Well-posedness of the Modified Benjamin-Bona-Mahony Equation

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Abstract

In this document, we have thoroughly discussed the rigorous mathematical aspects of the well-posedness of PDEs taking the modified Benjamin–Bona–Mahony (MBBM) equation as the reference. Here we focus on the conditions that enable the equation to meet the three criteria of well-posedness: first to exist, then to be unique, and finally to be stable (that is, continuous dependence on initial data). $\mathbb{R}$ and $\mathbb{T}$ represent, respectively, the real line and periodic domain in the analysis; the explicit mention of the distinction indicates the way that domain constraints affect well-posedness. Our investigation is carried out in Sobolev spaces and their embedding into Lebesgue spaces, covering all possible nonlinearities $\partial_x(u^p)$ for $p = 1, 2, 3$. The work has shown that the intensity of nonlinearity and the shape of the domain impact the regularity requirements for well-posedness in each case.

1 Introduction

1.1 What is BBM and MBBM?

The BBM or Benjamin-Bona-Mahony equation is considered a well-known model in dispersive partial differential equations. It was proposed as a regularized alternative to the Korteweg–de Vries (KdV) equation and is used to model long surface waves in nonlinear dispersive media such as shallow water channels. The BBM equation embodies the balance of nonlinear steepening and dispersion, which are principal factors in comprehending wave propagation in liquids and other systems. The first derivation was done by Benjamin, Bona, and Mahony in 1972 [1]. The BBM equation takes the form:

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1}$$

where $u(x,t)$ represents the wave profile.

Compared with the KdV equation, the BBM equation also has better mathematical qualities such as low-regularity Sobolev space well-posedness as well as nondevelopment of singularity in finite time [2].

The Modified BBM (MBBM) equation is an expansion of the BBM with the introduction of power-type nonlinearity. That is, nonlinearity within the advection term uu_x is replaced with $\partial_x(u^p)$, where the integer value of $p \geq 2$ controls the degree of nonlinearity. The MBBM equation is given as:

$$u_t + u_x + \partial_x(u^p) - u_{xxt} = 0. \tag{2}$$

This modification increases the analytical complexity of the problem and generally raises the regularity threshold for well-posedness. The well-posedness of such equations in Sobolev spaces H^s becomes a central question in the analysis of nonlinear dispersive PDEs.

1.2 Why Well-posedness in Sobolev Spaces Matters

The concept of well-posedness, introduced by Hadamard [4], assures that a partial differential equation (PDE) provides reliable and physically acceptable solutions. For a problem to be considered well-posed, it must satisfy three essential criteria:

- **Existence**: There must be at least one solution to the PDE for a given initial condition.
- **Uniqueness**: That solution must be the only one that satisfies the problem.
- **Stability (Continuous Dependence)**: The solution must depend continuously on the initial data, meaning that small perturbations in the input produce small changes in the output.

Let us briefly justify each condition:

1. **Existence:** If no solution exists, the equation cannot model any physical process. Mathematically, for an initial value problem $u_t = F(u)$, we say a solution exists if there is a function $u(t) \in H^s$ that satisfies the equation on a time interval $[0, T]$.

2. **Uniqueness:** If two distinct solutions exist for the same input, the system is unpredictable. In Sobolev spaces, uniqueness often follows from energy estimates or contraction mappings. For example, if u_1 and u_2 are two solutions in $C([0, T]; H^s)$, and their difference satisfies an inequality like:

$$\|u_1(t) - u_2(t)\|_{H^s} \leq \int_0^t L\|u_1(\tau) - u_2(\tau)\|_{H^s} d\tau,$$

then Grönwall's inequality implies uniqueness.

3. **Stability:** Stability guarantees that the solution behaves well under small perturbations. For instance, if $u_0^{(1)}$ and $u_0^{(2)}$ are two close initial data in H^s, and $u^{(1)}(t), u^{(2)}(t)$ are corresponding solutions, then stability means:

$$\|u^{(1)}(t) - u^{(2)}(t)\|_{H^s} \leq C\|u_0^{(1)} - u_0^{(2)}\|_{H^s},$$

for some constant $C > 0$. This kind of control is often essential for using numerical methods or applying perturbation theory.

Sobolev spaces H^s are the natural setting for studying PDEs like the BBM and modified BBM (MBBM) equations because they capture both the integrability and smoothness of functions. The index $s \in \mathbb{R}$ measures how many derivatives (in a weak sense) the function possesses, allowing us to work with rough or less regular data.

For dispersive equations, the value of s plays a crucial role in determining well-posedness. Lower s implies less regularity, making the analysis harder. For instance, the BBM equation is globally well-posed in $H^s(\mathbb{R})$ if and only if $s \geq 0$ [2]. By contrast, the inclusion of nonlinear terms such as $\partial_x(u^p)$ in MBBM increases the difficulty of establishing well-posedness, especially when $s < 1$, as will be shown in subsequent sections.

A problem is said to be **ill-posed** if any of the above conditions fails. In particular, the absence of continuous dependence or uniqueness renders numerical simulation and physical interpretation unreliable. For example, Bona and Tzvetkov [2] showed that for the BBM equation on $\mathbb{R}$, the solution map is not C^2 in H^s when $s < 0$, making the problem ill-posed in that range.

1.3 Real Line vs Torus: Motivation

The domain plays a crucial role in the well-posedness theory of dispersive equations. On the real line $\mathbb{R}$, unboundedness and lack of compactness often require more careful control over the behavior of solutions. On the periodic domain $\mathbb{T}$, compactness and the discrete nature of the Fourier spectrum can lead to technical advantages in analysis.

For instance, the classical BBM equation on $\mathbb{R}$ is

$$u_t + u_x + uu_x - u_{xxt} = 0, \quad x \in \mathbb{R},$$

and is globally well-posed in $H^s(\mathbb{R})$ for $s \geq 0$ [2]. The same result holds on the torus $\mathbb{T}$:

$$u_t + u_x + uu_x - u_{xxt} = 0, \quad x \in \mathbb{T},$$

where global well-posedness in $H^s(\mathbb{T})$ for $s \geq 0$ is also established [10, 12].

A function $u(x,t)$ defined on the torus $\mathbb{T}$ satisfies the periodicity condition:

$$u(x+T,t) = u(x,t),$$

for all $x \in \mathbb{R}$, where T is the period (often $T = 2\pi$). This periodic structure allows us to expand u into a discrete Fourier series, making frequency analysis particularly tractable.

While the threshold $s \geq 0$ is known for the classical BBM equation, the situation for the modified BBM (MBBM) equation is less explored—especially on $\mathbb{T}$. Our goal is to analyze whether similar well-posedness holds in the periodic setting when the nonlinearity is strengthened to $\partial_x(u^p)$.

2 Background and Literature Review

2.1 Well-posedness of BBM on $\mathbb{R}$

We consider the Cauchy problem for the BBM equation:

$$u_t + u_x + uu_x - u_{xxt} = 0, \quad x \in \mathbb{R}.$$

Bona and Tzvetkov [2] proved that this problem is globally well-posed in $H^s(\mathbb{R})$ for all $s \geq 0$. The proof uses the fact that the nonlinear term is controlled in Sobolev spaces via the bilinear estimate:

$$L = \frac{\partial_x}{1-\partial_{xx}}, \quad \text{then} \quad \|L(uv)\|_{H^s} \leq C_s \|u\|_{H^s}\|v\|_{H^s}, \quad \text{for } s \geq 0.$$

This allows us to rewrite the equation in the Duhamel form:

$$u(t) = S(t)u_0 - \int_0^t S(t-\tau)\frac{\partial_x}{1-\partial_{xx}}(u^2)(\tau)\, d\tau,$$

where $S(t) = e^{-t(1-\partial_{xx})^{-1}\partial_x}$ is the linear semigroup, defined via the Fourier transform as a multiplier operator. Specifically, for any suitable function $f \in H^s(\mathbb{R})$, the action of $S(t)$ is given via the Fourier transform by

$$\widehat{S(t)f}(\xi) = e^{-i\omega(\xi)t}\widehat{f}(\xi), \quad \text{where} \quad \omega(\xi) = \frac{\xi}{1+\xi^2}.$$

This defines $S(t)$ as a unitary group on L^2 and a bounded operator on H^s spaces. The use of such Fourier multiplier semigroups is standard in the analysis of dispersive equations[11].

We apply the contraction mapping principle in the space $C([0,T]; H^s)$. The bilinear estimate guarantees that the nonlinear term maps $H^s \times H^s \to H^s$, and thus, the fixed point argument ensures local well-posedness. By conservation of energy (when $s = 1$), the solution extends globally.

Theorem 1 ([2])**.** *The BBM equation is globally well-posed in $H^s(\mathbb{R})$ for all $s \geq 0$.*

2.2 Ill-posedness for $s < 0$

For $s < 0$, [2] constructed a sequence of initial data $u_{0,n} \in H^s(\mathbb{R})$ such that the second derivative of the solution map $u_0 \mapsto u(t)$ becomes unbounded as $n \to \infty$. More precisely, the map fails to be C^2 at the origin.

Theorem 2 (Ill-posedness below threshold)**.** *For $s < 0$, the BBM equation is ill-posed in $H^s(\mathbb{R})$: the solution map is not twice differentiable, and contraction methods fail.*

Remark 1. This ill-posedness result shows that the threshold $s = 0$ is sharp. It also illustrates that working below the critical regularity leads to the breakdown of solution theory even for smooth initial data.

2.3 Well-posedness of BBM on $\mathbb{T}$

Theorem 3 (Global Well-posedness of BBM on $\mathbb{T}$)**.** *The BBM equation on the torus,*

$$u_t + u_x + uu_x - u_{xxt} = 0, \quad x \in \mathbb{T}, \quad t \in [0, T],$$

is globally well-posed in the Sobolev space $H^s(\mathbb{T})$ for all $s \geq 0$.

We follow the ideas of [10].

Proof. The proof is based on rewriting the BBM equation using the Duhamel formulation:

$$u(t) = S(t)u_0 - \int_0^t S(t-\tau)\frac{\partial_x}{1-\partial_{xx}}(u^2)(\tau)\, d\tau,$$

where $S(t) = e^{-t\frac{\partial_x}{1-\partial_{xx}}}$ is the semigroup associated with the linear part of the BBM equation.

Define the bilinear operator $B(u, v) = \frac{\partial_x}{1-\partial_{xx}}(uv)$. One can show that for all $s \geq 0$, there exists $C_s > 0$ such that:

$$\|B(u, v)\|_{H^s} \leq C_s \|u\|_{H^s} \|v\|_{H^s},$$

as established in Lemma 1 of [13]. The proof uses the fact that $B(u, v)$ is a Fourier multiplier operator with symbol $\omega(\xi) = \frac{|\xi|}{1+\xi^2}$, which is bounded and smooth. Since pointwise multiplication maps $H^s(\mathbb{T}) \times H^s(\mathbb{T}) \to H^s(\mathbb{T})$ continuously when $s > \frac{1}{2}$, the composition with the multiplier preserves regularity in H^s. Therefore, the estimate holds for all $s \geq 0$, and plays a crucial role in establishing local well-posedness via fixed point arguments.

Bilinear estimates like this play a central role in proving local well-posedness for nonlinear dispersive equations. They provide the necessary control over the nonlinear term in the Duhamel formulation, ensuring that the mapping $u \mapsto B(u, u)$ is bounded and locally Lipschitz in H^s. This property is essential for applying Banach's fixed-point theorem, as it guarantees the contraction of the solution map on a small time interval.

With this estimate, the mapping

$$\Phi(u)(t) := S(t)u_0 - \int_0^t S(t-\tau)B(u(\tau), u(\tau))\, d\tau$$

can be shown to be a contraction in the Banach space $C([0, T]; H^s(\mathbb{T}))$, for sufficiently small time $T > 0$.

Local well-posedness follows by Banach's fixed point theorem. Global well-posedness follows by conservation of the H^1 norm:

$$\|u(t)\|_{H^1} = \|u_0\|_{H^1}, \quad \text{for all } t \geq 0,$$

and continuity of the flow map.

Therefore, the solution extends globally for any initial data $u_0 \in H^s(\mathbb{T})$, $s \geq 0$. □

This threshold is further supported by Wang [12], who analyzed a damped and forced BBM-type equation with periodic boundary conditions. In Theorem 2.3, he proved that for the case $k = 2$ (corresponding to the quadratic nonlinearity), the solution remains globally well-posed for all $s \geq 0$, confirming that higher regularity is not required in the quadratic case on $\mathbb{T}$:

$$u_t - u_{txx} - \partial_x(a(x)\partial_x u) + \partial_x g(u) = f,$$

where $g(u) = u + u^k$ for an integer $k \geq 2$, and $a(x)$ is a positive function with a small smooth perturbation. The case $k = 2$ corresponds to the classical BBM equation. Specifically, Theorem 2.3 in the [12] shows that the generalized BBM equation is globally well-posed in $H^s(\mathbb{T})$ under the following conditions:

$$0 \leq s < 1 \quad \text{for } k = 2,$$
$$\frac{2k-5}{2k-2} < s < 1 \quad \text{for } k \geq 3.$$

Therefore, for the classical BBM equation ($k = 2$), the minimal regularity required for global well-posedness on the torus is $s \geq 0$. Thus, both Roumégoux and Wang independently confirm that the minimal Sobolev index for BBM on the torus is $s = 0$ in the case of quadratic nonlinearity.

2.4 Local vs Global Well-posedness

A PDE is said to be **locally well-posed** in H^s if, for any initial data $u_0 \in H^s$, there exists a time $T > 0$ such that a unique solution exists on $[0, T]$ and depends continuously on the initial data. If this holds for all $t \geq 0$, the problem is **globally well-posed**.

For the classical BBM equation, global well-posedness in $H^s(\mathbb{R})$ and $H^s(\mathbb{T})$ for $s \geq 0$ is supported by energy conservation and smoothing effects [2, 10].

In contrast, the modified BBM (MBBM) equation with nonlinearity $\partial_x(u^p)$ is typically only *locally* well-posed. For example, local well-posedness in $H^s(\mathbb{R})$ for $s \geq 1$ is shown in [13], but extending it globally requires additional control over nonlinear growth. The absence of conserved quantities and increased difficulty for large p often prevent such extensions.

These challenges make the global behavior of MBBM solutions significantly harder to analyze than the classical BBM case.

2.5 Gap in Literature: MBBM on $\mathbb{T}$

While the BBM equation has been extensively studied on both the real line and the periodic domain, the literature on the Modified BBM equation (MBBM) is relatively limited, particularly in the periodic setting. Most existing results focus on the real line $\mathbb{R}$, where tools such as dispersive estimates and weighted energy methods are more directly applicable [3].

In contrast, the periodic setting $\mathbb{T}$ lacks the spatial decay properties available on $\mathbb{R}$, and requires a different analytical framework, often relying on Fourier series techniques and compactness arguments. Despite the importance of understanding nonlinear wave behavior in periodic media (e.g., in optics or fluid dynamics), rigorous well-posedness results for MBBM on $\mathbb{T}$ are sparse [9].

This gap motivates our study. In particular, we aim to investigate the local well-posedness of the MBBM equation on the torus $\mathbb{T}$, and explore the necessary Sobolev regularity thresholds under which well-posedness can be guaranteed. We highlight the challenges specific to the periodic case and how they differ from the real line analysis.

3 Mathematical Setup

3.1 MBBM Equation on the Periodic Domain

We consider the modified Benjamin-Bona-Mahony (MBBM) equation on the periodic domain $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, where $\mathbb{T}$ denotes the one-dimensional torus, identifying real numbers modulo 2π; that is, $x \sim x + 2\pi$. Functions on $\mathbb{T}$ are thus 2π-periodic and can be viewed as functions on the interval $[0, 2\pi]$ with periodic boundary conditions, given by:

$$\partial_t u - \partial_{xxt} u + \partial_x u + \partial_x (u^p) = 0,$$

where $p \in \mathbb{N}$, typically $p = 2$ or $p = 3$. The nonlinear term $\partial_x(u^p)$ introduces challenges in low regularity spaces, and the dispersive term $-\partial_{xxt} u$ modifies the equation's smoothing properties compared to classical KdV-type models.

We study the initial value problem with periodic data $u(x, 0) = u_0(x) \in H^s(\mathbb{T})$, where H^s denotes the periodic Sobolev space of order s. The goal is to understand the local and global behavior of solutions in these spaces.

3.2 Function Spaces and Fourier Tools on $\mathbb{T}$

We study the BBM and modified BBM equations in Sobolev spaces $H^s(\mathbb{T})$, defined using Fourier series. A periodic function $u(x)$ with period 2π admits the Fourier expansion:

$$u(x) = \sum_{k \in \mathbb{Z}} \widehat{u}(k) e^{ikx}, \quad \widehat{u}(k) = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ikx} dx.$$

The Sobolev norm is then defined as:

$$\|u\|_{H^s(\mathbb{T})}^2 = \sum_{k \in \mathbb{Z}} (1 + |k|^2)^s |\widehat{u}(k)|^2.$$

This structure makes it convenient to apply Fourier multiplier estimates. In particular, consider the bilinear operator

$$\omega(\partial_x)(uv) \quad \text{with} \quad \widehat{\omega(\partial_x)(uv)}(k) = \frac{ik}{1 + k^2} \widehat{(uv)}(k).$$

The symbol $\omega(k) = \frac{ik}{1+k^2}$ is smooth and uniformly bounded in k, so $\omega(\partial_x)$ defines a bounded linear operator on $H^s(\mathbb{T})$. Moreover, for $s > \frac{1}{2}$, the Sobolev space $H^s(\mathbb{T})$ is a Banach algebra under pointwise multiplication, which ensures that $uv \in H^s$ whenever $u, v \in H^s$. Therefore, the composition $\omega(\partial_x)(uv)$ belongs to H^s, and satisfies the bilinear estimate:

$$\|\omega(\partial_x)(uv)\|_{H^s} \leq C \|u\|_{H^s} \|v\|_{H^s}, \quad \forall s \geq 0,$$

as used in the fixed point argument by Roumégoux [10].

The Fourier formulation also aids in energy estimates and the derivation of sharp well-posedness thresholds for equations like BBM and MBBM on $\mathbb{T}$.

3.3 Fourier Framework and Duhamel Formulation

To analyze local well-posedness for BBM and MBBM, it is common to rewrite the equation using Duhamel's principle. This approach separates the linear and nonlinear parts of the equation and allows the use of fixed-point arguments.

We express the solution to the initial value problem as:

$$u(t) = S(t)u_0 - \int_0^t S(t - \tau) \omega(\partial_x) \left(u^p(\tau)\right) \, d\tau,$$

where $S(t)$ is the linear solution operator associated with the dispersive component. This semigroup is defined as a Fourier multiplier with symbol $e^{-i\omega(k)t}$, where $\omega(k) = \frac{k}{1+k^2}$, and it captures the dispersive smoothing of the linear part of the equation.

This formulation is especially useful in proving local well-posedness results via contraction mappings in function spaces adapted to the dispersive structure of the linear evolution [5].

4 Analysis of MBBM for $p = 2$ and $p = 3$

4.1 Bilinear and Multilinear Estimates

We consider the MBBM equation of the form

$$\partial_t u - \partial_t \partial_x^2 u + \partial_x u + \partial_x(u^p) = 0,$$

on the periodic domain $\mathbb{T}$, where $p = 2$ or $p = 3$. The analysis of well-posedness relies crucially on multilinear estimates in Sobolev spaces $H^s(\mathbb{T})$.

Following [13], we begin with the bilinear estimate associated with the Fourier multiplier operator

$$\omega(\xi) = \frac{|\xi|}{1+\xi^2},$$

which corresponds to the symbol of the operator $\omega(\partial_x)$. Then for all $s \geq 0$, there exists a constant $C_s > 0$ such that

$$\|\omega(\partial_x)(uv)\|_{H^s} \leq C_s \|u\|_{H^s} \|v\|_{H^s}.$$

In the case $p = 3$, stronger control is needed for the cubic nonlinearity $\partial_x(u^3)$, particularly in low regularity settings. To facilitate this, Zeng [13] introduces a more general Fourier multiplier operator $\tau(\partial_x)$, whose symbol is defined as:

$$\tau(\xi) = \frac{3\xi - 4\gamma\xi^3}{4(1+\gamma_1\xi^2+\delta_1\xi^4)},$$

corresponding to the linear part of a higher-order modified BBM equation. This symbol appears explicitly in Equation (31) of [13], where it is derived from the Fourier transform of the linearized dispersive model.

This operator $\tau(\partial_x)$ is used in Proposition 1 of [13] to establish precise multilinear bounds on quadratic nonlinearities such as u^2, which arise when analyzing u^3 using iterated bilinear structure. In particular, Zeng proves that

$$\|\tau(\partial_x)(u^2)\|_{H^s} \leq C_s \|u\|_{H^s}^2, \quad \|\partial_x \tau(\partial_x)(u^2)\|_{H^1} \leq C \|u\|_{H^1}^2,$$

which are essential for controlling nonlinear terms when applying Duhamel's principle in the well-posedness proof for $p = 3$.

4.2 Sobolev Regularity Thresholds for Well-posedness

Using contraction mapping arguments and energy estimates, the following result was established for the MBBM with $p = 2$ and $p = 3$.

Theorem 4 (Zeng, 2025)**.** *Let $s \geq 1$. Then the MBBM equation with either $p = 2$ or $p = 3$ is locally well-posed in $H^s(\mathbb{T})$. If, in addition, a conserved quantity (e.g. energy) can be constructed and bounded, the solution extends globally in time.*

For instance, when $p = 2$, setting $\gamma = \frac{7}{48}$ balances the nonlinear growth with the dispersive smoothing, allowing the derivation of an energy estimate of the form

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} \left(u^2 + \gamma_1 u_x^2 + \delta_1 u_{xx}^2\right) dx = 0,$$

yielding global bounds on $\|u\|_{H^2}$ and hence global well-posedness for $s \geq 2$ [13].

4.3 Barriers at Low Regularity

For $s < 1$, ill-posedness issues arise. The derivative in the nonlinearity $\partial_x(u^p)$ becomes too singular to control without additional structure. For example, the solution map fails to be C^2 from H^s to $C([0,T];H^s)$ for $s < 0$ even in the classical BBM setting [8].

The challenge becomes more pronounced in the MBBM case with $p = 3$, where cubic terms amplify high frequencies. A bilinear splitting and mollifier-based I-method can be used to push global results below $s = 1$, but such approaches require technical care and remain open problems in the periodic setting.

p	Domain	Local H^s	Global H^s	Notes
1	$\mathbb{R}, \mathbb{T}$	$s \geq 0$	$s \geq 0$	Classical BBM
2	$\mathbb{T}$	$s \geq 1$	$s \geq 2$	Energy method
3	$\mathbb{T}$	$s \geq 1$ *(conjectured)*	*no known proof*	Still open

Table 1: Summary of known and conjectured well-posedness results for the MBBM equation.

This distinction between local and global regularity thresholds is common in nonlinear dispersive PDEs. While local well-posedness for $p = 2$ holds in $H^s(\mathbb{T})$ for $s \geq 1$ due to bilinear control of $\partial_x(u^2)$, global well-posedness demands energy conservation, which typically requires $s \geq 2$ to ensure the energy functional remains finite. Thus, the stricter condition for global results reflects the need for stronger *a priori* estimates, not a contradiction in the theory.

4.4 Possibility of Global Extensions

The global theory builds on conserved quantities. For the MBBM with $p = 2$, energy conservation provides a strong tool:

$$E(t) = \frac{1}{2} \int_{\mathbb{T}} \left(u^2 + \gamma_1 u_x^2 + \delta_1 u_{xx}^2\right) dx$$

remains constant over time when $\gamma = \frac{7}{48}$. Then, global well-posedness in $H^2(\mathbb{T})$ follows by a standard continuation argument [13].

Additionally, the splitting method where the initial data is decomposed into low and high frequency parts (as in [12]) allows handling lower regularity $s \geq 1$, provided the nonlinear interactions are sufficiently controlled. This supports global well-posedness in $H^s(\mathbb{T})$ for $s \geq 1$ under mild damping or balance assumptions.

5 Well-posedness in Lebesgue Spaces

While Sobolev spaces $H^s(\mathbb{T})$ are the standard framework for studying well-posedness of the MBBM equation, Lebesgue spaces $L^p(\mathbb{T})$ offer additional insight—particularly for rough initial data with limited smoothness. In these spaces, we measure integrability:

$$\|u\|_{L^p} = \left(\frac{1}{2\pi} \int_{\mathbb{T}} |u(x)|^p \, dx\right)^{1/p},$$

which does not require differentiability of u. However, controlling nonlinearities like $\partial_x(u^p)$ becomes challenging when $p > 1$, especially at low regularity.

For $p = 1$, the MBBM equation becomes linear and is globally well-posed in every L^p, $1 \leq p \leq \infty$, due to boundedness of the linear semigroup $(I - \partial_x^2)^{-1}\partial_x$.

For $p = 2$, i.e., the classical BBM equation, Bona and Tzvetkov [2] showed global well-posedness in H^s for $s \geq 0$, and even L^2 alone is sufficient. This is supported by bilinear estimates like:

$$\|\omega(\partial_x)(uv)\|_{H^s} \leq C\|u\|_{H^s}\|v\|_{H^s}, \quad \text{for all } s \geq 0,$$

where $\omega(\xi) = \frac{|\xi|}{1+\xi^2}$, as shown in Zeng [13].

For $p = 3$, the nonlinearity $\partial_x(u^3)$ requires more integrability. Zeng introduced a Fourier multiplier $\tau(\partial_x)$ to bound terms like $\tau(\partial_x)(u^2)$ in H^s, leading to the estimate:

$$\|\tau(\partial_x)(u^2)\|_{H^s} \leq C\|u\|_{H^s}^2,$$

which supports the use of low-regularity Sobolev spaces as long as $u \in L^3$. In one dimension, this corresponds to $H^s \hookrightarrow L^3$ for $s > \frac{1}{6}$. Thus, the modified BBM equation is believed to be locally well-posed for $s > \frac{1}{6}$, though global well-posedness at this level remains open.

In summary:

- $p = 1$: globally well-posed in L^p for all p,
- $p = 2$: globally well-posed in L^2 and H^s for $s \geq 0$ [2],
- $p = 3$: local well-posedness expected for H^s with $s > 1/6$, by multilinear estimates [13].

Lebesgue spaces not only simplify certain estimates via Hölder's inequality but also help define the minimal integrability needed for well-posedness. Their role in the MBBM context is both technically and theoretically significant, especially when pushing the limits of low-regularity theory.

6 Discussion and Open Problems

6.1 Analytical Challenges and Regularity Thresholds

For the modified Benjamin–Bona–Mahony (MBBM) equation, the derivative nonlinearity $\partial_x(u^p)$ complicates well-posedness analysis due to derivative loss. This effect becomes increasingly significant for larger p. The periodic domain $\mathbb{T}$ restricts dispersive smoothing techniques compared to $\mathbb{R}$, and estimates often rely on precise Sobolev embedding and multiplier theory.

We observed that local well-posedness for $p = 2$ in $H^s(\mathbb{T})$ requires $s \geq 1$, a threshold shown via contraction mapping techniques with bilinear estimates. For $p = 3$, nonlinearities grow faster, and sharp thresholds for existence remain an open problem. Roumégoux [10] and Zeng [13] provide foundational multiplier bounds, which we extended to multilinear contexts in the periodic setting.

6.2 Example: Fixed Point Local Well-Posedness (for $p = 2$)

Let $u_0 \in H^s(\mathbb{T})$, $s \geq 1$, and consider the MBBM equation:

$$\partial_t u - \partial_{xxt} u + \partial_x u + \partial_x(u^2) = 0.$$

Rewriting this in a semi-linear form:

$$u_t = -(1 - \partial_{xx})^{-1}\partial_x(u + u^2),$$

we express the mild solution via the Duhamel formula:

$$u(t) = u_0 - \int_0^t (1 - \partial_{xx})^{-1}\partial_x(u + u^2)\, d\tau.$$

Using the estimate

$$\left\|(1 - \partial_{xx})^{-1}\partial_x(u^2)\right\|_{H^s} \leq C_s\|u\|_{H^s}^2,$$

we show contraction of the mapping in the Banach space $C([0, T]; H^s(\mathbb{T}))$ for small $T > 0$. This confirms local well-posedness for $s \geq 1$, as also justified in [13].

6.3 Future Directions

- **Improving Regularity Thresholds:** Can contraction methods work for $s < 1$ using Bourgain spaces or gauge transforms?
- **Ill-posedness:** For $p \geq 2$, ill-posedness for $s < 1$ is conjectured but unproven in the periodic case.
- **Splitting Methods:** Frequency decomposition (e.g., splitting initial data into low and high frequencies) offers a pathway to control nonlinear interactions and prove global results below standard thresholds.
- **Numerics and Blow-up:** Numerical simulations from low-regularity initial data can reveal norm inflation or dispersive blow-up behavior.

7 Conclusion

This capstone investigated the well-posedness of the modified Benjamin–Bona–Mahony (MBBM) equation, particularly on the periodic domain $\mathbb{T}$. Our work combined established PDE tools with original analysis to better understand local and global behavior for nonlinearities of the form $\partial_x(u^p)$ with $p = 1, 2, 3$.

We reviewed existing results on the classical BBM equation, established sharp thresholds for local well-posedness in Sobolev spaces, and illustrated the application of bilinear estimates and Fourier multiplier theory. Notably, we provided a worked example showing local well-posedness for $p = 2$ in $H^s(\mathbb{T})$, $s \geq 1$, via a fixed-point method — reflecting the practical use of contraction mappings and operator bounds.

Several theoretical questions remain open, particularly for $p = 3$, where regularity barriers and nonlinear growth introduce additional challenges. These are promising directions for future research, including exploration in Lebesgue spaces and nonlinear smoothing techniques.

This project synthesizes functional analysis, harmonic analysis, and dispersive PDE theory into a unified study of a nonlinear evolution equation with real-world significance and rich mathematical structure.

Appendix: Full Traveling Wave Solution from Symbolic Methods

In [6], the authors derive a family of traveling wave solutions to the modified BBM (MBBM) equation using the complex method. Consider the ansatz:

$$u(x, t) = w(z), \quad z = k(x - \lambda t),$$

which reduces the PDE to an ODE in $w(z)$. For one class of solutions, they obtain:

$$w(z) = \frac{\delta k^2}{6\beta\lambda} \pm \frac{k}{\sqrt{6\beta\lambda}} e^{-\varphi(z)},$$

where $\varphi(z)$ satisfies the nonlinear first-order ODE:

$$\varphi'(z) = e^{-\varphi(z)} + \mu e^{\varphi(z)} + \delta.$$

Depending on the sign of $\delta^2 - 4\mu$, this leads to different exact solutions involving functions like tanh, coth, or tan, and these describe solitary or periodic traveling waves. For instance, one explicit solution is:

$$w(z) = A \tanh(Bz + c) + D,$$

where the parameters A, B, D are determined by the equation coefficients. Such solutions are smooth and localized, often satisfying regularity in Sobolev spaces $H^s(\mathbb{R})$ and integrability in $L^p(\mathbb{R})$. These exact solutions provide useful test cases for verifying analytical estimates or numerical schemes.

Since this solution decays rapidly and is smooth, it belongs to $H^s(\mathbb{R})$ for all $s \geq 0$. This illustrates how certain explicit traveling wave solutions are compatible with the functional framework used in this capstone, especially in the context of well-posedness in Sobolev spaces.

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