2) at $\xi_2 = 1$. The final result for $N=3$ case will have following form

$$\boxed{\oint_C \oint_C e^{t\left(\frac{1}{\xi_1} + \frac{1}{\xi_3} - 2\right)} \xi_1^{x-y_1-1} \xi_3^{x-y_3-1} \times T_{31} \frac{1}{\xi_3 - 1} d\xi_1 d\xi_3.} \quad (2.3)$$

Chapter 3

N=4 Particle System

The probability that the leftmost particle of species 1 is at position x after time t for four particles system is given by

$$\begin{aligned}
& \sum_{i=1}^{\infty} \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \mathbf{P}_{(y_1, y_2, y_3, y_4, 1112)}(x-i, x, x+j_1, x+j_2+j_1, 2111; t) \\
&= \sum_{i=1}^{\infty} \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \oint_C \oint_C \oint_C \oint_C e^{t\left(\frac{1}{\xi_1} + \frac{1}{\xi_2} + \frac{1}{\xi_3} + \frac{1}{\xi_4} - 4\right)} \\
&\times [A_{\pi}]_{\sigma, \eta} \xi_{\pi(1)}^{x-i-y_{\pi(1)}-1} \xi_{\pi(2)}^{x-y_{\pi(2)}-1} \xi_{\pi(3)}^{x+j_1-y_{\pi(3)}-1} \xi_{\pi(4)}^{x+j_1+j_2-y_{\pi(4)}-1} d\xi_1 d\xi_2 d\xi_3 d\xi_4. \quad (3.1)
\end{aligned}$$

We compute the transition from 1112 to 2111 with particles located at positions x_1, x_2, x_3, x_4 . Similar to N=3 case, we define the restricted form of simple transposition matrices $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ acting on the basis $\{1112, 1121, 1211, 2111\}$ as follows

$$\mathcal{T}_1(2, 1) = \begin{pmatrix} S_{21} & 0 & 0 & 0 \\ 0 & S_{21} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & T_{21} & S_{21} \end{pmatrix}, \quad \mathcal{T}_2(3, 2) = \begin{pmatrix} S_{32} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & T_{32} & S_{32} & 0 \\ 0 & 0 & 0 & S_{32} \end{pmatrix},$$

$$\mathcal{T}_3(4, 3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ T_{43} & S_{43} & 0 & 0 \\ 0 & 0 & S_{43} & 0 \\ 0 & 0 & 0 & S_{43} \end{pmatrix}.$$

All the heavy matrix multiplication calculations, needed to evaluate the nonzero amplitudes $[A_\pi]_{2111, 1112}$, is carried out in Appendix B. From those computations, only six permutations out of $4! = 24$ give nonzero contributions

π	$[A_\pi]_{2111, 1112}$
4123	$T_{41}T_{42}T_{43}$
4132	$S_{32}T_{41}T_{42}T_{43}$
4213	$S_{21}T_{41}T_{42}T_{43}$
4231	$S_{21}S_{31}T_{41}T_{42}T_{43}$
4312	$S_{31}S_{32}T_{41}T_{42}T_{43}$
4321	$S_{21}S_{31}S_{32}T_{41}T_{42}T_{43}$

(3.2)

In total 18 permutations has value of zero and 6 permutations has nonzero value, which is exactly $(N-1)!$ number of permutations. Full table of all values can be seen in Appendix A. As previously, we substitute (x_1, x_2, x_3, x_4) with $(x-i, x, x+j_1, x+j_1+j_2)$ and for simplicity set

$$B := e^{t\left(\frac{1}{\xi_1} + \frac{1}{\xi_2} + \frac{1}{\xi_3} + \frac{1}{\xi_4} - 4\right)} \left(\xi_1^{x-y_1-1} \xi_2^{x-y_2-1} \xi_3^{x-y_3-1} \xi_4^{x-y_4-1} \right). \quad (3.3)$$

Substituting the nonzero amplitudes and factoring out common terms, we obtain

$$\begin{aligned} & \sum_{i=1}^{\infty} \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \mathbf{P}_{(y_1, y_2, y_3, y_4, 1112)}(x-i, x, x+j_1, x+j_1+j_2, 2111; t) \\ &= \sum_{i=1}^{\infty} \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \oint_C \oint_C \oint_C \oint_C B \times T_{41}T_{42}T_{43} \cdot \xi_4^{-i} \left[\xi_2^{j_1} \xi_3^{j_1+j_2} + S_{32} \xi_3^{j_1} \xi_2^{j_1+j_2} + S_{21} \xi_1^{j_1} \xi_3^{j_1+j_2} \right. \\ & \quad \left. + S_{21}S_{31} \xi_3^{j_1} \xi_1^{j_1+j_2} + S_{31}S_{32} \xi_1^{j_1} \xi_2^{j_1+j_2} + S_{21}S_{31}S_{32} \xi_2^{j_1} \xi_1^{j_1+j_2} \right] d\xi_1 d\xi_2 d\xi_3 d\xi_4. \end{aligned}$$

We can put the sums inside the integrals and evaluate them term by term. From here, we will treat each term separately labeling them $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5, \mathcal{I}_6$.

$$\mathcal{I}_1 = \oint_C \oint_C \oint_C \oint_C B \cdot T_{41} T_{42} T_{43} \cdot \frac{1}{\xi_4 - 1} \sum_{j_1}^{\infty} (\xi_2 \xi_3)^{j_1} \sum_{j_2}^{\infty} \xi_3^{j_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4$$

We can shrink the contour for ξ_3 to evaluate the sum for j_2 , giving us

$$\oint_C \oint_c \oint_C \oint_C B \cdot T_{41} T_{42} T_{43} \cdot \frac{1}{\xi_4 - 1} \sum_{j_1}^{\infty} (\xi_2 \xi_3)^{j_1} \frac{\xi_3}{1 - \xi_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4$$

Expand back the contour to the big one with residue term at $\xi_3 = 1$

$$\begin{aligned} \oint_C \oint_C \oint_C \oint_C B \cdot T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1}^{\infty} (\xi_2 \xi_3)^{j_1} \frac{\xi_3}{1 - \xi_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\ + \oint_C \oint_C \oint_C B \cdot T_{41} T_{42} \frac{1}{\xi_4 - 1} \sum_{j_1}^{\infty} (\xi_2)^{j_1} d\xi_1 d\xi_2 d\xi_4 \end{aligned}$$

Now, we need to deal with the summation over j_1 in triple integral. We shrink the contour for ξ_2

$$\oint_C \oint_c \oint_C B \cdot T_{41} T_{42} \frac{1}{\xi_4 - 1} \frac{\xi_2}{1 - \xi_2} d\xi_1 d\xi_2 d\xi_4$$

Expand the contour back to big one with the residue term for $\xi_2 = 1$

$$\oint_C \oint_C \oint_C B \cdot T_{41} T_{42} \cdot \frac{1}{\xi_4 - 1} \frac{\xi_2}{1 - \xi_2} d\xi_1 d\xi_2 d\xi_4 + \oint_C \oint_C B \cdot T_{41} \frac{1}{\xi_4 - 1} d\xi_1 d\xi_4$$

The overall result for the first term is

$$\begin{aligned} \mathcal{I}_1 = \oint_C \oint_C \oint_C \oint_C B \cdot T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1}^{\infty} (\xi_2 \xi_3)^{j_1} \frac{\xi_3}{1 - \xi_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\ + \oint_C \oint_C \oint_C B \cdot T_{41} T_{42} \cdot \frac{1}{\xi_4 - 1} \frac{\xi_2}{1 - \xi_2} d\xi_1 d\xi_2 d\xi_4 \\ + \oint_C \oint_C B \cdot T_{41} \frac{1}{\xi_4 - 1} d\xi_1 d\xi_4 \end{aligned}$$

Moving forward to second term, which has the following form

$$\mathcal{I}_2 = \oint_C \oint_C \oint_C \oint_C B \cdot S_{32} T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1} \sum_{j_2} (\xi_2 \xi_3)^{j_1} \xi_2^{j_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Evaluate summation for j_2 by shrinking the contour for ξ_2

$$\mathcal{I}_2 = \oint_C \oint_C \oint_C \oint_C B \cdot \frac{\xi_3(1 - \xi_2)}{\xi_2(\xi_3 - 1)} T_{41} T_{42} T_{43} \cdot \frac{1}{\xi_4 - 1} \sum_{j_1} (\xi_2 \xi_3)^{j_1} \frac{\xi_2}{1 - \xi_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Expanding the contour back to the big one without a residue term, since after cancellation of similar terms there are no singularities and we obtain

$$\mathcal{I}_2 = \oint_C \oint_C \oint_C \oint_C B \cdot S_{32} T_{41} T_{42} T_{43} \cdot \frac{1}{\xi_4 - 1} \sum_{j_1} (\xi_2 \xi_3)^{j_1} \frac{\xi_2}{1 - \xi_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Combining Terms $\mathcal{I}_1$ and $\mathcal{I}_2$, using the cancellation identity $\frac{\xi_3}{1 - \xi_3} + S_{32} \frac{\xi_2}{1 - \xi_2} = 0$, the quadruple integrals cancel each other leaving only

$$\begin{aligned} \mathcal{I}_1 + \mathcal{I}_2 &= \oint_C \oint_C \oint_C B \cdot T_{41} T_{42} \cdot \frac{\xi_2}{1 - \xi_2} \cdot \frac{1}{\xi_4 - 1} d\xi_1 d\xi_2 d\xi_4 \\ &\quad + \oint_C \oint_C B \cdot T_{41} \cdot \frac{1}{\xi_4 - 1} d\xi_1 d\xi_4. \end{aligned}$$

For the third term, we have

$$\mathcal{I}_3 = \oint_C \oint_C \oint_C \oint_C B \cdot S_{21} T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1} \sum_{j_2} (\xi_1 \xi_3)^{j_1} \xi_3^{j_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Evaluate summation for j_2 by shrinking the contour for ξ_3

$$\mathcal{I}_3 = \oint_C \oint_C \oint_C \oint_C B \cdot S_{21} T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1} (\xi_1 \xi_3)^{j_1} \frac{\xi_3}{1 - \xi_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Expanding back the contour to the big one with the residue term at $\xi_3 = 1$ and we obtain

$$\begin{aligned}\mathcal{I}_3 = & \oint_C \oint_C \oint_C \oint_C B \cdot S_{21} T_{41} T_{42} T_{43} \cdot \frac{1}{\xi_4 - 1} \sum_{j_1} (\xi_1 \xi_3)^{j_1} \frac{\xi_3}{1 - \xi_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\ & + \oint_C \oint_C \oint_C B \cdot S_{21} T_{41} T_{42} \frac{1}{\xi_4 - 1} \sum_{j_1} \xi_1^{j_1} d\xi_1 d\xi_2 d\xi_4.\end{aligned}$$

Evaluating the summation over j_1 in the triple integral by shrinking the contour for ξ_1 we get

$$\begin{aligned}\mathcal{I}_3 = & \oint_C \oint_C \oint_C \oint_C B \cdot S_{21} T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1} (\xi_1 \xi_3)^{j_1} \frac{\xi_3}{1 - \xi_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\ & + \oint_C \oint_C \oint_C B \cdot \frac{\xi_2(1 - \xi_1)}{\xi_1(\xi_2 - 1)} T_{41} T_{42} \frac{1}{\xi_4 - 1} \frac{\xi_1}{1 - \xi_1} d\xi_1 d\xi_2 d\xi_4.\end{aligned}$$

Expanding the contour back to the big one without a residue term, since after cancellation of similar terms there are no singularities and we get

$$\begin{aligned}\mathcal{I}_3 = & \oint_C \oint_C \oint_C \oint_C B \cdot S_{21} T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1} (\xi_1 \xi_3)^{j_1} \frac{\xi_3}{1 - \xi_3} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\ & + \oint_C \oint_C \oint_C B \cdot T_{41} T_{42} \frac{\xi_2}{\xi_2 - 1} \cdot \frac{1}{\xi_4 - 1} d\xi_1 d\xi_2 d\xi_4.\end{aligned}$$

Forth term of the equation will have following form

$$\mathcal{I}_4 = \oint_C \oint_C \oint_C \oint_C B \cdot S_{21} S_{31} T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1} \sum_{j_2} (\xi_1 \xi_3)^{j_1} \xi_1^{j_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Evaluating the summation for j_2 by shrinking the contour for ξ_1 we get

$$\oint_C \oint_C \oint_C \oint_C B \cdot \frac{\xi_2(1 - \xi_1)}{\xi_1(\xi_2 - 1)} S_{31} T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1} (\xi_1 \xi_3)^{j_1} \frac{\xi_1}{1 - \xi_1} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Expanding the contour back to the big one without a residue term, since after the cancellation of similar terms there are no singularities, so we obtain

$$\oint_C \oint_C \oint_C \oint_C B \cdot S_{21} S_{31} T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1} (\xi_1 \xi_3)^{j_1} \frac{\xi_1}{1 - \xi_1} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Combining Terms 3 and 4, using the cancellation identity $\frac{\xi_3}{1 - \xi_3} + S_{31} \frac{\xi_1}{1 - \xi_1} = 0$, the quadruple integrals cancel each other leaving only

$$\mathcal{I}_3 + \mathcal{I}_4 = \oint_C \oint_C \oint_C B \cdot T_{41} T_{42} \frac{\xi_2}{\xi_2 - 1} \frac{1}{\xi_4 - 1} d\xi_1 d\xi_2 d\xi_4.$$

Now we need to deal with last two terms of the equation. For the fifth term we have

$$\mathcal{I}_5 = \oint_C \oint_C \oint_C \oint_C B \cdot S_{31} S_{32} T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1} \sum_{j_2} (\xi_1 \xi_2)^{j_1} \xi_2^{j_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Evaluating summation for j_2 by shrinking the contour for ξ_2 to get

$$\oint_C \oint_C \oint_C \oint_C B \cdot S_{31} \frac{\xi_3(1 - \xi_2)}{\xi_2(\xi_3 - 1)} T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1} (\xi_1 \xi_2)^{j_1} \frac{\xi_2}{1 - \xi_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Expanding the contour back to the big one without a residue term, since after the cancellation of similar terms there are no singularities, so we get

$$\oint_C \oint_C \oint_C \oint_C B \cdot S_{31} S_{32} T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1} (\xi_1 \xi_2)^{j_1} \frac{\xi_2}{1 - \xi_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

The last term of the equation is sixth one and it has following form

$$\mathcal{I}_6 = \oint_C \oint_C \oint_C \oint_C B \cdot S_{21} S_{31} S_{32} T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1} \sum_{j_2} (\xi_1 \xi_2)^{j_1} \xi_1^{j_2} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Evaluating summation for j_2 by shrinking the contour for ξ_1 to obtain

$$\oint_C \oint_C \oint_C \oint_C B \cdot \frac{\xi_2(1 - \xi_1)}{\xi_1(\xi_2 - 1)} S_{31} S_{32} T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1} (\xi_1 \xi_2)^{j_1} \frac{\xi_1}{1 - \xi_1} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Expanding the contour back to the big one without a residue term, since after the cancellation of similar terms there are no singularities, so as the result we get

$$\oint_C \oint_C \oint_C \oint_C B \cdot S_{21} S_{31} S_{32} T_{41} T_{42} T_{43} \frac{1}{\xi_4 - 1} \sum_{j_1} (\xi_1 \xi_2)^{j_1} \frac{\xi_1}{1 - \xi_1} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Combining Terms 5 and 6, using the cancellation identity $\frac{\xi_2}{1 - \xi_2} + S_{21} \frac{\xi_1}{1 - \xi_1} = 0$, these two terms cancel entirely, and we obtain

$$\mathcal{I}_5 + \mathcal{I}_6 = 0.$$

Finally, we can now combine all terms and substitute back B at $\xi_2 = 1, \xi_3 = 1$ to calculate the probability for N=4 particle case, so we get

$$\boxed{\oint_C \oint_C e^{t\left(\frac{1}{\xi_1} + \frac{1}{\xi_4} - 2\right)} \xi_1^{x-y_1-1} \xi_4^{x-y_4-1} \times T_{41} \frac{1}{\xi_4 - 1} d\xi_1 d\xi_4} \quad (3.4)$$

Chapter 4

General N Particle System

Based on the results for $N = 3$ and $N = 4$, we now state and prove the general formula for arbitrary N . The pattern observed in both cases suggests that only the $(N - 1)!$ permutations with N in the first position contribute nonzero amplitudes, and that the coefficients are products of S -factors determined by the inversions of the remaining permutation.

4.1 General Amplitude

The conjecture of general amplitude formula is given by the following

$$[A_{N, \sigma(1), \dots, \sigma(N-1)}]_{21 \dots 11, 11 \dots 12} = \begin{cases} \left(\prod_{k=1}^{N-1} T_{Nk} \right) \cdot \prod_{\substack{i < j \\ i \text{ appears after } j \text{ in } \sigma}} S_{ji} & \text{if } \sigma(1) = N, \\ 0 & \text{otherwise.} \end{cases} \quad (4.1)$$

The coefficient of each $\sigma \in S_{N-1}$ is thus:

$$\text{coefficient of } \sigma = \prod_{\substack{i < j \\ i \text{ appears after } j \text{ in } \sigma}} S_{ji}. \quad (4.2)$$

We verify this formula for $N = 3$ and $N = 4$.

4.2 Verification via N=3 and N=4

4.2.1 Verification for N=3

Verification for $N = 3$. Here $\sigma \in S_2$ acts on $\{1, 2\}$ and the common prefactor is $T_{31}T_{32}$. We go through each σ :

- $\sigma = (1, 2)$: No pair is out of order. The product over inversions is empty, so the coefficient is 1. Thus

$$[A_{312}]_{211, 112} = T_{31}T_{32} \cdot 1 = T_{31}T_{32}.$$

- $\sigma = (2, 1)$: The pair $(1, 2)$ has 1 appearing after 2. The coefficient is S_{21} . Thus

$$[A_{321}]_{211, 112} = T_{31}T_{32} \cdot S_{21} = T_{31}T_{32}S_{21}.$$

Both match the values computed directly from the matrix products $\mathcal{T}_1\mathcal{T}_2$ and $\mathcal{T}_1\mathcal{T}_2\mathcal{T}_1$ in the $N = 3$ section.

4.2.2 Verification for N=4

Verification for $N = 4$. Here $\sigma \in S_3$ acts on $\{1, 2, 3\}$ and the common prefactor is $T_{41}T_{42}T_{43}$. We go through each $\sigma \in S_3$:

- $\sigma = (1, 2, 3)$: No pair is out of order. The product over inversions is empty, so the coefficient is 1. Thus

$$[A_{4123}]_{2111, 1112} = T_{41}T_{42}T_{43} \cdot 1 = T_{41}T_{42}T_{43}.$$

- $\sigma = (1, 3, 2)$: The pair $(2, 3)$ has 2 appearing after 3. The coefficient is S_{32} . Thus

$$[A_{4132}]_{2111, 1112} = T_{41}T_{42}T_{43} \cdot S_{32} = S_{32}T_{41}T_{42}T_{43}.$$

- $\sigma = (2, 1, 3)$: The pair $(1, 2)$ has 1 appearing after 2. The coefficient is S_{21} .

Thus

$$[A_{4213}]_{2111, 1112} = T_{41}T_{42}T_{43} \cdot S_{21} = S_{21}T_{41}T_{42}T_{43}.$$

- $\sigma = (2, 3, 1)$: We check all pairs:
 - Pair $(1, 2)$: 1 appears after 2. Contributes S_{21} .
 - Pair $(1, 3)$: 1 appears after 3. Contributes S_{31} .
 - Pair $(2, 3)$: 2 appears before 3. No contribution.

The coefficient is $S_{21}S_{31}$. Thus

$$[A_{4231}]_{2111, 1112} = T_{41}T_{42}T_{43} \cdot S_{21}S_{31} = S_{21}S_{31}T_{41}T_{42}T_{43}.$$

- $\sigma = (3, 1, 2)$: We check all pairs:
 - Pair $(1, 2)$: 1 appears before 2. No contribution.
 - Pair $(1, 3)$: 1 appears after 3. Contributes S_{31} .
 - Pair $(2, 3)$: 2 appears after 3. Contributes S_{32} .

The coefficient is $S_{31}S_{32}$. Thus

$$[A_{4312}]_{2111, 1112} = T_{41}T_{42}T_{43} \cdot S_{31}S_{32} = S_{31}S_{32}T_{41}T_{42}T_{43}.$$

- $\sigma = (3, 2, 1)$: We check all pairs:
 - Pair $(1, 2)$: 1 appears after 2. Contributes S_{21} .
 - Pair $(1, 3)$: 1 appears after 3. Contributes S_{31} .
 - Pair $(2, 3)$: 2 appears after 3. Contributes S_{32} .

The coefficient is $S_{21}S_{31}S_{32}$. Thus

$$[A_{4321}]_{2111, 1112} = T_{41}T_{42}T_{43} \cdot S_{21}S_{31}S_{32} = S_{21}S_{31}S_{32}T_{41}T_{42}T_{43}.$$

All six nonzero amplitudes match the values computed directly from the matrix products in the $N = 4$ section and listed in Appendix B. All remaining 18 permutations $\pi \in S_4$ that do not have 4 in the first position give $[A_\pi]_{2111,1112} = 0$, also consistent with the general formula since those permutations do not have N leading.

4.3 General N case

Theorem. For all $N \geq 2$,

$$\begin{aligned} & \sum_{i=1}^{\infty} \sum_{j_1=1}^{\infty} \cdots \sum_{j_{N-2}=1}^{\infty} \oint_C \cdots \oint_C e^{t\left(\frac{1}{\xi_1} + \cdots + \frac{1}{\xi_N} - N\right)} \sum_{\sigma \in S_{N-1}} [A_\sigma]_{21^{N-1}, 1^{N-1}2} \xi_N^{-i} \\ & \quad \times (\xi_{\sigma(1)} \cdots \xi_{\sigma(N-1)})^{j_1} (\xi_{\sigma(2)} \cdots \xi_{\sigma(N-1)})^{j_2} \cdots (\xi_{\sigma(N-1)})^{j_{N-2}} d\xi_1 \cdots d\xi_N \\ & \quad = \oint_C \oint_C e^{t\left(\frac{1}{\xi_1} + \frac{1}{\xi_N} - 1\right)} \cdot T_{N1} \cdot \frac{\xi_1^{x-y_1-1} \xi_N^{x-y_N-1}}{\xi_N - 1} d\xi_1 d\xi_N. \quad (4.3) \end{aligned}$$

Proof. We proceed by strong induction on N . Let $L(N)$ denote the left-hand side and $R(N)$ the right-hand side. Throughout the proof we write

$$B = e^{t\left(\frac{1}{\xi_1} + \cdots + \frac{1}{\xi_N} - N\right)} \xi_1^{x-y_1-1} \cdots \xi_N^{x-y_N-1}$$

Base case ($N = 2$). There are no gap variables j_k . The only nonzero amplitude is $[A_{21}]_{12,21} = T_{21}$, so

$$L(2) = \oint_C \oint_C B \cdot T_{21} \cdot \sum_{i=1}^{\infty} \xi_2^{-i} d\xi_1 d\xi_2 = \oint_C \oint_C B \cdot T_{21} \cdot \frac{1}{\xi_2 - 1} d\xi_1 d\xi_2 = R(2),$$

where we used $\sum_{i=1}^{\infty} \xi^{-i} = \frac{1}{\xi-1}$ on the contour $|\xi| > 1$.

Inductive hypothesis. Assume the theorem holds for all systems of size $2, 3, \dots, N-1$, i.e. $L(N-1) = R(N-1)$.

Inductive step The left-hand side is:

$$L(N) = \oint \cdots \oint B \cdot \sum_{\tau \in S_{N-1}} \left[\left(\prod_{k=1}^{N-1} T_{Nk} \right) \prod_{\substack{i < j \\ i \text{ after } j \text{ in } \tau}} S_{ji} \right] \\ \times \xi_N^{-i} \cdot (\xi_{\tau(1)} \cdots \xi_{\tau(N-1)})^{j_1} \cdots (\xi_{\tau(N-1)})^{j_{N-2}} d\xi_1 \cdots d\xi_N. \quad (4.4)$$

We start by summing over $j_{N-2} \geq 1$ on the small contour gives

$$\sum_{j_{N-2}=1}^{\infty} \xi_{\tau(N-1)}^{j_{N-2}} = \frac{\xi_{\tau(N-1)}}{1 - \xi_{\tau(N-1)}}, \quad |\xi_{\tau(N-1)}| < 1.$$

Expanding from the small contour back to the large contour together with residue term

$$\oint_{\text{small}} f(\xi_a) d\xi_a = \oint_{\text{large}} f(\xi_a) d\xi_a - 2\pi i \text{Res}_{\xi_a=1} f(\xi_a).$$

Each term pairs up according to the permutation. They pair up by swapping the last two positions: $(\dots, a, b) \leftrightarrow (\dots, b, a)$. Each pair after expanding to the big contour can be summed up as

$$C \cdot \left(\frac{\xi_a}{1 - \xi_a} + S_{ba} \cdot \frac{\xi_b}{1 - \xi_b} \right) = C \cdot 0 = 0,$$

since

$$\frac{\xi_a}{1 - \xi_a} + S_{ba} \cdot \frac{\xi_b}{1 - \xi_b} = \frac{\xi_a}{1 - \xi_a} - \frac{\xi_b(\xi_a - 1)}{\xi_a(\xi_b - 1)} \cdot \frac{\xi_b}{1 - \xi_b} = 0.$$

So all summation will vanish completely.

The next step is to deal with residue terms. Each time the residue at $\xi_a = 1$ eliminates ξ_a variable from the N -fold integral and reduces it to an $(N - 1)$ -fold integral. We get the integral form exactly like in equation (4.4)

$$B^{(N-1)} \cdot \left(\prod_{k=1}^{N-2} T_{N-1,k} \right) \prod_{\substack{i < j \\ i \text{ after } j \text{ in } \tau'}} S_{ji} \cdot \xi_N^{-i} \cdot (\xi_{\tau'(1)} \cdots \xi_{\tau'(N-2)})^{j_1} \cdots (\xi_{\tau'(N-2)})^{j_{N-3}},$$

where $\tau' \in S_{N-2}$ is the permutation induced by τ on the remaining indices, and

$$B^{(N-1)} = e^{t\left(\frac{1}{\xi_1} + \dots + \frac{1}{\xi_{N-1}} - (N-1)\right)} \xi_1^{x-y_1-1} \dots \xi_{N-1}^{x-y_{N-1}-1}.$$

This is precisely the structure of $L(N-1)$.

By the inductive hypothesis $L(N-1) = R(N-1)$, so the residue terms give:

$$\sum_{i, j_1, \dots, j_{N-3}} (\text{residue terms}) = \oint_C \oint_C e^{t\left(\frac{1}{\xi_1} + \frac{1}{\xi_{N-1}} - 1\right)} T_{N-1,1} \frac{\xi_1^{x-y_1-1} \xi_{N-1}^{x-y_{N-1}-1}}{\xi_{N-1} - 1} d\xi_1 d\xi_{N-1}.$$

We perform the same procedure each time to each of the gap variables $j_{N-3}, j_{N-4}, \dots, j_1$ reducing the problem one dimension at a time. After all $N-2$ reductions, and after evaluating the sum over $-i$, we get the final result

$$L(N) = \oint_C \oint_C e^{t\left(\frac{1}{\xi_1} + \frac{1}{\xi_N} - 1\right)} \cdot T_{N1} \cdot \frac{\xi_1^{x-y_1-1} \xi_N^{x-y_N-1}}{\xi_N - 1} d\xi_1 d\xi_N = R(N).$$

The result shows that regardless of the number of particles N , the probability of the left-most particle 1 moving to the right of particle of species 2 reduces to a two-dimensional contour integral depending only on ξ_1 and ξ_N .

Appendix A

Full Amplitude Table for $N=3$, $N=4$

Table A.1: All values of $[\mathbf{A}_\pi]_{211,112}$ for $\pi \in S_3$.

π	$[\mathbf{A}_\pi]_{211,112}$
123	0
132	0
213	0
231	0
312	$T_{31}T_{32}$
321	$T_{31}T_{32}S_{21}$

Table A.2: All values of $[\mathbf{A}_\sigma]_{2111,1112}$ for $\sigma \in S_4$.

σ	$[\mathbf{A}_\sigma]_{1112,2111}$
1234	0
1243	0
1324	0
1342	0
1423	0
1432	0
2134	0
2143	0
2314	0
2341	0
2413	0
2431	0
3124	0
3142	0
3214	0
3241	0
3412	0
3421	0
4123	$T_{41}T_{42}T_{43}$
4132	$S_{32}T_{41}T_{42}T_{43}$
4213	$S_{21}T_{41}T_{42}T_{43}$
4231	$S_{21}S_{31}T_{41}T_{42}T_{43}$
4312	$S_{31}S_{32}T_{41}T_{42}T_{43}$
4321	$S_{21}S_{31}S_{32}T_{41}T_{42}T_{43}$

Appendix B

S-matrix Products for the N=4 System

The rows and columns of all matrices below are ordered as 1112, 1121, 1211, 2111.

Pairwise products:

$$\mathcal{T}_1(3, 1)\mathcal{T}_2(3, 2) = \begin{pmatrix} S_{31}S_{32} & 0 & 0 & 0 \\ 0 & -S_{31} & 0 & 0 \\ 0 & -T_{32} & -S_{32} & 0 \\ 0 & T_{31}T_{32} & T_{31}S_{32} & S_{31}S_{32} \end{pmatrix}.$$

Triple products:

$$\mathcal{T}_1(4, 1)\mathcal{T}_2(4, 2)\mathcal{T}_3(4, 3) = \begin{pmatrix} -S_{41}S_{42} & 0 & 0 & 0 \\ -S_{41}T_{43} & -S_{41}S_{43} & 0 & 0 \\ -T_{42}T_{43} & -T_{42}S_{43} & -S_{42}S_{43} & 0 \\ T_{41}T_{42}T_{43} & T_{41}T_{42}S_{43} & T_{41}S_{42}S_{43} & S_{41}S_{42}S_{43} \end{pmatrix}.$$

Quadruple products:

$$\mathcal{T}_3(3, 2)\mathcal{T}_1(4, 1)\mathcal{T}_2(4, 2)\mathcal{T}_3(4, 3) = \begin{pmatrix} S_{41}S_{42} & 0 & 0 & 0 \\ -T_{32}S_{41}S_{42} - S_{32}S_{41}T_{43} & -S_{32}S_{41}S_{43} & 0 & 0 \\ -S_{32}T_{42}T_{43} & -S_{32}T_{42}S_{43} & -S_{32}S_{42}S_{43} & 0 \\ S_{32}T_{41}T_{42}T_{43} & S_{32}T_{41}T_{42}S_{43} & S_{32}T_{41}S_{42}S_{43} & S_{32}S_{41}S_{42}S_{43} \end{pmatrix},$$

$$\mathcal{T}_2(2, 1)\mathcal{T}_1(4, 1)\mathcal{T}_2(4, 2)\mathcal{T}_3(4, 3) = \begin{pmatrix} -S_{21}S_{41}S_{42} & 0 & 0 & 0 \\ S_{41}T_{43} & S_{41}S_{43} & 0 & 0 \\ -T_{43}(T_{21}S_{41} + S_{21}T_{42}) & -S_{43}(T_{21}S_{41} + S_{21}T_{42}) & -S_{21}S_{42}S_{43} & 0 \\ S_{21}T_{41}T_{42}T_{43} & S_{21}T_{41}T_{42}S_{43} & S_{21}T_{41}S_{42}S_{43} & S_{21}S_{41}S_{42}S_{43} \end{pmatrix}.$$

Bibliography

- [1] M. Alimohammadi, V. Karimipour, and M. Khorrami. Exact solution of a one-parameter family of asymmetric exclusion processes. *Physical Review E*, 57:6370–6376, 1998.
- [2] Eunghyun Lee. Integrability of the multi-species asymmetric simple exclusion processes with long-range jumps on $\mathbb{Z}$. *Symmetry*, 16:1164, 2024.
- [3] Eunghyun Lee and T. Raimbekov. Simplified forms of the transition probabilities of the two-species ASEP with some initial orders of particles. *SIGMA*, 18:008, 2022.
- [4] Eunghyun Lee and Temirlan Raimbekov. Simplified forms of the transition probabilities of the two-species ASEP with some initial orders of particles. *Symmetry, Integrability and Geometry: Methods and Applications*, 18:008, 2022. 24 pages.
- [5] T. Sasamoto and M. Wadati. Exact results for one-dimensional totally asymmetric diffusion models. *Journal of Physics A: Mathematical and General*, 31:605, 1998.
- [6] G. M. Schütz, R. Ramaswamy, and M. Barma. Pairwise balance and invariant measures for generalized exclusion processes. *Journal of Physics A: Mathematical and General*, 29:837–843, 1996.