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Math 499 Capstone Project

Analytic Solutions for a Nonlinear Transport Equation

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April, 2019

Abstract

We prove that the Cauchy problem for a transport equation with algebraic nonlinearity of degree p with initial data in Gevrey spaces is locally well-posed. In particular, we show that the analyticity of solutions persists for a short time and we derive a sufficient condition for solutions to be analytic for all times.

Keywords: Transport equation, well-posedness, analytic spaces, Gevrey spaces
MSC 2010: 35F25

1 Introduction

Solutions to partial differential equations often behave in unexpected ways. An interesting example of this is the Korteweg-de Vries equation $u_t + u_{xxx} + uu_x = 0$, which has solutions which are analytic, even for initial conditions which are not differentiable. Observations such as this have led many mathematicians to consider the following question: suppose we have a partial differential equation, and an initial condition $u(x, 0) = f(x)$ which has an analytic continuation on the subset of the complex plane

$$S_\sigma = \big\{x + iy \in \mathbb{C} : \, |y| < \sigma\big\}.$$

What will happen to the analyticity of the solution u as time progresses? Many authors have considered this question for many different equations [1, 2, 3]. In almost all cases, the authors obtained estimates on how quickly the width σ can shrink over time. In this project, we will consider this question for the initial value problem

$$\begin{cases} u_t - u_x = u^p, \\ u(x,0) = f(x). \end{cases} \tag{1.1}$$

where $p > 1$ is an integer, the unknown $u(x, t)$, and the datum $f(x)$ are real-valued. In particular, we will try to obtain a simple condition on the solution to ensure that the width σ does not decay at all over time, as was done in [4, 5].

Next, we introduce some notations and definitions. We use the Lebesgue norms

$$\|f\|_{L^q(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^q dx\right)^{1/q},$$

for $1 \le q < \infty$, with the usual convention

$$\|f\|_{L^\infty(\mathbb{R})} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)|.$$

We define the mixed Lebesgue norms $L_t^q L_x^r(I \times \mathbb{R})$ for any time interval I as the space of all functions $u(x, t)$ with norm

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R})} = \left(\int_I \|u(t)\|^q_{L_x^r(\mathbb{R})} dt\right)^{1/q} = \left(\int_I \left(\int_{\mathbb{R}} |u(x,t)|^r dx\right)^{q/r} dt\right)^{1/q}$$

with the usual modifications when $q = \infty$ or $r = \infty$. Then the vector space $C(I)$ can be defined by the norm

$$\|u\|_{C(I)} = \sup_{x\in I} |u(x)|.$$

Suppose we have a function $u(x,t)$. Then the combination of $L^\infty(\mathbb{R})$ and $H^s(\mathbb{R})$ spaces is $L_t^\infty H_x^s(\mathbb{R}\times\mathbb{R})$ with the norm

$$\|u\|_{L_t^\infty H_x^s(\mathbb{R}\times\mathbb{R})} = \sup_{t\in I} \|u(\cdot,t)\|_{H_x^s(\mathbb{R})}.$$

The Fourier transform of f, denoted by $\hat{f}$, is given by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx.$$

Here, $i=\sqrt{-1}$, and ξ is the frequency variable. The inverse Fourier transform of g, denoted by $\check{g}$, is given by

$$(\mathcal{F}^{-1}g)(x) = \check{g}(x) = \int_{-\infty}^{\infty} g(\xi) e^{ix\xi}\, d\xi.$$

The Sobolev space $H^s(\mathbb{R})$ for $s\in\mathbb{R}$ is defined by the norm

$$\|f\|_{H^s(\mathbb{R})} = \|\langle\xi\rangle^s \hat{f}(\xi)\|_{L_\xi^2(\mathbb{R})},$$

where $\langle\xi\rangle = (1+\xi^2)^{1/2}$.

Definition 1.1. Let $u(x)$ be a function with Fourier transform $\hat{u}(\xi)$. Let $p(\xi)$ be some locally integrable function of ξ. Then the expression

$$p(\nabla/i)u = \mathcal{F}^{-1}(p(\xi)\hat{u}(\xi))$$

is called a Fourier multiplier. The function $p(\xi)$ is called the symbol of the multiplier.

We define the operator $p(\nabla/i) = |D_x| = |d/dx|$ as the Fourier multiplier with symbol $|\xi|$.

In the present work, we will consider the problem (1.1) with initial data in Gevrey spaces $G^{\sigma,s} = G^{\sigma,s}(\mathbb{R})$, with norm defined by

$$\|f\|_{G^{\sigma,s}(\mathbb{R})} = \|e^{\sigma|D_x|}\langle D_x\rangle^s f\|_{L_x^2(\mathbb{R})} = \|e^{\sigma|\xi|}\langle\xi\rangle^s \hat{f}(\xi)\|_{L_\xi^2(\mathbb{R})},$$

where $D_x = -i\partial_x$, ξ is the Fourier variable corresponding to x. These spaces have the following property

Theorem 1.2 (Paley-Wiener theorem (see [6], p. 209))**.** *Let $\sigma>0$ and $s\in\mathbb{R}$. Then, the following statements are equivalent:*
(i) $f\in G^{\sigma,s}(\mathbb{R})$.
(ii) f is the restriction to the real line of a function F which is holomorphic in the strip

$$S_\sigma = \{x+iy : x,y\in\mathbb{R},\ |y|<\sigma\}$$

and satisfies

$$\sup_{|y|<\sigma} \|F(x+iy)\|_{H_x^s(\mathbb{R})} < \infty.$$

A function $f:\mathbb{R}\longrightarrow\mathbb{C}$ is said to be rapidly decreasing if we have

$$\|\langle x\rangle^N f(x)\|_{L^\infty(\mathbb{R})} < \infty$$

for all $N\ge 0$, where $\langle x\rangle = (1+x^2)^{1/2}$. We say that a function is a Schwartz function if it is smooth and all of its derivatives are rapidly decreasing. We use $S(\mathbb{R})$ to denote the space of all Schwartz functions. The space $S(\mathbb{R})$ has a dual $S'(\mathbb{R})$, the space of tempered distributions.

Definition 1.3. A tempered distribution on $\mathbb{R}$ is a continuous linear functional on $S(\mathbb{R})$.

Definition 1.4. Let $q \in [1, \infty]$. The Fourier Lebesgue space $\mathcal{F}L^q(\mathbb{R})$ is the inverse Fourier image of $L^q(\mathbb{R})$, i.e., $\mathcal{F}L^q(\mathbb{R})$ consists of all $f \in S'(\mathbb{R})$ such that

$$\|f\|_{\mathcal{F}L^q(\mathbb{R})} = \|\hat{f}(\xi)\|_{L^q(\mathbb{R})}$$

is finite ([7], p. 378).

Thus, any function f in $G^{\sigma,s}(\mathbb{R})$ is analytic in a strip whose width is at least σ. We will consider the following question: will the solution of the problem (1.1) also belong to the space $G^{\sigma,s}(\mathbb{R})$ at a time $t > 0$? Similar issues were considered in [4, 5, 8].

Our main results in this paper are the following theorems

Theorem 1.5. *Let $\sigma > 0$ and $s > 1/2$. Then the Cauchy problem* (1.1) *is locally well-posed for initial data f in $G^{\sigma,s}(\mathbb{R})$. That is, there exists $T > 0$ such that there exists a unique solution $u(x,t)$ of problem* (1.1) *in $C([0,T]; G^{\sigma,s}(\mathbb{R}))$. Moreover, the solution depends continuously on the initial data. Thus, a solution which is analytic in S_σ at $t = 0$ will continue to be analytic in S_σ at least in $[0,T]$.*

This first main result states that analycity persists, at least for a short time. The proof of Theorem 1.5 is contained in Section 3.

Theorem 1.6. *Let u be a solution of the Cauchy problem* (1.1) *for initial data $f \in G^{\sigma,s}(\mathbb{R})$, $\sigma > 0$ and $s > 1/2$. Define*

$$T^* = \sup\{T > 0 : \ \|u(\cdot,t)\|_{G^{\sigma,s}} < \infty \ \text{for} \ t \in [0,T]\}.$$

Then either $T^* = \infty$ or there exists at least one k such that $1 \le k \le p-1$ and

$$e^{\sigma|D_x|}u^k \notin L_t^\infty \mathcal{F}L_x^1\big([0,T^*] \times \mathbb{R}\big).$$

This second main result gives us a sufficient condition to determine if analyticity persists for all time. The proof is contained in Section 4.

2 Preliminaries

We will need the following inequality later (see [9, Lemma A8, p. 338])

Lemma 2.1. *If f, $g \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $s \ge 0$, then*

$$\|fg\|_{H^s(\mathbb{R})} \le C_s(\|f\|_{H^s(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})}\|g\|_{H^s(\mathbb{R})}), \tag{2.1}$$

if f, $g \in H^s(\mathbb{R})$ and $s > 1/2$, then

$$\|fg\|_{H^s(\mathbb{R})} \le C\|f\|_{H^s(\mathbb{R})}\|g\|_{H^s(\mathbb{R})}. \tag{2.2}$$

Next we will prove the following product estimates

Lemma 2.2. *If $f, g \in G^{\sigma,s}(\mathbb{R})$ and $s > 1/2$, then*

$$\|fg\|_{G^{\sigma,s}} \le C\|f\|_{G^{\sigma,s}}\|g\|_{G^{\sigma,s}}, \tag{2.3}$$

if $e^{\sigma|D|}f, e^{\sigma|D|}g \in H^s(\mathbb{R}) \cap \mathcal{F}L^1(\mathbb{R})$, and $s \ge 0$ then

$$\|fg\|_{G^{\sigma,s}} \le C(\|f\|_{G^{\sigma,s}}\|e^{\sigma|D|}g\|_{\mathcal{F}L^1(\mathbb{R})} + \|e^{\sigma|D|}f\|_{\mathcal{F}L^1(\mathbb{R})}\|g\|_{G^{\sigma,s}}), \tag{2.4}$$

Proof. Indeed, if $s > 1/2$, then by (2.2) and Plancherel's theorem [6, p.156], we have

$$\begin{aligned}
\|fg\|_{G^{\sigma,s}} &= \|e^{\sigma|D|}\langle D\rangle^s fg\|_{L^2(\mathbb{R})} = \|e^{\sigma|\xi|}\langle\xi\rangle^s \widehat{(fg)}(\xi)\|_{L^2(\mathbb{R})} \\
&= \Big(\int_{\mathbb{R}} e^{2\sigma|\xi|}\langle\xi\rangle^{2s}\Big|\int_{\mathbb{R}} \hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta\Big|^2 d\xi\Big)^{1/2} \\
&\le \Big(\int_{\mathbb{R}} \langle\xi\rangle^{2s}\Big[\int_{\mathbb{R}} \big(e^{\sigma|\xi-\eta|}|\hat{f}(\xi-\eta)|\big)\big(e^{\sigma|\eta|}|\hat{g}(\eta)|\big)\,d\eta\Big]^2 d\xi\Big)^{1/2} \\
&= \big\|\langle D\rangle^s\big(e^{\sigma|D|}\mathcal{F}^{-1}(|\hat{f}(\xi)|)\big)\big(e^{\sigma|D|}\mathcal{F}^{-1}(|\hat{g}(\xi)|)\big)\big\|_{L^2(\mathbb{R})} \\
&= \big\|\big(e^{\sigma|D|}\mathcal{F}^{-1}(|\hat{f}(\xi)|)\big)\big(e^{\sigma|D|}\mathcal{F}^{-1}(|\hat{g}(\xi)|)\big)\big\|_{H^s(\mathbb{R})} \\
&\le C\big\|e^{\sigma|D|}\mathcal{F}^{-1}(|\hat{f}(\xi)|)\big\|_{H^s(\mathbb{R})}\big\|e^{\sigma|D|}\mathcal{F}^{-1}(|\hat{g}(\xi)|)\big\|_{H^s(\mathbb{R})} \\
&= C\Big(\int_{\mathbb{R}} \langle\xi\rangle^{2s}e^{2\sigma|\xi|}|\hat{f}(\xi)|^2\,d\xi\Big)^{1/2}\Big(\int_{\mathbb{R}} \langle\xi\rangle^{2s}e^{2\sigma|\xi|}|\hat{g}(\xi)|^2\,d\xi\Big)^{1/2} \\
&= C\|f\|_{G^{\sigma,s}}\|g\|_{G^{\sigma,s}},
\end{aligned}$$

if $e^{\sigma|D|}f, e^{\sigma|D|}g \in H^s(\mathbb{R}) \cap \mathcal{F}L^1(\mathbb{R})$, and $s \ge 0$ then by (2.1)

$$\begin{aligned}
\|fg\|_{G^{\sigma,s}} &\le \big\|\big(e^{\sigma|D|}\mathcal{F}^{-1}(|\hat{f}(\xi)|)\big)\big(e^{\sigma|D|}\mathcal{F}^{-1}(|\hat{g}(\xi)|)\big)\big\|_{H^s(\mathbb{R})} \\
&\le C\Big(\big\|e^{\sigma|D|}\mathcal{F}^{-1}(|\hat{f}(\xi)|)\big\|_{H^s(\mathbb{R})}\big\|e^{\sigma|D|}\mathcal{F}^{-1}(|\hat{g}(\xi)|)\big\|_{L^\infty(\mathbb{R})} \\
&\quad+\big\|e^{\sigma|D|}\mathcal{F}^{-1}(|\hat{f}(\xi)|)\big\|_{L^\infty(\mathbb{R})}\big\|e^{\sigma|D|}\mathcal{F}^{-1}(|\hat{g}(\xi)|)\big\|_{H^s(\mathbb{R})}\Big) \\
&\le C\Big(\|f\|_{G^{\sigma,s}}\int_{\mathbb{R}} e^{\sigma|\xi|}|\hat{g}(\xi)|\,d\xi + \int_{\mathbb{R}} e^{\sigma|\xi|}|\hat{f}(\xi)|\,d\xi\,\|g\|_{G^{\sigma,s}}\Big) \\
&= C\big(\|f\|_{G^{\sigma,s}}\|e^{\sigma|D|}g\|_{\mathcal{F}L^1(\mathbb{R})} + \|e^{\sigma|D|}f\|_{\mathcal{F}L^1(\mathbb{R})}\|g\|_{G^{\sigma,s}}\big).
\end{aligned}$$

in the last inequality we used the following

$$\big\|e^{\sigma|D|}\mathcal{F}^{-1}(|\hat{f}(\xi)|)\big\|_{L^\infty(\mathbb{R})} \le \|e^{\sigma|D|}f\|_{\mathcal{F}L^1(\mathbb{R})}. \tag{2.5}$$

Indeed,

$$\begin{aligned}
|e^{\sigma|D|}\mathcal{F}^{-1}(|\hat{f}(\xi)|)| &= |\mathcal{F}^{-1}(e^{\sigma|\xi|}|\hat{f}(\xi)|)| \\
&= \Big|\int_{\mathbb{R}} e^{i\xi x}e^{\sigma|\xi|}|\hat{f}(\xi)|d\xi\Big| \le \int_{\mathbb{R}} e^{\sigma|\xi|}|\hat{f}(\xi)|d\xi \\
&= \|e^{\sigma|\xi|}\hat{f}(\xi)\|_{L^1(\mathbb{R})} = \|e^{\sigma|D|}f\|_{\mathcal{F}L^1(\mathbb{R})}.
\end{aligned}$$

Hence we have (2.5). Thus, Lemma 2.2 is proved. □

3 Proof of Theorem 1.5

For the proof of Theorem 1.5, we will use Picard iteration. Define a sequence of functions $\{u_n\}_{n=0}^{\infty}$ according to

$$\partial_t u_0 - \partial_x u_0 = 0, \;\; u_0(x,0) = f(x), \;\; f \in G^{\sigma,s}(\mathbb{R}),$$

and

$$\partial_t u_n - \partial_x u_n = u_{n-1}^p, \;\; u_n(x,0) = f(x),\, n \ge 1.$$

Using Duhamel's principle [9, p. 67], it is easy to see that

$$\begin{aligned} u_0(x,t) &= f(x+t), \\ u_n(x,t) &= f(x+t) + \int_0^t u_{n-1}^p(x+(t-\tau),\tau)\,d\tau. \end{aligned} \tag{3.1}$$

We must first show that $u_n \in C([0,T];\, G^{\sigma,s})$ for all $n \geq 0$. We will give a proof by induction on n. Indeed, first observe that

$$\begin{aligned} \|u_0\|_{C([0,T];\,G^{\sigma,s})} &= \sup_{t\in[0,T]} \|u_0(\cdot,t)\|_{G^{\sigma,s}} \\ &= \sup_{t\in[0,T]} \left(\int_{\mathbb{R}} e^{2\sigma|\xi|} \langle\xi\rangle^{2s} |\widehat{f(x+t)}|^2\, d\xi \right)^{1/2} \\ &= \sup_{t\in[0,T]} \left(\int_{\mathbb{R}} e^{2\sigma|\xi|} \langle\xi\rangle^{2s} |e^{i\xi t}\hat{f}(\xi)|^2\, d\xi \right)^{1/2} \\ &= \|f\|_{G^{\sigma,s}} \leq 2\|f\|_{G^{\sigma,s}}. \end{aligned}$$

Now assume that $u_{n-1} \in C([0,T];\, G^{\sigma,s})$, with

$$\|u_{n-1}\|_{C([0,T];\,G^{\sigma,s})} \leq 2\|f\|_{G^{\sigma,s}}.$$

We will show that the same is true for u_n. By (3.1), we have

$$\|u_n\|_{G^{\sigma,s}} \leq \|f\|_{G^{\sigma,s}} + \|\int_0^t u_{n-1}^p(x+(t-\tau),\tau)\,d\tau\|_{G^{\sigma,s}}.$$

By the formula (3.1), Minkowski inequality [6, p. 273], and using inequality (2.3)

$$\begin{aligned} \Big\| \int_0^t u_{n-1}^p(x+(t-\tau),\tau)\,d\tau \Big\|_{G^{\sigma,s}} &\leq \int_0^t \|u_{n-1}^p(x+(t-\tau),\tau)\|_{G^{\sigma,s}}\,d\tau \\ = \int_0^t \|u_{n-1}^p(\cdot,\tau)\|_{G^{\sigma,s}}\,d\tau &\lesssim \int_0^t \|u_{n-1}^{p-1}(\cdot,\tau)\|_{G^{\sigma,s}} \|u_{n-1}(\cdot,\tau)\|_{G^{\sigma,s}}\,d\tau \\ \lesssim \int_0^t \|u_{n-1}(\cdot,\tau)\|_{G^{\sigma,s}}^p\,d\tau &\leq C\int_0^t \|u_{n-1}\|_{C([0,T];\,G^{\sigma,s})}^p\,d\tau \leq 2CT\|f\|_{G^{\sigma,s}}^p. \end{aligned}$$

Then we have

$$\|u_n\|_{C([0,T];\,G^{\sigma,s})} \leq (1+2CT\|f\|_{G^{\sigma,s}}^{p-1})\|f\|_{G^{\sigma,s}} \leq 2\|f\|_{G^{\sigma,s}},$$

if

$$T \leq \frac{1}{2C\|f\|_{G^{\sigma,s}}^{p-1}}.$$

We will now use this bound on the norms of the u_n to show convergence. Note that it suffices to show that

$$\|u_n - u_{n-1}\|_{C([0,T];\,G^{\sigma,s})} < \|u_{n-1} - u_{n-2}\|_{C([0,T];\,G^{\sigma,s})},$$

for all n. Using the inequality (2.3) it follows that

$$\begin{aligned}
\|u_n - u_{n-1}\|_{G^{\sigma,s}} &= \Big\| \int_0^t \big(u^p_{n-1}(x+(t-\tau),\tau) - u^p_{n-2}(x+(t-\tau),\tau)\big)\, d\tau \Big\|_{G^{\sigma,s}} \\
&\le \int_0^t \|u^p_{n-1}(\cdot,\tau) - u^p_{n-2}(\cdot,\tau)\|_{G^{\sigma,s}}\, d\tau \\
&\lesssim \int_0^t \|u^{p-1}_{n-1}(\cdot,\tau) + u^{p-2}_{n-1}(\cdot,\tau)u_{n-2}(\cdot,\tau) + u^{p-3}_{n-1}(\cdot,\tau)u^2_{n-2}(\cdot,\tau) + \dots \\
&\qquad + u^2_{n-1}(\cdot,\tau)u^{p-3}_{n-2}(\cdot,\tau) + u_{n-1}(\cdot,\tau)u^{p-2}_{n-2}(\cdot,\tau) \\
&\qquad + u^{p-1}_{n-2}(\cdot,\tau)\|_{G^{\sigma,s}} \|u_{n-1}(\cdot,\tau) - u_{n-2}(\cdot,\tau)\|_{G^{\sigma,s}}\, d\tau \\
&\lesssim \int_0^t \Big(\|u_{n-1}(\cdot,\tau)\|^{p-1}_{G^{\sigma,s}} + \|u_{n-1}(\cdot,\tau)\|^{p-2}_{G^{\sigma,s}} \|u_{n-2}(\cdot,\tau)\|_{G^{\sigma,s}} + \dots \\
&\qquad + \|u_{n-1}(\cdot,\tau)\|_{G^{\sigma,s}} \|u_{n-2}(\cdot,\tau)\|^{p-2}_{G^{\sigma,s}} \\
&\qquad + \|u_{n-2}(\cdot,\tau)\|^{p-1}_{G^{\sigma,s}} \Big) \|u_{n-1}(\cdot,\tau) - u_{n-2}(\cdot,\tau)\|_{G^{\sigma,s}}\, d\tau \\
&\le Cp2^{p-1}\|f\|^{p-1}_{G^{\sigma,s}} \int_0^t \|u_{n-1}(\cdot,\tau) - u_{n-2}(\cdot,\tau)\|_{G^{\sigma,s}}\, d\tau \\
&\le Cp2^{p-1}\|f\|^{p-1}_{G^{\sigma,s}} \int_0^t \|u_{n-1} - u_{n-2}\|_{C([0,T];\, G^{\sigma,s})}\, d\tau \\
&\le Cp2^{p-1}\|f\|^{p-1}_{G^{\sigma,s}} \|u_{n-1} - u_{n-2}\|_{C([0,T];\, G^{\sigma,s})} < \|u_{n-1} - u_{n-2}\|_{C([0,T];\, G^{\sigma,s})},
\end{aligned}$$

as

$$T < \frac{1}{Cp2^{p-1}\|f\|^{p-1}_{G^{\sigma,s}}}.$$

Thus, it was proved that the sequence $\{u_n\}_{n=0}^{\infty}$ as the Cauchy sequence in Banach space $C([0,T];\, G^{\sigma,s})$ converges to $u(x,t)$ in $C([0,T];\, G^{\sigma,s})$. By passing to the limit as $n \longrightarrow \infty$ in the equations (3.1), we see that $u(x,t)$ is the solution of the problem (1.1).

We will prove that the map $f \longrightarrow u$ is continuous as a mapping from $G^{\sigma,s}(\mathbb{R})$ to $C([0,T]; G^{\sigma,s})$. Suppose that for two initial conditions $f, g \in G^{\sigma,s}$, we have corresponding solutions u and v, respectively. Let us estimate the difference between the solutions u and v in the space $C([0,T];\, G^{\sigma,s})$.

$$\begin{aligned}
&\|u - v\|_{G^{\sigma,s}} \\
&\quad \le \|f(x+t) - g(x+t)\|_{G^{\sigma,s}} + \int_0^t \|u^p(x+(t-\tau),\tau) - v^p(x+(t-\tau),\tau)\|_{G^{\sigma,s}}\, d\tau \\
&\quad = \|f - g\|_{G^{\sigma,s}} + \int_0^t \|u^p(\cdot,\tau) - v^p(\cdot,\tau)\|_{G^{\sigma,s}}\, d\tau \\
&\quad \le \|f - g\|_{G^{\sigma,s}} + C\int_0^t \|u^{p-1}(\cdot,\tau) + u^{p-2}(\cdot,\tau)v(\cdot,\tau) + \dots \\
&\quad + u(\cdot,\tau)v^{p-2}(\cdot,\tau) + v^{p-1}(\cdot,\tau)\|_{C([0,T];\, G^{\sigma,s})} \|u(\cdot,\tau) - v(\cdot,\tau)\|_{C([0,T];\, G^{\sigma,s})}\, d\tau
\end{aligned}$$

$$\leq \|f-g\|_{G^{\sigma,s}} + C\int_0^t \Big(\|u(\cdot,\tau)\|_{G^{\sigma,s}}^{p-1} + \|u(\cdot,\tau)\|_{G^{\sigma,s}}^{p-2}\|v(\cdot,\tau)\|_{G^{\sigma,s}} + \dots$$

$$+\|u(\cdot,\tau)\|_{G^{\sigma,s}}\|v(\cdot,\tau)\|_{G^{\sigma,s}}^{p-2} + \|v(\cdot,\tau)\|_{G^{\sigma,s}}^{p-1}\Big)\|u(\cdot,\tau)-v(\cdot,\tau)\|_{C([0,T];\,G^{\sigma,s})}\,d\tau$$

$$\leq \|f-g\|_{G^{\sigma,s}} + 2^{p-1}CT\big(\|f\|_{G^{\sigma,s}}^{p-1} + \|f\|_{G^{\sigma,s}}^{p-2}\|g\|_{G^{\sigma,s}} + \dots$$

$$+\|f\|_{G^{\sigma,s}}\|g\|_{G^{\sigma,s}}^{p-2} + \|g\|_{G^{\sigma,s}}^{p-1}\big)\|u-v\|_{C([0,T];\,G^{\sigma,s})}.$$

Then

$$\|u-v\|_{C([0,T];\,G^{\sigma,s})} \leq \|f-g\|_{G^{\sigma,s}} + 2^{p-1}CT\big(\|f\|_{G^{\sigma,s}}^{p-1} + \|f\|_{G^{\sigma,s}}^{p-2}\|g\|_{G^{\sigma,s}} + \dots$$

$$+\|f\|_{G^{\sigma,s}}\|g\|_{G^{\sigma,s}}^{p-2} + \|g\|_{G^{\sigma,s}}^{p-1}\big)\|u-v\|_{C([0,T];\,G^{\sigma,s})}.$$

If

$$T < \frac{1}{2^{p-1}C\big(\|f\|_{G^{\sigma,s}}^{p-1} + \|f\|_{G^{\sigma,s}}^{p-2}\|g\|_{G^{\sigma,s}} + \dots + \|f\|_{G^{\sigma,s}}\|g\|_{G^{\sigma,s}}^{p-2} + \|g\|_{G^{\sigma,s}}^{p-1}\big)},$$

then

$$\|u-v\|_{C([0,T];\,G^{\sigma,s})}$$

$$\leq \frac{1}{1-2^{p-1}CT\big(\|f\|_{G^{\sigma,s}}^{p-1} + \|f\|_{G^{\sigma,s}}^{p-2}\|g\|_{G^{\sigma,s}} + \dots + \|f\|_{G^{\sigma,s}}\|g\|_{G^{\sigma,s}}^{p-2} + \|g\|_{G^{\sigma,s}}^{p-1}\big)}\|f-g\|_{G^{\sigma,s}}.$$

Thus, we have proven Theorem 1.5.

4 Proof of Theorem 1.6

For the proof of Theorem 1.6, we need to estimate the norm $\|u\|_{G^{\sigma,s}}$. We apply the operator $e^{\sigma|D_x|}\langle D_x\rangle^s$ to the both sides of the equation (1.1) to get

$$e^{\sigma|D_x|}\langle D_x\rangle^s u_t - e^{\sigma|D_x|}\langle D_x\rangle^s u_x = e^{\sigma|D_x|}\langle D_x\rangle^s u^p.$$

If we define

$$U = e^{\sigma|D_x|}\langle D_x\rangle^s u,$$

then the commutativity of Fourier multipliers lets us rewrite the result as

$$\frac{1}{2}\partial_t U^2 - \frac{1}{2}\partial_x U^2 = U e^{\sigma|D_x|}\langle D_x\rangle^s u^p.$$

We now integrate both sides of this equation in space. Then by the equality

$$\int_{\mathbb{R}} \partial_t U^2(x,t)\,dx = \frac{d}{dt}\int_{\mathbb{R}} U^2(x,t)\,dx,$$

we get

$$\frac{d}{dt}\int_{\mathbb{R}} U^2\,dx - \int_{\mathbb{R}} \partial_x U^2\,dx = 2\int_{\mathbb{R}} U e^{\sigma|D_x|}\langle D_x\rangle^s u^p\,dx.$$

The first integral on the left-hand side of the equation is nothing more than $\|u(\cdot,t)\|_{G^{\sigma,s}}^2$. The second integral on the left-hand side of the equation vanishes, since U must vanish at infinity [10, Corollary 7.9.4, p. 243]. On the right-hand side the integral can be estimated by using Hölder's inequality [11, Proposition 1.1, p 1] and the inequality (2.4)

$$\begin{aligned}
2\Big|\int_{\mathbb{R}} Ue^{\sigma|D_x|}\langle D_x\rangle^s u^p\,dx\Big| &\le 2\Big(\int_{\mathbb{R}} U^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}} \big|e^{\sigma|D_x|}\langle D_x\rangle^s u^p\big|^2\,dx\Big)^{1/2}\\
&= 2\|u\|_{G^{\sigma,s}}\|u^p\|_{G^{\sigma,s}}\\
&\lesssim 2\|u\|_{G^{\sigma,s}}\big(\|u\|_{G^{\sigma,s}}\|e^{\sigma|D_x|}u^{p-1}\|_{\mathcal{F}L^1(\mathbb{R})}+\|e^{\sigma|D_x|}u\|_{\mathcal{F}L^1(\mathbb{R})}\|u^{p-1}\|_{G^{\sigma,s}}\big)\\
&\lesssim 2\|u\|_{G^{\sigma,s}}\big(\|u\|_{G^{\sigma,s}}\|e^{\sigma|D_x|}u^{p-1}\|_{\mathcal{F}L^1(\mathbb{R})}+\|u\|_{G^{\sigma,s}}\|e^{\sigma|D_x|}u\|_{\mathcal{F}L^1(\mathbb{R})}\|e^{\sigma|D_x|}u^{p-2}\|_{\mathcal{F}L^1(\mathbb{R})}\\
&\quad+\|e^{\sigma|D_x|}u\|_{\mathcal{F}L^1(\mathbb{R})}\|e^{\sigma|D_x|}u\|_{\mathcal{F}L^1(\mathbb{R})}\|u^{p-2}\|_{G^{\sigma,s}}\big)\\
&\le 2C\|u\|^2_{G^{\sigma,s}}\big(\|e^{\sigma|D_x|}u^{p-1}\|_{\mathcal{F}L^1(\mathbb{R})}+\|e^{\sigma|D_x|}u\|_{\mathcal{F}L^1(\mathbb{R})}\|e^{\sigma|D_x|}u^{p-2}\|_{\mathcal{F}L^1(\mathbb{R})}\\
&\quad+\|e^{\sigma|D_x|}u\|^2_{\mathcal{F}L^1(\mathbb{R})}\|e^{\sigma|D_x|}u^{p-3}\|_{\mathcal{F}L^1(\mathbb{R})}+\ldots+\|e^{\sigma|D_x|}u\|^{p-2}_{\mathcal{F}L^1(\mathbb{R})}\|e^{\sigma|D_x|}u\|_{\mathcal{F}L^1(\mathbb{R})}\big)\\
&= 2C\|u\|^2_{G^{\sigma,s}}\sum_{j=0}^{p-2}\Big(\|e^{\sigma|D_x|}u\|^j_{\mathcal{F}L^1(\mathbb{R})}\|e^{\sigma|D_x|}u^{p-1-j}\|_{\mathcal{F}L^1(\mathbb{R})}\Big).
\end{aligned}$$

If we define

$$E(t)=\|u\|^2_{G^{\sigma,s}},$$

we have the inequality

$$\frac{d}{dt}E(t)\le 2C\sum_{j=0}^{p-2}\Big(\|e^{\sigma|D_x|}u\|^j_{\mathcal{F}L^1(\mathbb{R})}\|e^{\sigma|D_x|}u^{p-1-j}\|_{\mathcal{F}L^1(\mathbb{R})}\Big)E(t).$$

By Grönwall's inequality [9, Theorem 1.10, p. 11], we have that

$$\begin{aligned}
E(t)&\le E(0)\exp\Big(2C\int_0^t\sum_{j=0}^{p-2}\Big(\|e^{\sigma|D_x|}u\|^j_{\mathcal{F}L^1(\mathbb{R})}\|e^{\sigma|D_x|}u^{p-1-j}\|_{\mathcal{F}L^1(\mathbb{R})}\Big)\,d\tau\Big)\\
&\le \|f\|^2_{G^{\sigma,s}}\exp\Big(2C\sum_{j=0}^{p-2}\Big(\|e^{\sigma|D_x|}u\|^j_{L^\infty_t\mathcal{F}L^1_x(\mathbb{R})}\|e^{\sigma|D_x|}u^{p-1-j}\|_{L^\infty_t\mathcal{F}L^1_x(\mathbb{R})}\Big)t\Big).
\end{aligned}$$

If $T^*=\infty$, then we have proven Theorem 1.6. If $T^*<\infty$, then it must be the case that

$$\|u(\cdot,t)\|_{G^{\sigma,s}}\longrightarrow\infty,\ \text{ as }\ t\to T^*.$$

Otherwise, we would be able to apply Theorem 1.5 again to extend our solution to a slightly larger interval $[0,\,T^*+\varepsilon]$. But this is a contradiction, as it would violate the definition of T^*. Then from the resulting inequality

$$\|u\|^2_{G^{\sigma,s}}\le\|f\|^2_{G^{\sigma,s}}\exp\Big(2C\sum_{j=0}^{p-2}\Big(\|e^{\sigma|D_x|}u\|^j_{L^\infty_t\mathcal{F}L^1_x(\mathbb{R})}\|e^{\sigma|D_x|}u^{p-1-j}\|_{L^\infty_t\mathcal{F}L^1_x(\mathbb{R})}\Big)t\Big),$$

it follows that there exists at least one k such that $1\le k\le p-1$ and

$$e^{\sigma|D_x|}u^k\notin L^\infty_t\mathcal{F}L^1_x(\mathbb{R})\big([0,T^*]\times\mathbb{R}\big).$$

Thus, Theorem 1.6 is proved.

5 Conclusion

It was proven that the Cauchy problem for a transport equation with algebraic nonlinearity of degree p with initial data in Gevrey spaces is locally well-posed. In particular, the persistence of the analyticity of solutions for a short time was proven and a sufficient condition for solutions to be analytic for all times was derived.

Acknowledgements

I would like to express my deep gratitude to my research supervisor Dr. Daniel Oliveira da Silva for his guidance, cordial support, valuable advice and constant encouragement throughout the course of this project. I also thank Dr. Alejandro Javier Castro Castilla as the second reader of this project for critical remarks and valuable advice.

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