# Algorithmic properties of Rogers Semilattices

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A thesis presented by
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to

The School of Sciences and Humanities in partial fulfillment of the
requirements for the degree of

*Doctor of Philosophy in Mathematics*
*Nazarbayev University*

Astana, Kazakhstan
2024

# Acknowledgements

I would like to express my sincere appreciation to my supervisor, Professor Manat Mustafa, for his unwavering support, guidance, and patience throughout the completion of this work. His immense knowledge of mathematics research, inspiration, motivation, and enthusiasm have been instrumental in achieving the success of this project. His invaluable support and encouragement have extended beyond research and have been an honor to have him as my supervisor.

I would also like to thank my co-supervisor, Dr. Nikolay Bazhenov, for his insightful discussions and valuable guidance, which played a significant role in the successful publication of our scientific articles.

I wish to express my gratitude to my co-supervisor Professor Zhibek Kadyrsizova for her perceptive remarks and questions that significantly contributed to the enhancement of this thesis.

I am deeply grateful to the Department of Mathematics and School of Sciences and Humanities for their generous financial support that enabled me to attend the Graduate Summer School at the National University of Singapore during my graduate studies. Their dedication to promoting international academic opportunities has been essential in expanding my perspectives and enhancing my research journey. I am truly grateful for their continued support and for providing excellent working conditions.

I would also like to acknowledge the Grant of Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (AP19676989) and the Nazarbayev University Faculty Development Grant (N021220FD3851) for funding this research.

I owe a special thanks to my parents, whose unwavering support and belief in my abilities and potential have led me to where I am today. Their constant encouragement has been the driving force behind my academic pursuits.

Lastly, but by no means least, I wish to express my gratitude to my husband for his unconditional love, support, and personal sacrifices throughout my academic and

personal life. To my beloved daughters, Mereyli, Balausa, and Shugila, I am indebted for their constant smiles and support through the long hours, days, weeks, months, and years.

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# Abstract

The thesis uses various approaches to explore the algorithmic complexity of families of subsets of natural numbers. One of these approaches involves investigating upper semilattices of computable numberings of a given family and their complexity in different hierarchies. These semilattices, known as Rogers semilattices, can help distinguish different structural properties of families of partial computable functions and computably enumerable sets. As a result, by using Rogers semilattices of computable numberings, we can measure the algorithmic complexity of the corresponding family.

In Chapter 3, we focus on limitwise monotonic numberings for families of limitwise monotonic sets and define their Rogers semilattices. The chapter investigates global invariants that show differences in the algebraic and elementary properties of the Rogers semilattices of families of sets from arithmetical hierarchy and Rogers semilattices of limitwise monotonic numberings. Such invariants include cardinality, laticeness, and types of isomorphism.

Within Chapter 4, we explore the different forms of isomorphism exhibited by Rogers semilattices of families of sets in the analytical hierarchy. Additionally, we take into account various set-theoretic assumptions. Our research demonstrates that, when set-theoretic assumption known as Projective Determinacy is assumed, there exist an infinite number of non-isomorphic Rogers semilattices at each $\Sigma^1_n$-level of the analytical hierarchy.

*Keywords: theory of numberings, computable numbering, Rogers semilattice, limitwise monotonic, analytical hierarchy, projective determinacy, types of isomorphism.*

# Аңдатпа

Диссертацияда натурал сандардың ішкі жиындар үйірінің алгоритмдік күрделілігін зерттеу үшін әртүрлі тәсілдер қолданылады. Осы тәсілдердің бірі берілген үйірдің есептелімді нөмірлеулерінің жоғарғы жарты торларын және олардың әртүрлі иерархияларда есептеулерін зерттеуді қамтиды. Роджерс жарты торлары деп аталатын бұл жарты торлар жартылай анықталған функциялар үйірі мен есептелімді саналымды жиындар үйірінің әртүрлі құрылымдық қасиеттерін ажыратуға көмектеседі. Нәтиже-сінде аталған үйірлердің есептелімді нөмірлеулерлерінің Роджерс жартылай торларын қолдану арқылы алгоритмдік күрделілігін анықтауға болады.

3-тарауда шекті монотонды жиындар үйірлері үшін шекті монотонды нөмірлеулері қарастырылған және олардың Роджерс жарты торлары анықталған. Сонымен қатар, бұл тарауда арифметикалық иерархиясындағы жиындар үйірлерінің және шекті монотонды нөмірлеулерінің Роджерс жартылай торларының алгебралық және элементар қасиеттері арасындағы айырмашылықтарды көрсететін жаһандық инварианттары зерттелген.

4-тарауда аналитикалық иерархиясындағы жиындар үйірлері үшін Роджерс жарты торларының изоморфизм типтерінің әртүрлі формалары зерттелген. Сонымен қатар, Проекциялық Детерминация деп аталатын жиынтық-теориялық ұйғарым ескеріліп, келесі нәтиже алынды: аналитикалық иерархияның әрбір $\Sigma^1_n$-деңгейінде изоморфты емес Роджерс жартылай торларының шексіз саны бар екені көрсетілді.

*Түйінді сөздер: нөмірлеулер теориясы, есептелімді нөмірлеу, Роджерс жартылай торы, шекті монотондылық, аналитикалық иерархия, проекциялық детерминация, изоморфизм түрлері.*

# Contents

# Chapter 1

# Introduction

## 1.1 Background review

### 1.1.1 Computability theory

Computability theory has played a significant role in the foundations of mathematics and theoretical computer science. The concept of computability is a major innovation of the twentieth century, and over the past century, it has evolved from being unknown to becoming a well-developed theory.

The origin of computability theory dates back to the 1930s when Turing, Gödel, Church, and others made groundbreaking contributions to the field. During this time, scientific inventions such as the development of quantum physics, the discovery of the UTM (Universal Turing Machine), and the First and Second Gödel Incompleteness Theorems significantly impacted science.

One of the most important developments in the history of computability was the twenty-three mathematical problems of the famous mathematician David Hilbert, as they provided a framework for understanding the limits of mathematical computation [Hodges, 1992]. However, it wasn't until 1936 that a young researcher named Alan Turing published his most influential paper [Turing, 1936], in which he proposed the idea of designing an abstract machine known as a Turing machine (TM). The TM was a theoretical computing machine that could simulate any algorithmic computation by reading and writing symbols on an infinite tape in a systematic manner. Turing's work on the concept of TMs was fundamentally important because it provided a rigorous definition of what it meant for a function to be computable, and it laid the foundation for the development of real-world computers [Davis, 2002].

Universal Turing Machines (UTMs) are abstract machines capable of implementing all algorithms, and they are a crucial part of the theory of computation. The UTM is a machine that can simulate any TM given an appropriate input. The UTM concept is important because it shows that any computation that can be performed by any algorithm can also be performed by a single machine - the UTM. The concept of TMs and UTMs are essential in computer science and mathematical structures, providing a theoretical framework for understanding what is computable and what is not.

Slightly before Alan Turing, Alonzo Church formally defined a computable function by an algorithm based on his so-called lambda calculus [Cooper, 2004]. At almost the same time, Gödel, Herbrand, and Kleene introduced definitions based on recursion schemes [Gödel, 1931, Kleene, 1952]. All of these concepts are equivalent and have contributed to the general conviction that these definitions are all correct. This statement is essentially what we now call the *Church's Thesis*. The thesis states that a Universal Turing Machine can precisely define the concept of an effective computable procedure.

Incomputability is a theory that deals with problems that cannot be solved by any computer. From a contemporary point of view, this theory is still being developed to describe natural phenomena that are hard to compute, and it focuses on more than just complexity. Despite being theoretically well-equipped, computers still face relatively natural problems that they cannot solve.

The basic example of noncomputable environments was first introduced in a challenging article of Turing [Turing, 1939]. Using the "Oracle Turing Machine," Turing explored the world of incomputability, revealing the mathematical structure of degrees of unsolvability known as the "Turing Degrees," and outlining their properties. His work laid the foundation for several models of computation, including the "Nondeterministic Turing machine". The "Nondeterministic Turing machine" is a model of computation that can be used to compute partial information. Unlike ordinary machines, we cannot define whether or not our oracle will give output.

### 1.1.2 Theory of numberings

The study of computability phenomenon leads to several interesting directions in mathematics and applications ([Ershov, 1998], [Badaev and Goncharov, 2000]). One of the significant areas is the *theory of numberings*. The theory of numberings was developed using different methods of classical computability theory to examine the algorithmic properties of abstract objects by encoding information about them and defining their connections by using the properties of their numerical indices. In simple terms, numbering is the process of assigning natural numbers to objects like functions, rational numbers, graphs, or words in formal languages.

A *numbering* of an arbitrary countable set $A$ is a surjective mapping $\nu$ from the set of natural numbers $\omega$ onto a given set $A$. A classical example of this approach is the Gödel numbering. It is a function that assigns *a unique natural number* to each formula and symbol of a given formal language. This enables the representation of complex mathematical and logical concepts as numerical sequences. It's named after Kurt Gödel, a mathematician whose work had a profound impact on logic, mathematics, and the foundations of computation. Gödel's construction involved considering a list of all possible sentences in the formal system, where each sentence was coded by a unique Gödel number. He then constructed a new sentence by "diagonalizing" across this list. This means that he selected elements (symbols or their negations) from the diagonal of this list to form a new sentence. By creating a sentence that refers to itself through diagonalization, Gödel effectively proved a statement that is true but unprovable within the formal system. This realization showed that there is no consistent and complete formal system which is capable of containing all truths about arithmetic. Hence, the incompleteness theorem states that it's impossible for a consistent set of axioms to prove all statements about the arithmetic of natural numbers.

As the theory of numbering is developed, naturally occurring questions are raised and corresponding answers are found. Some of these questions include whether certain properties of a set are determined by its numbering, whether a certain numbering preserves certain properties, and etc. In the process of developing a theory, inherent problems arise, resulting in unexpected and interesting results. Among these notable results is Friedberg's classical theorem. This theorem states that the class of all computably enumerable sets can be enumerated without repetition [Friedberg, 1958].

Results from the theory of numbering can help understand difficulties in modern computer science. As an example, among the most important problems of contemporary programming is finding an effective way to build a program that computes a function on one computer with a program that computes the same function on another [Ershov, 1999]. Implementing these translations in practice for a pair of universal computers has proven to be highly challenging and, in many cases, has not yet been achieved. If we consider the challenges of translating programs that compute the same function from one computer to another, we can envision a situation where the effectiveness of such translations depends on the numbering used to represent these programs.

Different numbering schemes might lead to different degrees of difficulty in achieving translations, and this is where the concept of a semilattice of numberings could come into play. The semilattice of numberings is a mathematical structure that provides a way to classify and compare different ways of representing computable functions using numberings. The concept of numbering plays a crucial role in differentiating and comprehending the internal structural properties of classes of partial computable functions and computably enumerable sets. Through the use of suitable numberings, S. Kleene discovered the most general existence theorems in the theory of computable functions (Recursion Theorem).

The fundamentals of numbering theory were established by A. N. Kolmogorov in the mid-1950s. This system of concepts received its first meaningful development in the works of V. A. Uspensky [Uspensky, 1955]. Objects related to numberings of families of computably enumerable sets and partial computable functions, arose independently in the works of Dekker [Dekker and Myhill, 1958], Rice [Rice, 1956], Rogers [Rogers, 1958], and others. The study of constructive algebras, which began around the same years, revealed the need of studying not only the numberings of families of sets and functions, but also objects of an arbitrary nature.

### 1.1.3 Theory of computable numberings

In computability theory and recursive mathematics, we often encounter situations that naturally lead us to study classes of constructive objects. These classes are best analyzed using the techniques and concepts of the theory of computable numberings. This theory allows us to explore the algorithmic properties of these classes more effectively. Ershov's monograph [Ershov, 1977a] introduces an extension of the classical concept of numbering. This extension permits us to consider surjections defined not only on the set of natural numbers, but also on arbitrary sets. As a result, it has become possible to consider computability in arbitrary sets. The notion of a "computable numbering" has been suitably expanded to accommodate this.

Assigning natural indices to all the elements of a countable class is considered as arbitrary numbering. In [Goncharov and Sorbi, 1997], S. S. Goncharov and A. Sorbi presented a general approach to study classes of objects that can be described constructively using formal languages with a Gödel numbering for formulas. They suggested that a numbering is computable if there exists a computable function that can produce some Gödel index of its constructive description for every object and each index of that object in numbering. Therefore, the index of an object relative to any computable numbering can be considered as its constructive description. Such numberings are known as *generalized computable numberings.*

The general approach has led to the study of various notions of computability in different research areas of computer science and mathematical logic. These research areas involve investigations of computable numberings in the arithmetical, analytical, and Ershov hierarchies. These studies have resulted in the publication of numerous works in this direction, including the works of Yu. Ershov, S. S. Goncharov, S. A. Badaev, S. Lempp, V. L. Selivanov, S. Yu. Podzorov, R. Solomon, A. Sorbi and so on ([Ershov, 1977b], [Badaev and Goncharov, 2001], [Badaev and Podzorov, 2002], [Badaev et al., 2006], [Podzorov, 2003], [Selivanov, 1976], [Goncharov et al., 2002]). In these studies, main focus is on computable numberings of families of computably enumerable (c.e.) sets which is considered as classical case. The class of c.e. sets forms the first level in both arithmetical and Ershov's hierarchies. When we talk about families in arithmetical, analytical, or Ershov's hierarchies, we only consider levels above the first level. For the case of Ershov hierarchy, we can only consider first and second levels, since the computational properties established for simpler levels can be applied to higher levels through reductions between types of numberings[Herbert et al., 2019]. *These reductions preserve the Rogers Semilattice of the numberings reduced, simplifies some constructions of specific semilattices. It is shown that for the basic types of numberings, one can reduce the $k$-r.e. numberings to the $(k+1)$-r.e. numberings; all further reductions are obtained by concatenating these reductions.*

### 1.1.4 Rogers semilattices

In the study of numberings, the theory of computable numberings is a fundamnetal concept. One of the most significant contributions to computable numbering came from Kleene's investigation [Kleene, 1967] where he constructed a computable numbering of all partial computable functions. In other words, he defined a computable numbering for the entire set of partial computable functions. Kleene's result is of great importance in computability theory as it helps study the properties of computable objects. The research papers of H. Rogers [Rogers, 1958] and R. Friedberg [Friedberg, 1958] mark the beginning of the systematic investigation of computable numberings. H. Rogers

[Rogers, 1958] mainly focused on the computable numberings of the entire family of partial computable functions. He introduced the class of all possible computable numberings of the family of all partial computable functions along with the reducibility relation between them. This reducibility relation determines a preordering on the given class of all possible computable numberings, which leads to the generation of the upper semilattice known as the *Rogers semilattice* of the family of all partial computable functions. Rogers [Rogers, 1958] considered only those computable numberings whose degrees correspond to the largest element of the described semilattice.

Rogers semilattices are mathematical structures that represent the complexity of computations for a family of sets. Problems related to computable numberings often focus on studying the algebraic and elementary properties of these semilattices. Rogers semilattices are also useful in measuring the different computations of a family of sets. They serve as a tool to classify the properties of computable numberings for various families of sets.

Generally, most questions related to numberings can be better understood in terms of Rogers semilattices, such as:

- how to find elementary and algebraic properties of Rogers semilattices, such as cardinality, laticeness, initial segments, ideals, and isomorphism types?

- how to define the characterization of numberings that generates in the Rogers semilattice elements with some special algebraic properties (i.e. maximal, minimal, indecomposable elements, etc.)

- how to distinguish different Rogers semilattices by describing the number of maximal and minimal elements;

## 1.2 Problem formulation

The goals and objectives of this thesis are in the partial solution of aforementioned questions for the family of limitwise monotonic sets and for the sets of analytical hierarchies. For the most part of this thesis, among generalized computable numberings, we will focus primarily on numberings for a subclass of $\Sigma_2^0$ sets known as limitwise monotonic sets. A set is said to be *limitwise monotonic (l.m.)* if it is either an empty set or the range of a limitwise monotonic function. For families of l.m. sets, we consider *limitwise monotonic numberings.* The set of all limitwise monotonic numberings of a family of subsets of natural numbers induces Rogers semilattices $\mathcal{R}_{lm}(\mathcal{S})$. (The formal definitions can be found in Section 2.2, and the study of this topic is the subject of Chapter 3.) The set of all computable numberings induces Rogers semilattices for computable families, i.e. uniformly enumerable families of computably enumerable subsets of the set of natural numbers. Our findings indicate that Rogers l.m. semilattices $\mathcal{R}_{lm}(\mathcal{S})$ exhibit a unique behavior that places them "in-between" the classical Rogers semilattices for computable families and Rogers semilattices of $\Sigma^0_2$-computable families (formal definition can be found in Section 2.2 ).

In Chapter 4 of this work, another approach is used to study the Rogers semilattice in the analytical hierarchy. An intensive study of Rogers semilattices for the families of sets in the analytical hierarchy started in the mid-2010s. A distinctive feature of working in this hierarchy is the using of additional set-theoretic axioms to obtain non-trivial results for the families of sets at levels, starting at level 3. For example, in [Dorzhieva, 2014, Dorzhieva, 2016, Dorzhieva, 2019] M.V. Dorzhieva, assuming Gödel's axiom of constructability proved that there is no Friedberg $\Sigma^1_n$-computable numbering for the family of all $\Sigma^1_n$-sets if $n > 3$. It was showed in [Bazhenov and Mustafa, 2020, Bazhenov et al., 2020, Bazhenov et al., 2022], that under the additional assumption of the axiom of analytical determinacy, many well-known results obtained for numberings in the arithmetical hierarchy can be transferred to the case of the analytical hierarchy. In this work, it is proved that at each $\Sigma^1_n$-level of the analytical hierarchy there are infinitely many pairwise non-elementarily equivalent Rogers semilattices [Bazhenov et al., 2021]. This result is an adaptation of the result of S.A. Badaev, S.S. Goncharov and A. Sorbi [Badaev et al., 2005].

The structure of the thesis is as follows. The thesis consists of an introduction, 3 chapters divided into sections, a conclusion and a bibliography. Statements are numbered with three digits: the first digit indicates the number of the chapter, the second one is the number of the section in the chapter, and the third is the number of the statement in the section.

The first chapter is introductory which provides a comprehensive review of the relevant literature related to the topic being investigated. The second chapter contains the notation and terminology used in the thesis and basic information about theory of numberings (Section 2.2). In the Section 2.3, some well-known approaches to the generalization of the concept of computable numbering are outlined.

The third chapter is devoted to the study of the structural properties of the Rogers semilattices for the families of limitwise monotonic numberings, which is denoted by

$\mathcal{R}_{lm}(\mathcal{S})$. It contains 6 sections, has the largest volume and is the most complex fragment of the work from a technical point of view.

Section 3.2 contains essential background information about limitwise monotonic numberings. For families $\mathcal{S}$ that are $\Sigma^0_n$-computable, the corresponding Rogers semilattice is $\mathcal{R}^0_n(\mathcal{S})$. In Section 3.3, we compare Rogers semilattices of l.m. numberings with the Rogers semilattices of $\Sigma^0_2$-computable numberings:

(1) There exists a l.m. family $\mathcal{S}'$ for any $\Sigma^0_2$-computable family $\mathcal{S}$, such that the semilattices $\mathcal{R}_{lm}(\mathcal{S}')$ and $\mathcal{R}^0_2(\mathcal{S})$ are isomorphic (Theorem 3.3.2). In particular, this implies that there are infinitely many pairwise non-elementarily-equivalent Rogers semilattices for l.m. families (Corollary 3.3.5).

(2) In addition, the Rogers semilattices $\mathcal{R}_{lm}(\mathcal{S})$ and $\mathcal{R}^0_2(\mathcal{S})$ are *equal* if an infinite l.m. family $\mathcal{S}$ contains only infinite sets (Theorem 3.3.3).

With the help of these theorems, it is possible to transfer a number of well-known results regarding $\Sigma^0_2$-computable families into the families of l.m. sets (Subsection 3.3.3).

In Section 3.4, we explain the differences between l.m. families and $\Sigma^0_2$-computable families. We show that there are infinitely many Rogers semilattices that are pairwise non-isomorphic for $\Sigma^0_1$-computable families, and it is possible to realize them as Rogers semilattices for l.m. families. This is stated in Proposition 3.4.2 and Observation 3.4.1. Particularly, the Rogers semilattice of limitwise monotonic family is isomorphic to that of c.e. $m$-degrees $\mathbf{R}_m$. However, it is still unknown whether Rogers semilattice of c.e. $m$-degrees $\mathbf{R}_m$ can be represented as the Rogers semilattice of a $\Sigma^0_2$-computable family. Another significant difference, which is stated in Proposition 3.4.1, relates to the well-researched class of numberings known as Friedberg numberings.

In Section 3.5, we provide a comprehensive solution to two fundamental questions of numbering theory about possible cardinalities and latticeness (for the families of l.m. numberings). For the case of families computable in the classical sense, these questions were formulated by Ershov [Ershov, 1967] and solved by Khutoretsky [Khutoretsky, 1971] and Selivanov [Selivanov, 1976], respectively. For $\Sigma^0_{n+2}$-computable families, for $n \in \mathbb{N}$, the formulated questions were solved in [Goncharov and Sorbi, 1997]. It follows that in each instance, the Rogers semilattices are either single element or have infinite cardinality, and by extension, they are not lattices. It is proven that if a limitwise monotonic family $\mathcal{S}$ contains more than one element, then the Rogers semilattice

$\mathcal{R}_{lm}(\mathcal{S})$ has infinite cardinality and cannot be classified as a lattice, as stated in Theorem 3.5.1.

In Section 3.6, we prove the following: in the class of all $\Sigma^0_2$-computable numberings, the index set of l.m. numberings is $\Sigma^0_4$-complete (Theorem 3.6.1).

Within Chapter 4, we explore different forms of isomorphism exhibited by Rogers semilattices of families of sets in the analytical hierarchy assuming various set-theoretic assumptions. Our research demonstrates that, when Projective Determinacy is assumed, there exist an infinite number of non-isomorphic Rogers semilattices at each $\Sigma^1_n$-level of the analytical hierarchy. The aim of our research is to explore the concept of Rogers semilattices in the analytical hierarchy, which have been previously studied in the works of [Dorzhieva, 2016, Bazhenov et al., 2023, Bazhenov and Mustafa, 2020, Bazhenov et al., 2020]. Specifically, we are interested in exploring the problem of determining the number of isomorphism types realized by these semilattices. We are considering $\Sigma^1_n$-computable families $\mathcal{S}$, where $n$ is a non-zero natural number, and examining the Rogers semilattices associated with them. Our goal is to provide a detailed analysis of these semilattices and their isomorphism types, which will contribute to a deeper understanding of the analytical hierarchy.

It was established in [Bazhenov and Mustafa, 2020] that there are at least four pairwise non-isomorphic Rogers semilattices for $\Sigma^1_n$-computable families under the assumption of *Projective Determinacy*(**PD**). In this thesis, we continue the investigation of [Bazhenov and Mustafa, 2020], and the result about the number of isomorphism types of Rogers semilattices for the families of sets of the analytical hierarchy is generalized. We have found that there exist infinitely many pairwise elementarily non-equivalent Rogers semilattices of $\Sigma^1_n$-computable families (Theorem 4.5.1).

The necessary background and known results on the analytical hierarchy that will be used in our proofs are provided in Section 4.3. Additionally, we include proofs of several useful results on numberings, specifically Lemma 4.3.1 and Lemma 4.4.4. In Section 4.4, we offer a brief overview of some consequences of **PD** that will also be utilized in our proofs. The main result regarding isomorphism types of Rogers semilattices for $\Sigma^1_n$-computable families is proven in Section 4.5. Finally, in Section 4.6, we discuss additional problems.

# Chapter 2

# Preliminaries

## 2.1 Notations and terminology

Notations and terminology in the work are standard and in mostly consistent with monographs [Ershov, 1977b, Rogers, 1967] . We denote the set of all natural numbers by $\omega$ . $\emptyset$ is the empty set. Capital letters $A, B, C, \dots$ are used for subsets of $\omega$. Capital calligraphic letters $\mathcal{S}, \mathcal{T}, \dots$ are used for the family of sets. Variables are denoted by lower-case letters $x, y, z, \dots$ which range over $\omega$. For a set $A$, we denote its cardinality by $\mathrm{card}(A)$, and its characteristic function by $\chi_A$. For a set $A \subseteq \omega$, we denote the complement of $A$ by $\overline{A}$, i.e. $\overline{A} = \omega \setminus A$.

We will also use some generally accepted notations of the theory of computability. We shall use $\varphi, \psi, \dots$ to denote partial funtions on $\omega$. By $f, g, \dots$ we denote total functions from $\omega^n$ to $\omega$, for $n \geq 1$. By $\omega^\omega$ we denote the set of all total functions from $\omega$ to $\omega$. For a partial function $\varphi$, we denote its domain by $dom(\varphi)$, and by $rng(\varphi)$ we denote its range. We write $\varphi(x) \downarrow$ if $\varphi(x)$ is defined or $x \in dom(\varphi)$, otherwise we write $\varphi(x) \uparrow$. $\varphi_e$ is a partial function computable by a Turing machine (TM) with the $e$-th program $P_e$ where $e$ is called Gödel number. $\varphi_e(x) = y$ means that $\varphi_e$ on input $x$ converged to $y$. The standard enumeration of partial computable functions and the computably enumerable sets are denoted by $\{\varphi_x\}_{x\in\omega}$ and $\{W_x\}_{x\in\omega}$, respectively. As usual, we append $s$ to the index of functionals such as $\varphi_{e,s}(x)$ and $W_{e,s}(x)$ to indicate the stage of affairs at stage $s$. We write

$$\varphi_{e,s}(x) = y \Leftrightarrow_{defn} x, y, e < s \quad \text{and} \quad y \quad \text{is the output of} \quad \varphi_e(x) \quad \text{in}$$

$$< s \quad \text{computational steps of } P_e.$$

The limit on the number of computational steps is necessary to ensure that searches do not go on infinitely. In other words, the bound $x, y, e < s$ means to give only finitely many $x, y, e$ for a given $s$ for which $\varphi_{e,s}(x) = y$ holds.

Given $x$ and $y$, $(x, y)$ is the ordered pair consisting of $x$ and $y$ in that order, while $\langle x, y\rangle$ is the image of $(x, y)$ under the standard pairing function from $\omega \times \omega$ onto $\omega$, where pairing function is defined as $\langle x, y\rangle = \frac{(x+y)^2+3x+y}{2}$.

The relation $\leq$ is a reducibility relation which is reflexive and transitive. Every reducibility relation $\leq$ induces an equivalence relation $\equiv$ on numberings in which two numberings are equivalent if and only if each one is reducible to the other. We denote the standard ordering of natural numbers by $\leq_\omega$. For a given numbering $\nu$, we denote the equivalence class of $\nu$ by $[\nu]$.

In the field of computability theory, equivalence classes of the reducibility relation are commonly referred to as degrees. Given two sets $A$ and $B$, that are subsets of natural numbers, we can say that $B$ is Turing reducible to $A$ and write $B \leq_T A$ if there is an oracle machine that can compute the characteristic function of $B$ with the help of $A$. Turing degree of a set $A$ is $deg(A) = \{B : B \equiv_T A\}$ where $B \equiv_T A$ iff $B \leq_T A$ and $A \leq_T B$.

Formally, given a set $A$ and a Gödel numbering $\varphi_i^A$ of the A-computable functions, the Turing jump $A^{'}$ of $A$ is defined as

$$A^{'} = \{x| \quad \Phi_x^A(x) \quad \text{is defined}\}$$

The $n$-th Turing jump $A^{(n)}$ is defined inductively as

$$A^{(0)} = A, \quad A^{(n+1)} = (A^{(n)})'$$

The notation $0^{'}$ or $\emptyset^{'}$ is often used for the Turing jump of the empty set. Similarly, $0^{(n)}$ is the $n$-th jump of the empty set.

For a non-zero natural number $n$, a set $X \subseteq \omega^m$ is $\Pi_n^1$ iff there exists a recursive predicate $R(x_1, \ldots, x_m, y; f_1, \ldots, f_n)$ such that for all $\bar{a} \in \omega^m$, we have

$$\bar{a} \in X \;\Leftrightarrow\; (\forall f_1)(\exists f_2)(\forall f_3)\ldots(Qf_n)(\overline{Q}y)R(\bar{a}, y; f_1, \ldots, f_n), \tag{2.1.1}$$

where the last quantifiers are defined as follows:

$$Q = \begin{cases} \forall, & \text{if } n \text{ is odd,} \\ \exists, & \text{if } n \text{ is even;} \end{cases} \qquad \overline{Q} = \begin{cases} \exists, & \text{if } Q = \forall, \\ \forall, & \text{if } Q = \exists. \end{cases}$$

A set $X \subseteq \omega^m$ is $\Sigma_n^1$ if it can be represented as:

$$\bar{a} \in X \;\Leftrightarrow\; (\exists f_1)(\forall f_2)(\exists f_3)\ldots(Qf_n)(\overline{Q}y)R(\bar{a}, y; f_1, \ldots, f_n).$$

We will use standard logical notation, including the quantifiers $\exists^{\infty}$ for "there exist infinitely many", $\forall^{\infty}$ for "for all but finitely many", and $\exists^{\geq k}$ for "there exist at least $k$ many".

## 2.2 Basic notions

The most unchanging aspect among all the improvements made to the concept of an algorithm is represented by the class of partial computable(or recursive) arithmetic functions. These functions are computable under any refinements of algorithmic computability, making them a reliable and consistent foundation for algorithmic theory and its applications. Essentially, this means that all refinements of algorithmic computability can be viewed as equivalent, and that the class of partial computable functions aligns with the class of partial arithmetic functions that can be effectively computed. Therefore, all studies in algorithmic theory and its applications should be conducted on the basis of this class of partial computable functions.

There exist several definitions of recursive functions, which were given by some of the most renowned mathematicians, including Church, Gödel, Kleene, Post and Turing. The definition of recursive functions is based on the principle of induction: initially, we define a small set of basic functions called *initial functions* to be recursive, and then we set out several rules to derive new ones from those already defined by the inductive process. However, before defining recursive functions, we start determining a basic class of functions known as *primitive recursive functions.* A "primitive recursive" function is a type of function that can be obtained from a set of basic functions that operate on non-negative integers. These basic functions include the zero function, the successor function, and projector functions. To create a primitive recursive function, we can use a limited number of compositions (substitutions) and primitive recursions. Essentially, this means that we can obtain a function by applying a set of mathematical operations to the basic functions.

In simpler terms, a primitive recursive function is a function that can be computed by a computer program. To create more complex functions, we can use the initial functions and apply substitution, primitive recursion, and minimalisation. These functions make up the smallest class of functions known as "partial computable functions".

A remarkable fact about computability is that it exists independently of any language used to describe it. According to Church's Thesis, the partial computable functions are exactly those functions that can be computed by Turing machines (i.e. effectively computable) on the natural numbers.

Here, by *effectively* computable we mean that there exists *some* algorithm which compute any value $f(x)$ for which $f(x)$ is defined. In other words, Church's Thesis claims that all formalization of definition of the concept of partial computable functions are equivalent. In practical way, it says that if the description of an algorithm for computing $f$ is given, then one can define $f$ as a partial computable function.

Now, with Church's Thesis in our reach, let us apply these notions of computability to other mathematical concepts, such as sets.

**Definition 2.2.1.** We say that a set $A$ is *computable* if its characteristic function $\chi_A$ is computable. Intuitively, a set $A$ is computable if there exists an effective procedure for deciding, given any $x$, whether or not $x \in A$.

Trivial example of computable sets is the entire set of natural numbers. Every finite or co-finite subset of the natural numbers is also computable. Note that if the set is computable then the complement of that set is also computable.

**Definition 2.2.2.** A set $A \subseteq \mathbb{N}$ is called *computably enumerable (c.e.)* if there is an effective process for enumerating all the members of $A$.

More formally, if a set $A$ is the range(or domain) of some partial computable function. So, denote $e$-th c.e.set by

$$W_e = dom(\varphi_e) = \{x : \varphi_e(x) \downarrow \quad for \quad x \in A\}$$

where $\varphi_e(x)$ is the partial function computed by a Turing machine(TM) with $e$-th program.

Informally, a c.e. set is a set of natural numbers that can be effectively listed by a computer program. That is, a set for which there exists an algorithm that can list all of its elements in a finite amount of time.

The use of appropriate numbering is one of the methods of reducing to natural numbers and computable functions that has been successfully used in algorithm theory and mathematical logic. Numbering is the process of assigning natural numbers to the class of constructive objects such as formulas, graphs, matrices, words, etc. It enables to examine the algorithmic properties of abstract objects by encoding information about them and defining their connections by using properties of their numerical indices. For example, in graph theory, graphs can be encoded by numbers and they can define connections between graphs using arithmetic operations on their corresponding numbers.

**Definition 2.2.3.** A *numbering* $\nu$ of a countable nonempty set $A \subseteq \mathbb{N}$ is a a surjective function that maps the set of natural numbers $\omega$ onto $A$.

For example,

1. The identity mapping $id_N : \mathbb{N} \to \mathbb{N}$ is a numbering.
2. The mapping $c : \mathbb{N}^2 \to \mathbb{N}$ defined by
$$\langle x, y \rangle = c(x, y) \rightleftharpoons \frac{(x+y)^2 + 3x + y}{2}$$
is one-to-one and surjective. The inverse mapping $c^{-1} : \mathbb{N} \to \mathbb{N}^2$ is a numbering of the set $\mathbb{N}^2$.

This is a fundamental concept in mathematics, which plays a crucial role in various fields of study.

Once we have two numberings of the same family, we want to compare the complexity of these numberings, therefore we define the notion of so called *reducibility relation* between numberings. The concept of reducibility is a classical tool for comparing the complexity of different numberings.

Suppose that $\nu$, $\mu$ are numberings of a set $A$.

**Definition 2.2.4.** A numbering $\nu$ is *reducible* to a numbering $\mu$ (in symbol, $\nu \leq \mu$), if there is total computable function $f(x)$ such that $\nu(k) = \mu(f(k))$ for all $k \in \omega$. Numberings $\nu$ and $\mu$ are *equivalent* (in symbol, $\nu \equiv \mu$), if they are reducible to each other, i.e. $\nu \leq \mu$ and $\mu \leq \nu$.

In other words, a $\nu$-index of a given object from a set $A$ can be effectively transformed into a $\mu$-index of a given object, which determines the pre-order in the class of all numberings.

Denote by $Num(A)$ the set of all numberings of a set $A$. The relation $\equiv$ on elements of $Num(A)$ is an equivalence relation. Then, the reducibility relation $\leq$ induces an order on the factor set $Num(A)/_{\equiv}$ which will be denoted also by $\leq$.

Let $\nu$, $\mu \in Num(A)$. Define the join of two numberings $\nu \oplus \mu$ of the set $A$ by
$$(\nu \oplus \mu)(2k) = \nu(k), \quad (\nu \oplus \mu)(2k+1) = \mu(k), \quad k \in \omega.$$

It is easy to verify that:

- $\nu \leq \nu \oplus \mu, \quad \mu \leq \nu \oplus \mu$;
- if $\tau \in Num(A), \quad \nu \leq \tau, \mu \leq \tau$, then $\nu \oplus \mu \leq \tau$.

If we define $[\nu]$ as the class that contains all numberings equivalent to $\nu$, and $[\mu]$ as the class that contains all numberings equivalent to $\mu$, then it follows that $[\nu \oplus \mu]$ is the least upper bound of $[\nu]$ and $[\mu]$ in the poset $\langle Num(A)/_{\equiv}, \leq \rangle$. Therefore, the ordered set $\langle Num(A)/_{\equiv}, \leq \rangle$ is an *upper semilattice*. Recall that an upper semilattice is a partially ordered set (poset) in which every pair of elements has a least upper bound (also called a join) or isomorphic to the empty set. For example, the set of all subsets of a given set ordered by inclusion forms an upper semilattice. The join operation corresponds to taking the union of two sets.

A numbering $\nu : \mathbb{N} \to A$ induces an *equivalence relation* $\eta_\nu$ on $\mathbb{N}$:

$$\eta_\nu \risingdotseq \{\langle x, y \rangle | x, y \in \mathbb{N}, \quad \nu(x) = \nu(y)\}$$

A numbering $\nu$ is *decidable (positive)* if the relation $\eta_\nu$ is computable (c.e.). A numbering $\nu$ is called *negative* if the relation $\eta_\nu$ is co-c.e., i.e. $\eta_\nu$ is the complement of a c.e. set. A numbering $\nu$ is one-to-one (or *Friedberg*) if $\eta_\nu = \{\langle x, x \rangle | x \in \mathbb{N}\}$.

In fact, positive numberings generate positive equivalence relations, also called as *computably enumerable equivalence relations(ceers)*. Positive equivalence relation was first studied by Ershov [Ershov, 1973], while a thorough introduction to ceers as a structure was provided by S.Gao and P. Gerdes [Gao and Gerdes, 2001]. Since then, the theory of ceers has become a vast research area with a lot of interesting results and applications [Andrews et al., 2017, Andrews et al., 2014, Nies and Sorbi, 2018]. A recent survey on ceers can be found in Andrews, Badaev and Sorbi [Andrews et al., 2017] which includes references about this topic, as well as those of other complexities.

In the field of computability theory and recursive mathematics, we often encounter situations that naturally lead us to study classes of constructive objects. Such classes are best analyzed by using the techniques and concepts of the theory of computable numberings, which allow us to explore their algorithmic properties. Ershov's monograph [Ershov, 1977a] introduces an extension of the classical concept of numbering, which permits us to consider surjections defined not only on the set of natural numbers, but also on arbitrary sets. This extension has made it possible to consider computability in arbitrary countable sets, and the notion of a *computable numbering* has been suitably expanded to accommodate this.

Assigning natural indices to all the elements of a countable class is considered as arbitrary numbering. In their work, S. S. Goncharov and A. Sorbi [Goncharov and Sorbi, 1997] presented a general approach to study classes of objects that can be described constructively using formal languages with a Gödel numbering for formulas. They suggested that a numbering is computable if there exists a computable function that can produce some Gödel index of its constructive description for every object and each index of that object in numbering. Therefore, the index of an object relative to any computable numbering can be considered as its constructive description. Such numberings are known as *generalized computable numberings.*

The general approach has led to the study of various notions of computability in different research areas of computer science and mathematical logic. These research areas involve investigations of computable numberings in the arithmetical, analytical, and Ershov hierarchies. These studies have resulted in the publication of numerous works in this direction, including the works of Yu. Ershov, S. S. Goncharov, S. A. Badaev, S. Lempp, V. L. Selivanov, S. Yu. Podzorov, R. Solomon, A. Sorbi and so on ([Ershov, 1977b], [Badaev and Goncharov, 2001], [Badaev and Podzorov, 2002], [Badaev et al., 2006], [Podzorov, 2003], [Selivanov, 1976], [Goncharov et al., 2002]). In these studies, main focus is on computable numberings of families of computably enumerable (c.e.) sets which is considered as classical case. The class of c.e. sets forms the first level in both arithmetical and Ershov's hierarchies. When we talk about families in arithmetical, analytical, or Ershov's hierarchies, we only consider levels above the first level. For the case of Ershov hierarchy, we can only consider first and second levels, since the computational properties established for simpler levels can be applied to higher levels through appropriate reduction operator [Herbert et al., 2019].

Let $\mathcal{S}$ be a countable family of c.e. sets, each of which is a subset of the natural numbers. It follows that $\mathcal{S} \subset P(\omega)$.

**Definition 2.2.5.** A numbering $\nu$ of the family $\mathcal{S}$ is *computable* if its universal set, i.e.

$$\{\langle k, x\rangle : \ x \in \nu(k)\}$$

is a c.e.set.

The family $\mathcal{S}$ is *computable* if it has a computable numbering.

Informally speaking, a family $\mathcal{S}$ of c.e. sets is called computable if its elements can be enumerated as a uniformly computably enumerable sequence. The standard numbering of the family of all c.e.subsets $W_0, W_1, W_2, ...$ is the classical example of a computable numbering. Recall that $W_n$ represents the domain of the partial computable function $\varphi_n$.

Kleene's investigation in [Kleene, 1967] led to the development of computable numbering that made it more appealing. He created a universal partial computable function, which means he defined a computable numbering of all partial computable functions. Kleene's result holds significant importance in the study of computable objects and is a crucial contribution in computability theory.

H. Rogers later focused on the computable numberings of the family $\mathcal{S}$ of all partial computable functions in his study [Rogers, 1958]. In particular, Rogers introduced the concept of $Com(\mathcal{S})$ which represents the class of all possible computable numberings of the family $\mathcal{S}$. Rogers also defined the reducibility relation between the numberings in $Com(\mathcal{S})$. As it was mentioned before, the reducibility relation determines a preordering on the given class $Com(\mathcal{S})$. Factorization by equivalence relation gives the factor set $Com(\mathcal{S})/_{\equiv}$ , determined by the reducibility relation $\leq$. For two numberings $\nu$, $\mu \in Com(\mathcal{S})$, their join $\nu \oplus \mu$ is defined as before. It is possible to construct a partial ordered set

$$R(\mathcal{S}) = \langle Com(\mathcal{S})/_{\equiv}, \leq \rangle$$

by generating the upper semilattice of computable numberings from the family $\mathcal{S}$. This is commonly referred to as the *Rogers semilattice* of $\mathcal{S}$. Rogers [Rogers, 1958], [Rogers, 1967] studied computable numberings whose degrees correspond to the largest element of a described semilattice. Moreover, R. Friedberg [Friedberg, 1958], A. Mal'tsev [Mal'tsev, 1961], and M. Pour-El [Pour-El, 1964], [Pour-El and Putnam, 1965] later presented the minimal elements of $R(\mathcal{S})$ as well as other properties.

**Definition 2.2.6.** Any nonempty subset $J \subseteq A$ is called an *ideal* of subsemilattice (semilattice) if it satisfies the following conditions:

- $x, y \in J \quad \Rightarrow \quad x \cup y \in J$
- $x \in J \quad y \leq x \quad \Rightarrow \quad y \in J$

Denote the class of all numberings of a family $\mathcal{S}$ by $Num(\mathcal{S})$. Note that the upper semilattice $\langle Com(\mathcal{S})/_{\equiv}, \leq \rangle$ is a subsemilattice and an ideal in $\langle Num(\mathcal{S})/_{\equiv}, \leq \rangle$, since a subset $Com(\mathcal{S})/_{\equiv}$ of a set $Num(\mathcal{S})/_{\equiv}$ satisfies the following two conditions:

1. for any $[\nu]$, $[\mu]$ from $Num(\mathcal{S})/_{\equiv}$, their least upper bound $[\nu \oplus \mu]$ is also in $Num(\mathcal{S})/_{\equiv}$.

2. if $[\nu] \in Num(\mathcal{S})/_{\equiv}$ and $[\mu] \leq [\nu]$, then $[\mu] \in Num(\mathcal{S})/_{\equiv}$.

The idea of computable numberings is practically used in classical computability theory. The construction of families of computably enumerable sets with a limited number of computable Friedberg numberings, initially proposed by Goncharov in [Goncharov, 1980a], has practical applications in recursive mathematics, as stated in [Ershov, 1998]. It has been used as a starting point for the investigation of algorithmic dimensions of computable models ([Cholak et al., 1999], [Goncharov, 1980b], [Goncharov and Khoussanov, 1997], [Ventsov, 1989], [Ventsov, 1994]). For instance, Kummer [Kummer, 1990] applied Goncharov's theorem [Goncharov, 1980a] on the number of computable Friedberg numberings of families of computably enumerable sets to solve an unsolved problem related to the types of isomorphisms of partial computable functions ([Rogers, 1967]). Kummer demonstrated that for any $k \in \omega$, a computable function exists, and it has precisely $k$ types of isomorphisms.

## 2.3 Generalized computable numbering

There are many problems in computability theory which are centered around the question on the existence of computable numberings. Furthermore, in order to gain a better understanding of the structure and properties of computable numberings of different classes, it is important to examine the computability of classes of computable numberings of families of c.e. sets ([Goncharov, 1980b]). It is also possible to solve various problems of algorithm theory by using the numberings of many other classes of constructive objects that can be computed.

Starting from the 1990s, Goncharov and Sorbi [Goncharov and Sorbi, 1997] began developing the theory of "generalized computable numberings". In their work, they proposed a new general approach that uses the concept of a computable family of constructive objects. The Goncharov-Sorbi approach allows us to refer to the concept of computable families of constructive objects, such as c.e. sets, constructive models, families of computable morphisms, and more. This approach also enables us to introduce the concept of a computable family of sets for different hierarchies of sets.

As per the Goncharov-Sorbi perspective, the overall characterization of a computable numbering was established as the following: Let $A$ represent some set of objects.

We are only interested in objects that can be described constructively. Define a language $\mathcal{L}$ and its interpretation $i : \mathcal{L} \to A$ as a partial surjective mapping. For any object $a \in A$, each "formula" in $i^{-1}(a)$ is interpreted as a description of $a$. For example, if $A$ consists of partial computable functions then $i^{-1}(a)$ may be considered as a set of programs of Turing machines for $a$. Another example is, if $A$ is a set of c.e. sets then $a \in A$ is definable by $\Sigma_1^0$ -formulas in arithmetic and we could consider $i^{-1}(a)$ as a collection of such formulas.

For $\mathcal{L}$, we consider a Gödel numbering $G : \omega \to \mathcal{L}$.

**Definition 2.3.1.** In the context of a language $\mathcal{L}$, let $A$ be a set. A numbering $\nu : \omega \to A$ is said to be a *computable numbering* of $A$ with respect to the interpretation $i$ if there exists a computable function $f$ such that the formula $G(f(n))$ can identify an element $\nu(n)$ in $\mathcal{L}$ relative to $i$. That is, for all $n \in \omega$, we have $\nu(n) = i(G(f(n)))$.

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "i"] \arrow[d, "G"] & A \arrow[d, "\nu"] \\
\omega \arrow[r, "f"] & \omega
\end{tikzcd}$$

To put it simply, a numbering is considered computable if there is a computable function that can produce a Gödel index of a constructive description for every object and each index of this object in the numbering.

For instance, let's consider the family of partial computable functions denoted by $\mathcal{F}$ and the language of Turing machines as $\mathcal{L}$. Suppose $i(TM)$ represents the function computed by a Turing machine$(TM)$. With this definition, we can obtain a computable numbering $\nu : \omega \to \mathcal{F}$ for the family of partial computable functions $\mathcal{F}$. The numbering $\nu$ is computable (relative to interpretation $i$) if and only if there is a partial computable function $g(n;x)$ such that for any $n \in \omega$, the functions $\nu(n)$ and $g(n;x)$ match ([Rogers, 1958], [Ershov, 1977b]).

Consider a language $\mathcal{L}$ and interpretation $i$. Let $Com(A)$ be the set of all computable numberings of a set $A$. The relation $\equiv$ is an equivalence relation, and the reducibility $\leq$ induces a partial ordering on the equivalence classes of this relation. We denote the partially ordered set induced by the reducibility relation between numberings as $R(A)$,

$$\langle Com(A)/_{\equiv}, \leq \rangle.$$

which can be seen as the Rogers semilattice.

Note that the join of two numberings $\nu$ and $\mu$ is given by their supremum $\nu \oplus \mu$.

According to the concept introduced by Ershov[Ershov, 1977b], Rogers semilattices are used to represent the algorithmic complexity of computations for a given set. The theory of computable numberings focuses on the algebraic and elementary properties of these semilattices. Rogers semilattices provide a way to measure the different computations of a family of sets, and they also help to classify the properties of computable numberings for different sets.

An overview of the outcomes concerning computable numberings will be presented through the use of various recursion-theoretic hierarchies: 1) arithmetical hierarchy [Badaev et al., 2006], [Badaev and Goncharov, 2008], [Badaev and Goncharov, 2000], [Badaev et al., 2005], [Badaev and Goncharov, 2001], [Badaev et al., 2003b]; 2) Ershov hierarchy [Badaev and Lempp, 2009], [Goncharov et al., 2002], [Herbert et al., 2019]; 3) analytical hierarchy [Bazhenov et al., 2020], [Dorzhieva, 2019], [Bazhenov et al., 2019], [Dorzhieva, 2016], [Dorzhieva, 2018], [Bazhenov and Mustafa, 2020]. From now on, we will only consider countable families $\mathcal{S} \subset P(\omega)$.

Let $\mathcal{C}$ be a class in the recursion-theoretic hierarchy, such as Kleene's arithmetical hierarchy $\Sigma^0_n$, analytical hierarchy $\Sigma^1_n$, and Ershov's difference hierarchy $\Sigma^{-1}_n$ for $n \geq 1$.

**Definition 2.3.2.** A numbering $\nu$ of a family $\mathcal{S}$ is *$\mathcal{C}$-computable* if the universal set $\{\langle k, x\rangle : x \in \nu(k)\}$ belongs to the class $\mathcal{C}$. This implies that a family $\mathcal{S}$ is $\mathcal{C}$-computable if it can be effectively represented by a $\mathcal{C}$-computable numbering.

If we take $\mathcal{C} = \Sigma^0_1$, it represents the class of all computably enumerable sets. Then, Definition 2.3.2 is equivalent to the classical concept of a computable numbering [Ershov, 1999], and the universal set $\{\langle k, x\rangle : x \in \nu(k)\}$ is considered as a uniformly effective procedure for determining if a number $x$ is in the set $\nu(n)$.

In the case of $\mathcal{C} = \Sigma^0_{n+1}$, computability of numberings means that universal set of numbering is $0^{(n)}$-enumerable by the strong hierarchy theorem of Kleene and Post [Rogers, 1967]. This means that many constructions of the classical theory of numberings can be analysed as oracles within the arithmetic hierarchy.

For the case of Ershov hierarchy when $\mathcal{C} = \Sigma^{-1}_n$, any number $x$ may be put into $\nu(n)$ at some stage of a uniformly effective procedure, but, in contrast with the classical case, $x$ may later be removed from $\nu(n)$ . And in contrast with the case of the arithmetical hierarchy, for any $x$, the number of enumerations of $x$ into, and extractions of $x$ from, $\nu(n)$ , is bounded by $n$. Recall that a set $A \subseteq \omega$ belongs to Ershov's hierarchy class

$\Sigma_n^{-1}$ if it is $n$-computably enumerable ($n$-c.e.). This means that $A$ can be expressed as $lim_s A_s$, where $A_s$ is a uniformly computable sequence of sets, $A_0 = \emptyset$, and for each $x$, there are at most $n$ many $s$ such that $A_s(x) = A_{s+1}(x)$. Moreover, a set $A$ is a difference of computably enumerable sets (d.c.e.) if it is 2-c.e., that is, if it can be represented as $A_0 - A_1$, where $A_0$ and $A_1$ are computably enumerable sets.

The detailed explanation of analytical hierarchy when $\mathcal{C} = \Sigma_n^1$, will be given in Section 4.3.

In what follows, the Rogers semilattice of $\mathcal{C}$-computable numberings of a family $\mathcal{S}$ will be denoted by $R_n^i(\mathcal{S})$ if $\mathcal{C} = \Sigma_n^i$. For Rogers semilattice of families of computably enumerable sets $R_1^0(\mathcal{S})$ , i.e. classical case when $\mathcal{C} = \Sigma_1^0$, we will omit indices and write as $R(\mathcal{S})$.

Understanding the algebraic properties of Rogers semilattices of $\mathcal{C}$-computable families is a crucial concern in the field of numberings theory. One of the significant properties is regarding cardinality and laticeness of Rogers semilattices. For the computable families of sets in well-known hierarchies and in the classical case, it has been demonstrated that the cardinality of Rogers semilattices may either be infinite or have one element, depending on the complexity and variety of the computable families involved [Goncharov and Sorbi, 1997], [Khutoretsky, 1971].

Historically, Yu. Ershov[Ershov, 1977a] raised the first two problems regarding cardinality and laticeness on Rogers semilattices $R(\mathcal{S})$ of families of computably enumerable sets:

(1) "What is the cardinality of a Rogers semilattice?"

(2) "Can a Rogers semilattice be a lattice?".

There are two main theorems that have resulted from most computability theorists efforts to resolve these problems.

**Theorem 2.3.3.** *([Khutoretsky, 1971]) Let $\mathcal{S}$ be a family of c.e. sets. If the Rogers semilattice $R(\mathcal{S})$ contains two elements then it is infinite.*

In fact, according to A. Khutoretsky, a linear ordering of order type $\omega$ is embedded into $R(\mathcal{S})$ above each (nongreatest) Rogers semilattice element. Furthermore, in 1974, S. Badaev proved that any nontrivial Rogers semilattice contains infinite chains, which can include any given (nongreatest and nonminimal) element of $R(\mathcal{S})$ [Badaev, 1974].

**Theorem 2.3.4.** *([Selivanov, 1976]) Let $\mathcal{S}$ be a family of c.e. sets. If the Rogers semilattice $R(\mathcal{S})$ contains two elements then it is not a lattice.*

S.Goncharov and A.Sorbi [Goncharov and Sorbi, 1997] completely solved both of Ershov's problems on the arithmetical hierarchy by showing that $\Sigma^0_{n+1}$-computability is connected to computations relative to the $0^{(n)}$-oracle.

**Theorem 2.3.5.** *([Goncharov and Sorbi, 1997]) If a family* $\mathcal{S} \subseteq \Sigma^0_{n+1}$ *contains at least two elements then the Rogers semilattice* $R^0_{n+1}(\mathcal{S})$ *is infinite and is not a lattice.*

A question that has been open for a long time is whether Khutoretsky's result holds for classes in Ershov's hierarchy [Badaev and Goncharov, 2000]. S. Badaev and S. Lempp [Badaev and Lempp, 2009] showed that the statement of Khutoretsky's Theorem fails for a set of families in the second level of the Ershov hierarchy.

The Rogers semilattices $R(\mathcal{S})$ of computably enumerable sets have been extensively studied by researchers including A. Mal'tsev [Mal'tsev, 1961], Yu. Ershov [Ershov, 1977b], S. Denisov [Denisov, 1970], A. Khutoretsky [Khutoretsky, 1971], S. Badaev [Badaev, 1974], [Badaev, 1977], V. Selivanov [Selivanov, 1976] and others.

Another question about Rogers semilattices is related to their algebraic and elementary properties. Specifically, researchers are interested in studying the different *isomorphism types* of Rogers semilattices of $\mathcal{C}$-computable families. To define them: the Rogers semilattices that are equivalent under the $\equiv$ relation are grouped together into *isomorphism types*. This question has been extensively studied for different families of Rogers semilattices. In the following, we'll provide a brief summary of some of the results related to this question.

For a non-zero natural number $n$, it was proven by S. Badaev, S. Goncharov, and A. Sorbi [Badaev et al., 2006] that any non-trivial Rogers semilattice of $\Sigma^0_m$-computable families is not isomorphic to a Rogers semilattice of $\Sigma^0_n$-computable families for any $m \geq n+3$. Note that a semilattice is non-trivial if it contains more than one element. Later, Podzorov [Podzorov, 2008] generalized this theorem and demonstrated that a similar result holds for $m \geq n+2$. However, it is still an open question whether this theorem holds true for $m = n+1$.

Badaev and Goncharov [Badaev and Goncharov, 2008] extended the result of the paper [Badaev et al., 2006] for the families of sets in the hyperarithmetical hierarchy. They showed that for any computable ordinals $\alpha > 0$ and $\beta \geq \alpha + 3$, any non-trivial Rogers semilattice of $\Sigma^0_\beta$-computable families cannot be isomorphic to a Rogers semilattice of $\Sigma^0_\alpha$-computable families.

In contrast, the situation is different for the Ershov hierarchy. The class of all $n$-c.e. sets for a non-zero natural number $n$ is denoted by $\Sigma^{-1}_n$. The result of Herbert, Jain, Lempp, Mustafa, and Stephan [Herbert et al., 2019] asserts that every Rogers semilattice of $\Sigma^{-1}_n$-computable families is also a Rogers semilattice of $\Sigma^{-1}_{n+1}$-computable families.

The study of isomorphism types of Rogers semilattices of $\Sigma^1_n$-computable families began in [Bazhenov and Mustafa, 2020]. In that work, it was proven that there are at least four pairwise non-isomorphic Rogers semilattices for $\Sigma^1_n$-computable families. However, in Chapter 4 of this thesis, which was written in collaboration with M. Mustafa and N. Bazhenov [Bazhenov et al., 2022], we obtained that under the assumption of **PD**, there are infinitely many pairwise elementarily non-equivalent Rogers $\Sigma^1_n$-semilattices (Theorem 4.5.1). The theory of recursion for subsets of natural numbers $\omega$ belonging to the levels of analytical hierarchy was originally developed by [Tanaka, 1978], based on the assumption of Projective Determinacy (**PD**). *Projective Determinacy* (**PD**) is a fundamental principle in set theory that asserts the determinacy of games played between two players. Specifically, it states that for any projective set $A$, which is a collection of infinite sequences of natural numbers, there exists a winning strategy for one of the players in the game $G(A)$. In other words, the outcome of the game is predetermined and not left to chance. This principle has far-reaching consequences in various branches of mathematics and has led to significant advancements in the field of set theory. More information about **PD** can be found in Section 4.4.

Since the late 1960s, research in the theory of numberings has primarily focused on Rogers semilattices. The research problems raised by Yu. Ershov [Ershov, 1967, Ershov, 1968, Ershov, 1977b] for Rogers semilattices of the families of sets from arithmetical, analytical, and Ershov hierarchies were thoroughly investigated and mostly solved. However, the study of Rogers semilattices of punctual numberings, polynomial-time numberings, as well as limitwise monotonic numberings, has become a new area of research. Therefore, the next chapter will focus on investigating Rogers semilattices of limitwise monotonic numberings.

# Chapter 3

# Rogers semilattices of limitwise monotonic numbering

This chapter focuses on numberings for a specific subclass of $\Sigma^0_2$-sets called limitwise monotonic sets. We explore the concept of *limitwise monotonic numberings* for these sets, which form a crucial tool in the field of computable structure theory. Our primary objective is to investigate the Rogers semilattices $R_{lm}(\mathcal{S})$ induced by the set of all limitwise monotonic numberings of a family of subsets of natural numbers. We use the arithmetical hierarchy as a framework for our study.

## 3.1 Motivation

Khisamiev's research, as described in [Khisamiev, 1998], emphasizes the significance of not only the complexity of a set in arithmetic terms, but also how it is constructed. His theory proposes that computability deals with dynamic enumerations, which affects objects in computable structures. Although Khisamiev's example focuses on abelian p-groups, we can use his theory to analyze which equivalence structures can be computed. One noteworthy feature of the equivalence class of $x$, where $x$ is any element of a gicven set $X$, is that it can only grow in size if we have a computable equivalence relation. Once $x$ is included in the universe of the relation, we may find that it has $n$ elements. As time passes, the class only gains more elements, which is explained by the concept of limitwise monotonicity. Functions and sets that are limitwise monotonic play a crucial role in the theory of computable structures.

## 3.2 Limitwise monotonic numberings

For a non-zero natural number $n$, a numbering $\nu$ of a family $\mathcal{S}$ is said to be $\Sigma^0_n$-computable if the set

$$G_\nu = \{\langle k, x\rangle : x \in \nu(k)\}$$

belongs to the class $\Sigma^0_n$. It is worth noting that the classical notion of computable numberings is a synonym for $\Sigma^0_1$-computable numberings. If a family $\mathcal{S} \subset P(\omega)$ has a $\Sigma^0_n$-computable numberings, it is called $\Sigma^0_n$-computable.

In this section, we explore numbering systems for *limitwise monotonic sets*, which are a specific subclass of $\Sigma^0_2$-sets.

**Definition 3.2.1.** A total function $F\colon \omega \to \omega$ is *limitwise monotonic* (or an *s-function*) if there is a computable function $f(x, s)$ with the following properties:

- $f(x, s) \leq f(x, s+1)$ for all $x$ and $s$, and
- $F(x) = \lim_s f(x, s)$ for all $x$.

Such a function $f$ is often called a *limitwise monotonic approximation* of the function $F$.

**Definition 3.2.2.** A set $A \subseteq \omega$ is *limitwise monotonic* if either $A = \emptyset$, or $A$ is the range of a limitwise monotonic function.

The $s$-function is a mathematical tool that was first introduced by Khisamiev to study computable abelian $p$-groups [Khisamiev, 1998]. The function is used to approximate $\Sigma^0_2$-sets in a limitwise monotonic manner, which has led to the development of a new trend in the theory of computable structures. This trend involves encoding a given set $A$ based not only on its level of complexity, but also on the properties of a dynamic enumeration of the set.

In a paper published in 1997, Coles, Downey, and Khoussainov introduced the term "limitwise monotonic function" to describe the method of using limitwise monotonic sets in the construction of computable linear orders [Coles et al., 1997]. The paper used this technique to construct computable linear orders that have $\Pi^0_2$ initial segments that cannot be computably presented. This work has contributed significantly to the understanding of computable structures and their properties.

In modern computability theory, limitwise monotonic functions and their generalizations have become essential tools. They are widely used by computability theorists and are considered standard. To learn about the applications of limitwise monotonic sets and functions, one can refer to the survey [Frolov and Zubkov, 2021]. The authors of the survey demonstrate the connection between limitwise monotonic functions and computable presentations of linear orders with condensation. They also provide examples of the technique's applications in solving fundamental problems of the theory of computable linear orders. In the following text, we will refer to the term *limitwise monotonic* as *l.m.*.

We are studying families of limitwise monotonic sets $\mathcal{S}$ and their numberings called limitwise monotonic numbering. Essentially, we are using the same definition as in Definition 1.2 in [Kalimullin et al., 2013]. In this definition, we use the notation where $F$ is a function that acts from $\omega^{n+1}$ to $\omega$, for $n \geq 1$. For a set of natural numbers $k_1, k_2, \ldots, k_n$, the unary function

$$g \colon y \mapsto F(k_1, k_2, \ldots, k_n, y).$$

can be denoted by $F(k_1, k_2, \ldots, k_n, \cdot)$.

**Definition 3.2.3.** A numbering $\nu$ is *limitwise monotonic* if there exists a computable function $f(k, z, s)$ with the following properties:

1. $f(k, z, s) \leq f(k, z, s+1)$, for all $k, z, s$.
2. For every $k$ and $z$, there exists a finite limit $F(k, z) = \lim_s f(k, z, s)$.
3. For every $k$, the set $\nu(k)$ is equal to the range of the function $F(k, \cdot)$.

Such a function $f$ is called a *limitwise monotonic approximation* of the numbering $\nu$.

A family $\mathcal{S}$ is said to be *limitwise monotonic* if it has a limitwise monotonic numbering. In simpler terms, l.m. families $\mathcal{S}$ are those that have a uniform l.m. approximation. For a l.m. family $\mathcal{S}$, the Rogers semilattice induced by the l.m. numberings of $\mathcal{S}$ is denoted by $R_{lm}(\mathcal{S})$. The Rogers semilattice of $\Sigma^0_2$-computable families is denoted by $R^0_2(\mathcal{S})$.

There is a well-known result that states:

**Lemma 3.2.4.** *Every l.m. numbering $\nu$ is $\Sigma^0_2$-computable.*

*Proof.* Assume that $f(k,z,s)$ is a l.m. approximation of a numbering $\nu$. Then, we can infer that:

$$x \in \nu(k) \;\Leftrightarrow\; \exists z \exists s \forall t[t \geq s \rightarrow f(k,z,t) = f(k,z,s) = x].$$

Hence, it can be concluded that $\nu$ is $\Sigma^0_2$-computable. □

Lemma 3.2.4 implies the following:

*Remark* 3.2.5. If $\mathcal{S}$ is a l.m. family, then the structure $R_{lm}(\mathcal{S})$ forms an ideal within $R^0_2(\mathcal{S})$.

## 3.3 Reductions for $\Sigma^0_2$-computable numberings

We discuss the concept of reductions between different types of numberings, following the approach of the reference [Herbert et al., 2019]. Specifically, we focus on our first reduction $\Gamma$ described in Subsection 3.3.1, which transforms a $\Sigma^0_2$-computable family into a l.m. family. Despite its simplicity, this reduction is quite useful. In fact, we can immediately conclude that every Rogers $\Sigma^0_2$-semilattice $R^0_2(\mathcal{S})$ is isomorphic to the semilattice $R_{lm}(\Gamma(\mathcal{S}))$.

In Subsection 3.3.2, it is shown that there exist some families of l.m. numberings $\mathcal{S}$, for which the identity mapping $\nu \mapsto \nu$ can be considered as a surjective reduction from l.m. numberings to $\Sigma^0_2$-computable numberings. For these families $\mathcal{S}$, we have the following equality: $R_{lm}(\mathcal{S}) = R^0_2(\mathcal{S})$.

In the last subsection, we present several interesting corollaries that can be deduced using known results on numberings in the arithmetical hierarchy.

### 3.3.1 From $\Sigma^0_2$-computable to limitwise monotonic

It is widely acknowledged that the facts presented below are well-known. (Please refer to Theorem 2.1 in [Downey et al., 2012] for more information.)

- Every l.m. set is $\Sigma^0_2$.
- For every $\Sigma^0_2$ set $A \subseteq \omega$, the set $A \oplus \omega$ is l.m.

Our reduction, denoted by $\Gamma$, relies on the second fact.

**Definition 3.3.1.** Let $A \subseteq \omega$ and define $\Gamma(A) := A \oplus \omega$. For a given numbering $\nu$, let $\Gamma(\nu)$ be the numbering defined as follows: for $k \in \omega$, define

$$(\Gamma(\nu))(k) := \Gamma(\nu(k)).$$

We define $\Gamma(\mathcal{S}) = \{\Gamma(A) : A \in \mathcal{S}\}$ for a family $\mathcal{S}$.

**Theorem 3.3.2.** *Let $\mathcal{S}$ be a $\Sigma^0_2$-computable family. Then the following holds:*

1. *The family $\Gamma(\mathcal{S})$ is limitwise monotonic.*
2. *The operator $\Gamma$ is a bijection from the set of all $\Sigma^0_2$-computable numberings of $\mathcal{S}$ onto the set of all l.m. numberings of $\Gamma(\mathcal{S})$.*
3. *A numbering $\nu \in Com_{\Sigma^0_2}(\mathcal{S})$ is positive if and only if $\Gamma(\nu)$ is positive. A similar fact is true for Friedberg numberings.*
4. *For any $\nu, \mu \in Com_{\Sigma^0_2}(\mathcal{S})$, we have $\nu \leq \mu$ if and only if $\Gamma(\nu) \leq \Gamma(\mu)$.*

*Consequently, the semilattices $\mathcal{R}^0_2(\mathcal{S})$ and $\mathcal{R}_{lm}(\Gamma(\mathcal{S}))$ are isomorphic.*

*Proof.* (1) This is a straightforward corollary of Theorem 2.1 from [Downey et al., 2012].

Let $\nu$ be a computable numbering of a family $\mathcal{S}$, where $\nu$ is $\Sigma^0_2$-computable. Then, there exists a computable ternary predicate $R(k, x, y)$ such that

$$x \in \nu(k) \ \Leftrightarrow \ (\exists^{<\infty} y) R(k, x, y).$$

Assuming $R(k, x, 0)$ is true for all $k, x$, we construct the l.m. approximation $f(k, z, s)$ for the numbering $\Gamma(\nu)$.

We define

$$f(k, \langle x, t\rangle, s) = \begin{cases} 2x, & \text{if } \operatorname{card}(\{y \leq s : R(k, x, y)\}) \leq t, \\ 2x + 1, & \text{otherwise.} \end{cases}$$

(2)Let's establish the following: we need to construct a numbering $\nu$, belonging to $Com_{\Sigma^0_2}(\mathcal{S})$, such that it is equal to the given l.m. numbering $\mu$ of the family $\Gamma(\mathcal{S})$, i.e. $\mu = \Gamma(\nu)$.

We start by fixing a l.m. approximation $f(k, z, s)$ of the numbering $\mu$. Similarly to Lemma 3.2.4, we define a $\Sigma^0_2$-computable numbering $\nu$ in the following way:

$$x \in \nu(k) \ \Leftrightarrow \ \exists z \exists s \forall t [t \geq s \rightarrow f(k, z, t) = f(k, z, s) = 2x].$$

It is clear that $\nu(k) = \{x : 2x \in \text{range}(\lim_s f(k,\cdot,s))\}$. Therefore, $\mu = \Gamma(\nu)$.

(3) This property holds: $A \subseteq B$ iff $\Gamma(A) \subseteq \Gamma(B)$. Moreover, notice that this property implies that $\nu(k) = \nu(l)$ iff $(\Gamma(\nu))(k) = (\Gamma(\nu))(l)$.

(4) Observing that a computable function $f$ gives a reduction $\nu \le \mu$ if and only if $f$ provides reduction $\Gamma(\nu) \le \Gamma(\mu)$. □

### 3.3.2 From limitwise monotonic to $\Sigma_2^0$-computable

**Theorem 3.3.3.** *Suppose that a family $\mathcal{S}$ is limitwise monotonic and each set $A \in \mathcal{S}$ is infinite. Then its numbering $\nu$ is limitwise monotonic if and only if $\nu$ is $\Sigma_2^0$-computable. Hence, the semilattices $R_{lm}(\mathcal{S})$ and $R_2^0(\mathcal{S})$ are equal.*

*Proof.* Lemma 3.2.4 implies that it is sufficient to establish that every $\Sigma_2^0$-computable numbering $\nu$ of $\mathcal{S}$ admits a l.m. approximation $f(k,z,s)$.

Based on Theorem 2.1 in [Downey et al., 2012], we can choose a computable predicate $R(k,x,y)$ that satisfies the given conditions:

$$x \in \nu(k) \;\Leftrightarrow\; (\exists^{<\infty} y) R(k,x,y).$$

To simplify the construction, we will focus on creating a l.m. approximation for a single set $\nu(k)$. However, it's worth noting that once we have constructed this approximation, extending it to all sets $\nu(l)$, $l \in \omega$ is a straightforward process since the construction is uniform across all values of $k \in \omega$.

At each stage $s$, we define a finite non-empty set $F_{x,s}$ for every $x \le s$. It's important to note that the sets $F_{x,s}$ and $F_{x',s}$ do not intersect if $x$ and $x'$ are distinct. Additionally, the values $f(k,z,s)$ must satisfy the following conditions:

- if $z \in F_{x,s}$, then $f(k,z,s) = x$;
- if $z \notin \bigcup_{x \le s} F_{x,s}$, then $f(k,z,s) = 0$.

*Stage* 0. Put $F_{x,0} = \{0\}$.

*Stage* $s+1$. Find the least number $w \notin \bigcup_{x \le s} F_{x,s}$, and set $f(k,w,s+1) = s+1$ and $F_{s+1,s+1} = \{w\}$.

For each $x \le s$ in turn, proceed as follows.

1. If $\neg R(k,x,s+1)$, then for each $z \in F_{x,s}$, set $f(k,z,s+1) = x$. Furthermore, if the value of $x$ is equal to zero, then we define $F_{x,s+1}$ to be the same as $F_{x,s}$.

2. If $R(k, x, s+1)$ is true, then for every $z$ in the set $F_{x,s}$, assign the value $x+1$ to $f(k, z, s+1)$. Next, determine the smallest value of $u$ such that $f(k, u, s+1)$ has not been defined, and set $f(k, u, s+1) = x$. Finally, if $x = 0$, define $F_{0,s+1}$ as the set containing only $u$, i.e. $F_{0,s+1} = \{u\}$.

After that, for every non-zero $x \leq s$, we define

$$F_{x,s+1} = \{z : f(k, z, s+1) = x\}.$$

The construction is now complete. It is noteworthy that for any given values of $z$ and $s$, the function $f(k, z, s)$ is always less than or equal to the value $f(k, z, s+1)$.

It is evident that for every natural number $z$, there exists a corresponding number $x^*$ and a stage $s^*$ such that for all stages $s$ greater than or equal to $s^*$, $z$ is an element of $F_{x^*,s}$ and $R(k, x^*, s)$ is false. This leads to the conclusion that the limit of $f(k, z, s)$ as $s$ tends to infinity is equal to $x^*$, i.e. $\lim_s f(k, z, s) = x^*$ and $x^* \in \nu(k)$.

Notice that we use the fact that the set $\nu(k)$ is infinite. Roughly speaking, the sequence of values $f(k, z, 0), f(k, z, 1), f(k, z, 2), \ldots$ will increase:

$$0, 1, 2, \ldots, m, \ldots,$$

until a sequence stabilizes on a large enough number, denoted by $x^* \in \nu(k)$. This allows us to deduce that the function $F(k, z) = \lim_s f(k, z, s)$ is well-defined and that the range of $F(k, \cdot)$ is a subset of $\nu(k)$.

Conversely, for every $x \in \nu(k)$, there exists a stage $s'$ such that $F_{x,s'} \neq \emptyset$ and $\neg R(k, x, s)$ for all $s \geq s'$. This means that for every $z^* \in F_{x,s'}$, we have $\lim_s f(k, z^*, s) = x$. We can conclude that $\nu(k) = \text{range}(F(k, \cdot))$, and that $f(k, z, s)$ is a l.m. approximation of the numbering $\nu$.

□

The next result shows that Theorem 3.3.3 cannot be extended to more general families $\mathcal{S}$.

**Proposition 3.3.4.** *There exists an infinite l.m. family $\mathcal{S}$ such that:*

1. *$\mathcal{S}$ contains precisely one finite set $F$, and*
2. *there exist infinitely many pairwise non-equivalent, $\Sigma^0_2$-computable numberings of $\mathcal{S}$ which are not l.m.*

*Proof.* The family $\mathcal{S}$ contains the following sets:

- $F = \{0\}$, and
- $A_i = \{\langle i, l\rangle + 1 : l \in \omega\}$, for $i \in \omega$.

Let $\nu$ be an arbitrary numbering of the family $\mathcal{S}$. We will show that the index set

$$I_\nu(F) = \{k : \nu(k) = F\}$$

is co-c.e. It is essential to select a l.m. approximation $f(k, z, s)$ of the numbering $\nu$. This allows us to obtain the following result:

$$\nu(k) \neq F \ \Leftrightarrow\ \exists z \exists s [f(k, z, s) \geq 1].$$

One can define a $\Sigma^0_2$-computable numbering $\mu_X$ for a non-empty $\Delta^0_2$ set $X \subsetneq \omega$. We set $\mu_X(2i + 1) = A_{i+1}$ for $i \in \omega$, and

$$\mu_X(2i) = \begin{cases} F, & \text{if } i \in X, \\ A_0, & \text{if } i \notin X. \end{cases}$$

It is clear that $\mu_X \leq \mu_Y$ if and only if $X \leq_m Y$.

Notice that $I_{\mu_X}(F) = \{2i : i \in X\}$. Hence, if a $\Delta^0_2$ set $X$ is not co-c.e., then its numbering $\mu_X$ cannot be equivalent to a l.m. numbering. □

### 3.3.3 Consequences

**Corollary 3.3.5.** *There exist l.m. families $\mathcal{S}_i$, $i \in \omega$, such that the corresponding Rogers semilattices $R_{lm}(\mathcal{S}_i)$ are pairwise not elementarily equivalent. Consequently, there are infinitely many isomorphism types of Rogers l.m. semilattices.*

*Proof.* Badaev, Goncharov, and Sorbi [Badaev et al., 2005] proved the existence of $\Sigma^0_2$-computable families $\mathcal{T}_i$, $i \in \omega$, such that the theories $Th(\mathcal{R}^0_2(\mathcal{T}_i))$ are pairwise different. It follows from Theorem 3.3.2 that $\mathcal{S}_i := \Gamma(\mathcal{T}_i)$ can be chosen. □

**Corollary 3.3.6.** *Suppose that $\mathcal{S}$ is a l.m. family such that the Rogers semilattice $R_{lm}(\mathcal{S})$ is not one-element. Then for any $n \geq 4$ and any $\Sigma^0_n$-computable family $\mathcal{T}$, the semilattices $R^0_n(\mathcal{T})$ and $R_{lm}(\mathcal{S})$ are not isomorphic.*

*Proof.* Essentially, this corollary is a consequence of Theorem 4 of [Podzorov, 2008]. There exists a bounded distributive semilattice $\mathcal{L}$ which has a $\Sigma^0_5$-presentation but does not have a $\Sigma^0_4$-presentation, as proven in Lemma 8 of [Podzorov, 2008]. Boundedness means that $\mathcal{L}$ has both a least and a greatest element.

Assuming that $\mathcal{R}^0_n(\mathcal{T})$ is not a one-element set, Corollary 4 of [Podzorov, 2008] implies that the semilattice $\mathcal{R}^0_n(\mathcal{T})$ has an ideal which is isomorphic to $\mathcal{L}$.

We assume that $R^0_n(\mathcal{T})$ and $R_{lm}(\mathcal{S})$ are isomorphic. However, this leads to a contradiction. According to Remark 3.2.5, $R_{lm}(\mathcal{S})$ is an ideal within $R^0_2(\mathcal{S})$. Hence, $R^0_2(\mathcal{S})$ must have an ideal isomorphic to $\mathcal{L}$. On the other hand, Lemma 7 of [Podzorov, 2008] implies that for any $a, b \in R^0_2(\mathcal{S})$, the segment $[a, b]$ inside $R^0_2(\mathcal{S})$ is bounded distributive semilattice which has a $\Sigma^0_4$-presentation. Hence, the semilattice $\mathcal{L}$ (as a segment inside $R^0_2(\mathcal{S})$) admits a $\Sigma^0_4$-presentation; this gives a contradiction. Therefore, we can conclude that $R^0_n(\mathcal{T}) \ncong R_{lm}(\mathcal{S})$. $\square$

**Corollary 3.3.7.** *There exists a l.m. family $\mathcal{S}$ which admits infinitely many pairwise non-equivalent l.m. Friedberg numberings.*

*Proof.* Corollary 2.1 of [Goncharov and Sorbi, 1997] shows that the family $\mathcal{C}$ of all $\Sigma^0_2$ sets has infinitely many pairwise non-equivalent $\Sigma^0_2$-computable Friedberg numberings. By Theorem 3.3.2, the l.m. family $\mathcal{S} := \Gamma(\mathcal{C})$ satisfies the desired condition. $\square$

**Corollary 3.3.8.** *Let $\mathcal{S}$ be an infinite l.m. family such that every set $A \in \mathcal{S}$ is infinite. Then:*

*(a) The semilattice $R_{lm}(\mathcal{S})$ contains infinitely many minimal elements. Consequently, $R_{lm}(\mathcal{S})$ is infinite, and it is not a lattice.*

*(b) If $\mathcal{S}$ has a positive l.m. numbering, then it has a Friedberg l.m. numbering.*

*(c) If $\mathcal{S}$ has a positive l.m. numbering, then it has infinitely many pairwise non-equivalent, positive and undecidable l.m. numberings.*

## 3.4 Distinctions between Rogers $\Sigma^0_2$-semilattices and limitwise monotonic

Based on the findings of the previous section, the following question arises:

> Is it possible that all problems related to l.m.families can be reduced to the investigations of $\Sigma^0_2$-computable families?

However, this section aims to show that this is not true in general. To provide a better understanding, we present two examples.

Let's consider an example related to Corollary 3.3.8.(b). According to Proposition 2.7 of [Goncharov and Sorbi, 1997], if an infinite $\Sigma^0_2$-computable family has a positive $\Sigma^0_2$-computable numbering, then it also has a Friedberg $\Sigma^0_2$-computable numbering. However, it is important to note that this result is not valid for l.m. families.

**Proposition 3.4.1.** *There is an infinite l.m. family $\mathcal{S}$ with a positive l.m. numbering, but no Friedberg l.m. numberings.*

*Proof.* Khoussainov, Nies, and Shore constructed a d.c.e. set $D$ which is not limitwise monotonic. This is described in Lemma 2.6 of their paper [Khoussainov et al., 1997]. The similar result is also presented in Theorem 2.2 of a paper by Downey, Kach, and Turetsky [Downey et al., 2012]. It is important to note that $D$ is neither c.e., nor co-c.e. For the sake of simplicity, we can assume that $0 \notin D$. The family $\mathcal{S}$ we want to define is as follows:

$$\mathcal{S} = \{\{x\} : x \in D\} \cup \{\omega\}.$$

Since $D$ is d.c.e., there is a computable $\{0,1\}$-valued function $h(x,s)$ such that for all $x$,

- $h(x,0) = 0$,
- $\text{card}(\{s : h(x,s) \neq h(x,s+1)\}) \leq 2$, and
- $D(x) = \lim_s h(x,s)$.

We construct a numbering $\nu$ of the family $\mathcal{S}$ that is positive and l.m. by using its l.m. approximation function $f(k,z,s)$.

*Stage* 0. Put $f(k,z,0) = 0$ for all $k, z$.

*Stage* $s+1$. For each non-zero $m \leq s+1$, consider the following four cases.

*Case 1.* If $h(m,s) = 0$ and $h(m,s+1) = 1$, then find the least number $u_m$ such that $f(u_m,0,s) = 0$ and the value $f(u_m,0,s+1)$ is still undefined. Set $f(u_m,0,s+1) = m$.

*Case 2.* If $h(m,s) = h(m,s+1) = 1$, then we have already defined the number $u_m$ associated with $m$. We define $f(u_m,z,s+1) = m$ for all $z \leq s+1$ and $f(u_m,z',s+1) = 0$ for all $z' > s+1$.

*Case 3.* If $h(m, s) = 1$ and $h(m, s+1) = 0$, then find the least value $z_0$ such that $f(u_m, z_0, s) = 0$. Set

$$f(u_m, z, s+1) = \begin{cases} m, & \text{if } z < z_0, \\ z - z_0, & \text{if } z \geq z_0. \end{cases}$$

*Case 4.* If the value of $u_m$ has already been determined, then there is no need to modify the values of $f(u_m, z, \cdot)$.

The construction is now complete, and it is clear that the constructed function $f(k, z, s)$ is a l.m. approximation.

As the set $D$ is infinite, we can find the unique number $m > 0$ for each element $k \in \omega$, such that $k = u_m$. It is essential to note that this $m$ must satisfy $\mathrm{card}(\{s : h(m, s) \neq h(m, s+1)\}) \in \{1, 2\}$. When $k = u_m$, we get:

$$\nu(k) = \begin{cases} \{m\}, & \text{if } \mathrm{card}(\{s : h(m, s) \neq h(m, s+1)\}) = 1, \\ \omega, & \text{if } \mathrm{card}(\{s : h(m, s) \neq h(m, s+1)\}) = 2. \end{cases}$$

That means, the numbering $\nu$ indexes our family $\mathcal{S}$ with precision.

Suppose that $u_m = k \neq l = u_n$. Then the following conditions are equivalent: $\nu(k) = \nu(l)$ if and only if $\nu(k) = \nu(l) = \omega$ if and only if

$$\mathrm{card}(\{s : h(m, s) \neq h(m, s+1)\}) = \mathrm{card}(\{s : h(n, s) \neq h(n, s+1)\}) = 2.$$

Hence, we can conclude that the equivalence relation $\eta_\nu$ is c.e., and $\nu$ is a positive numbering.

Towards a contradiction, suppose that $\mu$ is a Friedberg l.m. numbering of the family $\mathcal{S}$. Without loss of generality, let $\mu(0) = \omega$. Choose an l.m. approximation $f_\mu(k, z, s)$ of the numbering $\mu$.

Since $\mu(k+1)$ only consists of one element for every $k \in \omega$, we can deduce that the limitwise monotonic function $F(k) := \lim_s f_\mu(k+1, 0, s)$ also has only one element and its range is equal to $D$. This contradicts the fact that $D$ is not limitwise monotonic. Therefore, the family $\mathcal{S}$ does not have Friedberg limitwise monotonic numberings. Thus, we can confirm the validity of Proposition 3.4.1. $\square$

Our second example demonstrates that certain Rogers semilattices of computable families can be realized as Rogers l.m. semilattices.

**Proposition 3.4.2.** *Consider the following finite families for a non-zero $n \in \omega$:*

$$\mathcal{S}_n = \{\emptyset, \{0\}, \{0,1\}, \{0,1,2\}, \dots, \{0,1,2,\dots,n-1\}\},$$
$$\mathcal{T}_n = \{\{0\}, \{1\}, \{2\}, \{3\}, \dots, \{n\}\}.$$

*The Rogers semilattices $\mathcal{R}^0_1(\mathcal{S}_n)$ and $\mathcal{R}_{lm}(\mathcal{T}_n)$ are isomorphic.*

*Proof.* We define an operator $\Delta$ between types of numberings, similarly to Section 3.3.1. We put

$$\Delta(\emptyset) = \{0\}, \text{ and } \Delta(\{0,1,\dots,m\}) = \{m+1\}.$$

If $\nu$ is a numbering of the family $\mathcal{S}_n$, then we define

$$(\Delta(\nu))(k) := \Delta(\nu(k)).$$

It is sufficient to establish the following two facts, similarly to Theorem 3.3.2:

(i) The $\Delta$ operator is a bijection from the set of all $\Sigma^0_1$-computable numberings of $\mathcal{S}_n$ onto the set of all l.m. numberings of $\mathcal{T}_n$.

(ii) For any $\nu, \mu \in Com_{\Sigma^0_1}(\mathcal{S}_n)$, we have $\nu \leq \mu$ if and only if $\Delta(\nu) \leq \Delta(\mu)$.

(i) Suppose $\nu$ is a $\Sigma^0_1$-computable numbering of the family $\mathcal{S}_n$. There exists a strongly computable sequence $(V_{k,s})_{k,s\in\omega}$ of finite sets such that $V_{k,s} \subseteq V_{k,s+1}$ and $\nu(k) = \bigcup_{s\in\omega} V_{k,s}$. This allows us to define an l.m. approximation $f(k,z,s)$ of the numbering $\Delta(\nu)$ in the following manner:

$$f(k,z,s) = \begin{cases} 0, & \text{if } V_{k,s} = \emptyset, \\ 1 + \max V_{k,s}, & \text{if } V_{k,s} \neq \emptyset. \end{cases}$$

Therefore, we can deduce that $\Delta(\nu)$ is limitwise monotonic numbering.

Let $\mu$ be a l.m. numbering of the family $\mathcal{T}_n$, and $f_\mu(k,z,s)$ be a l.m. approximation of $\mu$. We construct a $\Sigma^0_1$-computable numbering $\nu$ of $\mathcal{S}_n$ by:

$$x \in \nu(k) \;\Leftrightarrow\; \exists s[x < f_\mu(k,0,s)].$$

It is clear that $\Delta(\nu) = \mu$.

(ii) This is the result of the following observation. Let $\nu$ and $\nu'$ be numberings of $\mathcal{S}_n$. Then for all $k, l \in \omega$, we have $\nu(k) = \nu'(l)$ if and only if $(\Delta(\nu))(k) = (\Delta(\nu'))(l)$. Proposition 3.4.2 is proven. □

Recall that it is known that the Rogers semilattice $R^0_1(\mathcal{S}_1)$ is isomorphic to the upper semilattice of c.e. $m$-degrees.

*Observation* 3.4.1. Consider a non-zero number $n$. Let $P_n$ be the poset obtained by removing the greatest element from the finite poset $(\mathcal{S}_n, \subseteq)$. It is worth noticing that if $m$ and $n$ are different, then the posets $P_m$ and $P_n$ are not isomorphic. According to a result by Ershov [Ershov, 2003], this implies that the semilattices $R^0_1(\mathcal{S}_m)$ and $R^0_1(\mathcal{S}_n)$ are not isomorphic.

As a result, Proposition 3.4.2 transfers infinitely many isomorphism types of Rogers $\Sigma^0_1$-semilattices into the limitwise monotonic setting.

To the best of our knowledge, it is still unknown whether a computable family $\mathcal{T}$ exists such that the semilattice $R^0_2(\mathcal{T})$ is isomorphic to $R^0_1(\mathcal{S}_n)$ for some $n \geq 1$.

## 3.5 Cardinalities, and nonlattices

Ershov [Ershov, 1977b] brought up two problems regarding cardinalities and nonlatice on Rogers semilattices for computable families [Khutoretsky, 1971]. Khutoretsky, in 1971, proved that any Rogers semilattice $R^0_1(\mathcal{S})$ with more than one element must be infinite. Selivanov [Selivanov, 1976] established that an infinite $R^0_1(\mathcal{S})$ cannot be a lattice, in 1976.

This section considers these two problems in the limitwise monotonic setting. We obtain the following results:

**Theorem 3.5.1.** *[Bazhenov et al., 2022] Suppose that a l.m. family $\mathcal{S}$ contains at least two elements. Then the semilattice $\mathcal{R}_{lm}(\mathcal{S})$ is infinite, and it is not a lattice.*

The proof identifies and analyzes three distinct cases.

*Case (i).* The family $\mathcal{S}$ is finite.

*Case (ii).* The family $\mathcal{S}$ is infinite, and it contains a finite set.

*Case (iii).* Otherwise, the family $\mathcal{S}$ is infinite, and every set $A \in \mathcal{S}$ is infinite. This result was previously obtained in Corollary 3.3.8 (a).

**Proof for Case (i)**: Suppose that the family $\mathcal{S}$ is equal to

$$\{A, B\} \cup \{C_i : i < n\} \text{ for some } n \geq 0.$$

Fix limitwise monotonic approximations $f_A(z, s)$ and $f_B(z, s)$ of the sets $A$ and $B$.

One can assume without loss of generality that either of two conditions holds:

- either $B$ is infinite, or
- both sets $A$ and $B$ are finite, and $\max A \leq \max B$.

This assumption ensures that values $v_{z,s}$ and $p_{z,s}$ can be effectively found for any $z$ and $s$ such that

$$p_{z,s} \geq s+1, \text{ and } f_B(v_{z,s}, p_{z,s}) \geq f_A(z,s). \tag{3.5.1}$$

Let $U$ be a c.e. set such that $U \neq \emptyset$ and $U \neq \omega$. We define a numbering $\nu_U$ of the family $\mathcal{S}$ as follows:

$$\nu_U(i) = C_i, \text{ for } i < n; \quad \nu_U(k+n) = \begin{cases} A, & \text{if } k \notin U, \\ B, & \text{if } k \in U. \end{cases}$$

It is clear that the numberings $\nu_U$ satisfy the following properties:

(a) $\nu_U \leq \nu_W$ if and only if $U \leq_m W$.

(b) If $\mu \leq \nu_U$, then there exists a c.e. set $W$ such that $\mu \equiv \nu_W$. Indeed, suppose that a computable function $h$ provides a reduction $\mu \leq \nu_U$. Then one can define $W$ as $h^{-1}(\{k+n : k \in U\})$.

It is sufficient to demonstrate that the numbering $\nu_U$ is limitwise monotonic. Once we prove this, we can combine Properties (a) and (b) to deduce that the semilattice $R_{lm}(\mathcal{S})$ has an ideal that is isomorphic to the upper semilattice of computably enumerable $m$-degrees. This confirms that $R_{lm}(\mathcal{S})$ is infinite and not a lattice. It's worth noting that the semilattice of c.e. $m$-degrees is upwards dense [Denisov, 1970], and it is not a lattice [Ershov, 1969].

Fix a strongly computable sequence of finite sets $(U_s)_{s\in\omega}$. This sequence should satisfy the conditions that $U_0 = \emptyset$, $U_s \subseteq U_{s+1}$ and $U = \bigcup_{s\in\omega} U_s$. We also need to define an approximation $f(l,z,s)$ for the numbering $\nu_U$. It is important to note that we only need to consider indices $l \geq n$.

*Stage* 0. Put $f(l,z,0) = 0$ for all $l, z \in \omega$.

*Stage* $s+1$. For each $k \in \omega$, consider the following three cases.

*Case 1.* If $k \notin U_{s+1}$, then set $f(k+n, z, s+1) = f_A(z, s+1)$ for all $z \leq s+1$. If $z > s+1$, then $f(k+n, z, s+1) = 0$.

*Case 2.* If $k \in U_{s+1} \setminus U_s$, then for each $z \leq s$, define auxiliary values $m_{k,z} := v_{z,s}$ and $t_{k,z} := p_{z,s} - s - 1$ (the values $v_{z,s}$ and $p_{z,s}$ are taken from Eq. (3.5.1)). Also define $b_k := s$.

Set $f(k+n,z,s+1) = f_B(v_{z,s}, p_{z,s})$ for $z \leq s$, and $f(k+n,z,s+1) = f_B(z-s-1, s+1)$ for $z > s$.

*Case 3.* If $k \in U_{s+1} \cap U_s$, then define $f(k+n,z,s+1) = f_B(m_{k,z}, t_{k,z}+s+1)$ for $z \leq b_k$, and $f(k+n,z,s+1) = f_B(z-b_k-1, s+1)$ for $z > b_k$.

Now, it is easy to show that $f(k,z,s)$ is a l.m. approximation of the numbering $\nu_U$.

**Proof for Case (ii)**

Fix a finite set $F$ from the family $\mathcal{S}$. Then, consider the new family

$$\widehat{\mathcal{S}} = \{A \in \mathcal{S} : \exists y (y \in A \ \&\ y > \max F)\}.$$

It is clear that the set $\mathcal{S} \setminus \widehat{\mathcal{S}}$ is finite. To simplify things, we assume that $\mathcal{S} \setminus \widehat{\mathcal{S}}$ only contains one element, denoted by $F$. However, we can easily prove the general case where $\mathcal{S} \setminus \widehat{\mathcal{S}} = \{F\} \cup \{C_i : i < n\}$, for some $n \geq 0$, by using a similar construction.

Suppose $f_\xi(k,z,s)$ is a l.m. approximation of a fixed l.m. numbering $\xi$ of a family $\mathcal{S}$. We recover a l.m. numbering $\widehat{\mu}$ of the family $\widehat{\mathcal{S}} = \mathcal{S} \setminus \{F\}$.

The set $W = \{k : \exists z \exists s [f_\xi(k,z,s) > \max F]\}$ is an infinite c.e. set. Therefore, we can choose a computable and injective function $g(x)$ such that $\mathrm{range}(g) = W$. This means that the numbering $\widehat{\mu} := \xi \circ g$ is a l.m. numbering of the family $\widehat{\mathcal{S}}$.

Consider a set $B \in \widehat{\mathcal{S}}$. Note that the definition of the family $\widehat{\mathcal{S}}$ implies the following: if $B$ is finite, then $\max B > \max F$.

Now, let us assume that $U$ is a c.e. set such that it is neither empty nor equal to $\omega$, i.e. $U \neq \emptyset$ and $U \neq \omega$. We can define a numbering $\nu_U$ for the family $\mathcal{S}$ as follows:

$$\nu_U = \mu_U \oplus \widehat{\mu}, \text{ where } \mu_U(k) = \begin{cases} F, & \text{if } k \notin U, \\ B, & \text{if } k \in U. \end{cases}$$

A similar argument as in Case (i) demonstrates that the numbering $\nu_U$ is limitwise monotonic.

Note that $\nu_U \leq \nu_W$ if and only if $\overline{U} \leq_m \overline{W}$, which is equivalent to $U \leq_m W$. This observation implies that the semilattice $\mathcal{R}_{lm}(\mathcal{S})$ is infinite.

In contrast to Case (i), it is not clear whether the degrees of the numberings $\nu_U$, for c.e. $U$, form an ideal inside $\mathcal{R}_{lm}(\mathcal{S})$. Nevertheless, we can still prove that $\mathcal{R}_{lm}(\mathcal{S})$ is not a lattice.

According to Proposition X.5.6 in [Odifreddi, 1989], there exists an infinite ascending chain of c.e. $m$-degrees that have a c.e. exact pair. This means that there is a sequence of c.e. sets $(U_i)_{i \in \omega}$, along with two c.e. sets $V$ and $W$. These sets have the following properties:

- $U_i \lneq_m U_{i+1}$ for all $i \in \omega$;
- $U_i \leq_m V$ and $U_i \leq_m W$ for all $i$; and
- if $X \leq_m V$ and $X \leq_m W$, then there exists $i \in \omega$ such that $X \leq_m U_i$.

We prove that $\nu_V$ and $\nu_W$ have no infimum in $\mathcal{R}_{lm}(\mathcal{S})$. Towards a contradiction, let $\beta$ be the greatest lower bound of $\nu_V$ and $\nu_W$. Consider the index set

$$J := \{k : \beta(k) = F\}.$$

Since $\beta \leq \nu_V$ and $\beta \leq \nu_W$, we deduce that $J \leq_m \overline{V}$ and $J \leq_m \overline{W}$.

The definition of exact pair then implies that there exists an index $i \in \omega$ such that $J \leq_m \overline{U_i}$. However, as $\beta$ serves as an infimum for $\nu_V$ and $\nu_W$, it follows that $\nu_{U_{i+1}} \leq \beta$ and $\overline{U_{i+1}} \leq_m J \leq_m \overline{U_i}$, leading to a contradiction.

Thus, $\mathcal{R}_{lm}(\mathcal{S})$ is not a lattice, which concludes the proof of Theorem 3.5.1.

## 3.6 The index set of limitwise monotonic numberings

We previously demonstrated in Subsection 3.3.2 that there exist many $\Sigma^0_2$-computable numberings which cannot be approximated by a l.m. function. As a result, it is logical to consider the following problem:

> Find the complexity of the index set of l.m. numberings within the class of all $\Sigma^0_2$-computable numberings.

McLaughlin [McLaughlin, 1971, McLaughlin, 1972] initiated investigations of index sets in the theory of numberings. For more information on this topic, please refer to [Wehner, 1995].

Let $\{\varphi^X_e\}_{e\in\omega}$ be Kleene's numbering of the family of all unary partial $X$-computable functions. If $X = \emptyset$, we write $\varphi_e$ for $\varphi^{\emptyset}_e$. For $e \in \omega$, $W^X_e$ denotes the $X$-c.e. set $\mathrm{dom}(\varphi^X_e)$.

Consider the following listing: For $i, k \in \omega$, let

$$\theta_i(k) = \begin{cases} W^{\emptyset'}_{\varphi_i(k)}, & \text{if } \varphi_i(k) \text{ is defined,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is known that the list $\{\theta_i\}_{i\in\omega}$ enumerates *all* $\Sigma^0_2$-computable numberings of *all* $\Sigma^0_2$-computable families (Proposition 2.6 in [Goncharov and Sorbi, 1997]). In addition, there exists a total computable function $h_0(x, y)$ such that $\theta_i(k) = W^{\emptyset'}_{h_0(i,k)}$ for all $i$ and $k$.

Here we prove the following result:

**Theorem 3.6.1.** *The index set of limitwise monotonic numberings*

$$I_{lm} = \{i \in \omega : \text{the numbering } \theta_i \text{ is l.m.}\}$$

*is $\Sigma^0_4$-complete.*

*Proof.* Firstly, we prove that the index set $I_{lm}$ is $\Sigma^0_4$. A numbering $\theta_i$ is l.m. if and only if it admits an l.m. approximation $\varphi_e(k, z, s)$ — that is, there exists an index $e \in \omega$ with the following properties:

- The ternary function $\varphi_e(k, z, s)$ is total. This fact is equivalent to a $\Pi^0_2$ condition.
- For all $k, z, s$, we have $\varphi_e(k, z, s) \leq \varphi_e(k, z, s + 1)$, this is a $\Pi^0_1$ condition.
- For every $k$ and $z$, there exists a finite limit $m^*(k, z) = \lim_s \varphi_e(k, z, s)$ such that $m^*(k, z) \in \theta_i(k)$. Or more formally, we have

  $$\forall k \forall z \exists s[(\forall t \geq s)(\varphi_e(k, z, t) = \varphi_e(k, z, s)) \ \& \ \varphi_e(k, z, s) \in \theta_i(k)],$$

  this is a $\Pi^0_3$ condition.
- For every $k$ and every $x \in \theta_i(k)$, there exists an index $z$ such that $m^*(k, z) = x$. More formally,

  $$\forall k \forall x[x \in \theta_i(k) \to \exists z \exists s(\forall t \geq s)(\varphi_e(k, z, t) = x)].$$

  Again, this is a $\Pi^0_3$ condition.

We deduce that $I_{lm} \in \Sigma^0_4$.

Let $X \subseteq \omega$ be an arbitrary $\Sigma^0_4$ set. By Theorem 4.3.3 in [Soare, 2016], there exists a computable function $g(i, j)$ such that

$$X = \{i : \exists j[W_{g(i,j)} \text{ is coinfinite}]\}.$$

Assuming that for every $i$ and $j$, we have $j \in \overline{W_{g(i,j)}}$. Replace $g(i, j)$ with $g_1(i, j)$ if necessary to satisfy this condition:

$$W_{g_1(i,j)} := W_{g(i,j)} \setminus \{j\}.$$

In order to prove $\Sigma^0_4$ completeness of $I_{lm}$, we need to construct a uniform sequence $(\nu_i)_{i\in\omega}$ of $\Sigma^0_2$-computable numberings such that

$$i \in X \ \Leftrightarrow \ \nu_i \text{ is limitwise monotonic.} \qquad (3.6.1)$$

Fix a d.c.e. set $D$ as in [Khoussainov et al., 1997], which is not limitwise monotonic. We may assume that $0 \in D$. For every $i$ and $k$, we define

$$\nu_i(k) := \left\{ x : \exists y \exists z \left[ z \geq y \geq x \ \& \ z \in \bigcup_{j \leq k} \overline{W_{g(i,j)}} \ \& \ y \in D \right] \right\}.$$

It is clear that the numberings $\nu_i$ are $\Sigma^0_2$-computable, uniformly in $i \in \omega$. Note that $0 \in \nu_i(k)$ for all $i$ and $k$. We now consider the following two cases.

*Case 1.* Suppose that $i \notin X$. Then for every $j \in \omega$, the set $\overline{W_{g(i,j)}}$ is finite. This implies that for every $k$, we have $\nu_i(k) = \{0, 1, 2, \ldots, b_k\}$ for some $b_k \in D$.

Notice that if $k \in D$, then $k \in D \cap \overline{W_{g(i,k)}} \subseteq \nu_i(k)$. Hence, $b_k \geq k$, and the set $B := \{b_k : k \in \omega\}$ is an infinite subset of $D$.

Towards a contradiction, assume that the numbering $\nu_i$ is limitwise monotonic, and fix a l.m. approximation $f(k, z, s)$ of $\nu_i$. Then we have

$$\{0, 1, \ldots, b_k\} = \{\lim_s f(k, z, s) : z \in \omega\}.$$

Consider a computable function

$$h(k, s) = \max\{f(k, z, s) : z \leq s\}.$$

It is clear that $\lim_s h(k, s) = b_k$, and thus, $h(k, s)$ is a l.m. approximation of the set $B$. Therefore, $B$ is an infinite l.m. subset of $D$. By Lemma 2.2 in [Kalimullin et al., 2013]), this implies that the set $D$ is also limitwise monotonic, which contradicts the choice of $D$. Hence, we can conclude that the numbering $\nu_i$ is not limitwise monotonic.

*Case 2.* Suppose that $i \in X$. To begin with, we need to choose the smallest $j_0$ such that the set $\overline{W_{g(i,j_0)}}$ is infinite. After we have determined $j_0$, we can conclude that for all $k < j_0$, the set $\nu_i(k)$ is finite. Similarly, for any $k' \geq j_0$, $\nu_i(k')$ is infinite. With this information, we can easily find an approximation $f(k, z, s)$ for $\nu_i$ by defining it in the following way:

$$f(k, z, s) = \begin{cases} z, & \text{if } k < j_0 \text{ and } z \in \nu_i(k), \\ 0, & \text{if } k < j_0 \text{ and } z \notin \nu_i(k), \\ z, & \text{if } k \geq j_0. \end{cases}$$

Thus, the sequence $(\nu_i)_{i \in \omega}$ satisfies Eq. (3.6.1). Theorem 3.6.1 is proved. □

# Chapter 4

# Rogers semilattice of analytical numberings

In this chapter, we discuss numberings in the analytical hierarchy, and specifically, the Rogers semilattices formed by numberings of the families of subsets of natural numbers. This chapter is based on the joint work with N.Bazhenov and M.Mustafa [Bazhenov et al., 2021]. We consider typical problems of the theory of numberings, especially, relative to the theory of Rogers semilattices and what results have been obtained for the particular case of analytical hierarchy, and how are these results compared with known results of Rogers semilattices in the arithmetical hierarchy. In particular, we investigate different forms of isomorphism found in Rogers semilattices of families of sets in the analytical hierarchy. We explore various set-theoretic assumptions. Our investigation reveals that in the presence of Projective Determinacy, there exist infinite pairwise non-isomorphic Rogers semilattices at each $\Sigma^1_n$-level of the analytical hierarchy. We aim to expand the existing knowledge on Rogers semilattices in the analytical hierarchy. These semilattices were already studied in previous works, such as [Dorzhieva, 2016, Bazhenov et al., 2023, Bazhenov and Mustafa, 2020, Bazhenov et al., 2020]. Our specific objective is to investigate the isomorphism types that these semilattices can assume for $\Sigma^1_n$-computable families $\mathcal{S}$, where $n$ is a non-zero natural number. By analyzing a wide range of results of the theory of numberings, we aim to provide a comprehensive understanding of the properties and behavior of such semilattices.

A recent research paper [Bazhenov and Mustafa, 2020] explored families of sets in the analytical hierarchy that are $\Sigma^1_n$-computable and found that there are at least four Rogers semilattices that are not pairwise isomorphic, assuming Projective Determinacy **PD**. This thesis builds on the previous research and extends the findings to generalize the number of isomorphism types of Rogers semilattices for these families. Assuming **PD**, Theorem 4.5.1 shows that there are infinitely many Rogers semilattices that are pairwise elementarily non-equivalent for $\Sigma^1_n$-computable families. When $n = 1$ and $n = 2$, this result holds even without assuming **PD** (Corollary 4.5.6).

The necessary background and known results on the analytical hierarchy that will be used in our proofs are provided in Section 4.3. Additionally, we include proofs of several useful results on numberings, specifically Lemma 4.3.1 and Lemma 4.4.4. In Section 4.4, we offer a brief overview of some consequences of **PD** that will also be utilized in our proofs. The main result regarding isomorphism types of Rogers semilattices for $\Sigma^1_n$-computable families is proven in Section 4.5. Finally, in Section 4.6, we discuss additional problems.

## 4.1 Motivation

The current chapter is a continuation of investigations of Rogers semilattices of families of sets in the analytical hierarchy developed in [Dorzhieva, 2016, Bazhenov et al., 2023]. These papers concentrated on Rogers semilattices of $\Pi^1_n$-computable families. It was established by Dorzhieva [Dorzhieva, 2016] that for a family $\mathcal{S}$ of $\Pi^1_n$-computable numbering, one of the following two conditions holds: 1)either the cardinality of the Rogers semilattices for $\Pi^1_n$-computable families is equal to one, or 2) the first-order theory of Rogers semilattices for families of $\Pi^1_n$-computable numberings $Th(R_{\Pi^1_n}(\mathcal{S}))$ is undecidable.

According to a reference [Bazhenov et al., 2023], it has been proven that if there are multiple elements in the Rogers semilattices for families of $\Pi^1_n$-computable numberings, then the structures of the Rogers semilattices for $\Pi^1_n$-computable families $\mathcal{S}$ and $\Pi^1_m$-computable families $\mathcal{T}$ are not isomorphic for any nonzero $m \neq n$. Further related work is discussed in 4.2.

The paper [Bazhenov et al., 2023] left an open problem concerning the number of isomorphism types realized by Rogers semilattices for $\Sigma^1_n$-computable families $\mathcal{S}$. The aforementioned paper did not provide a definitive answer to this question.

In studying this question, we discovered that additional set-theoretic assumptions are required in order to apply the known numbering-theoretic techniques in this scenario. To address this, we establish several results related to the number of isomorphism types realized by Rogers semilattices for $\Sigma^1_n$-computable families $\mathcal{S}$, assuming *Projective*

*Determinacy* (**PD**). More information about **PD** can be found in 4.4. We chose **PD** as an additional axiom because Tanaka [Tanaka, 1978] has already developed a theory for the levels of analytical hierarchy under this assumption. We found Tanaka's approach well-suited to our objectives.

In set theory, the Axiom of Dependent Choices (**DC**) states that for any non-empty set $A$ and any set of pairs $P$ that belongs to $A \times A$, if for every $x$ that belongs to $A$ there exists a $y$ in $A$ such that the predicate $P(x, y)$ is true, then there exists a function $f$ from the natural numbers $\omega$ to $A$ such that for all $n, P(f(n), f(n+1))$ is true. In chapter 4, we assume that the set theory used is **ZF**+**DC**.

## 4.2 Related work

Studying the isomorphism types of Rogers semilattices of computable families is a significant topic in the field of generalized computable numberings. Despite extensive research on Rogers semilattices of families for $\Sigma^0_n$-computable numberings, the results are not straightforward. Ershov and Lavrov[Ershov and Lavrov, 1973] discovered that there exist finite families $\mathcal{S}_i$, $i \in \omega$, of c.e. sets such that the their Rogers semilattices of $\Sigma^0_1$-computable numberings are not isomorphic to one another. This implies that there are infinitely many isomorphism types of Rogers semilattices of the $\Sigma^0_1$-computable family. V'yugin [V'yugin, 1973] further demonstrated that there are infinitely many Rogers semilattices of $\Sigma^0_1$-computable numberings that are elementarily non-equivalent to one another. Badaev, Goncharov, and Sorbi [Badaev et al., 2005] established that for any natural number $n \geq 2$, there are infinitely many pairwise elementarily non-equivalent Rogers semilattices of $\Sigma^0_n$-computable numberings. For additional information on Rogers semilattices of $\Sigma^0_n$-computable numberings, please refer to [Goncharov and Sorbi, 1997, Badaev et al., 2003a, Badaev et al., 2003c].

Dorzhieva previously conducted a study on numberings in the analytical hierarchy, and specifically, the Rogers semilattices formed by these analytical numberings [Dorzhieva, 2014, Dorzhieva, 2016]. She proved that the first order theory of any non-trivial Rogers semilattice of a computable family $\mathcal{S}$ of $\Pi^1_n$-computable numberings $R_{\Pi^1_n}(\mathcal{S})$ is undecidable, for any non-zero $n$. A semilattice is considered non-trivial if it contains more than one element. The Rogers semilattices of $\Pi^1_n$-computable families were also the subject of a study by authors in [Bazhenov et al., 2023]. In that paper, they proved that if the semilattice $R_{\Pi^1_n}(\mathcal{S})$ of a computable family $\mathcal{S}$ contains more than one element, it is not isomorphic to $R_{\Pi^1_m}(\mathcal{T})$ for any non-zero $m \neq n$ and for any $\Pi^1_m$-computable family $\mathcal{T}$.

In the same way as $\Pi^1_n$-computable families, $\Sigma^1_n$-computable families have also been studied with regard to Rogers semilattices in a previous work [Bazhenov and Mustafa, 2020]. The research conducted in [Bazhenov and Mustafa, 2020] revealed that there are at least four distinct Rogers semilattices for $\Sigma^1_n$-computable families that are not isomorphic to one another. In this chapter, the study of Rogers $\Sigma^1_n$-semilattices is continued.

Following [Addison and Moschovakis, 1968], we use the following notations: for a number $k \in \omega$,

- $E^1_{2k+1}$ is the (lightface) class $\Pi^1_{2k+1}$, and $\Upsilon^1_{2k+1}$ is the class $\Sigma^1_{2k+1}$;
- $E^1_{2k+2} = \Sigma^1_{2k+2}$ and $\Upsilon^1_{2k+2} = \Pi^1_{2k+2}$.

[Tanaka, 1978] developed a theory for subsets of $\omega$, belonging to the levels of analytical hierarchy, under the assumption of Projective Determinacy (**PD**). One of his results (given below) will be especially useful for us.

Let $n$ be a non-zero natural number. A set $A \subseteq \omega$ is called *$E^1_n$-maximal* if $A$ satisfies the following:

(a) $A \in E^1_n$, and the complement $\overline{A} = \omega \setminus A$ is infinite.

(b) For every $E^1_n$ set $C$, either $\overline{A} \cap C$ or $\overline{A} \setminus C$ is finite.

**Theorem 4.2.1** (Tanaka, Theorem 3.1 and Corollary 3.4 of [Tanaka, 1978]). *Assume* ***PD****. There is a $E^1_n$-maximal set.*

*Remark* 4.2.2. Without assuming **PD**, one can prove the existence of $\Pi^1_1$-maximal and $\Sigma^1_2$-maximal sets: the $\Pi^1_1$-case is due to Kreisel and Sacks (item (C) on p. 332 in [Kreisel and Sacks, 1965]); the $\Sigma^1_2$-case is due to Tanaka (see p. 113 of [Tanaka, 1978] — he does not assume **PD** for $\Sigma^1_2$).

## 4.3 Background on the analytical hierarchy

Here we discuss known results on the analytical hierarchy, which will be employed in our proofs. Furthermore, the section includes proofs of useful (but a little bit technical) results on numberings: Lemma 4.3.1 and Lemma 4.4.4.

Recall that a predicate $R(x_1, \ldots, x_m; f_1, \ldots, f_n)$, where the variables $f_i$ denote elements from $\omega^\omega$, is *recursive* if there is an index $e \in \omega$ such that for all $f_1, \ldots, f_n$ and $x_1, \ldots, x_m$, the following conditions hold:

(a) the value $\varphi_e^{f_1 \oplus \cdots \oplus f_n}(x_1, \ldots, x_m)$ is defined;

(b) the predicate $R(x_1, \ldots, x_m; f_1, \ldots, f_n)$ is true if and only if $\varphi_e^{f_1 \oplus \cdots \oplus f_n}(x_1, \ldots, x_m) = 1$.

If an index $e$ satisfies these conditions, then we say that $e$ *witnesses the recursiveness* of the predicate $R$.

Consider arbitrary recursion-theoretic hierarchy $\mathcal{C}$ (e.g., $\Sigma^0_1$, $\Sigma^{-1}_2$, $\Sigma^0_n$, or $\Pi^1_n$) and recall that $\mathcal{S}$ is the family of subsets of natural numbers $\omega$. Then a numbering $\nu$ of a family $\mathcal{S}$ is $\mathcal{C}$*-computable* if the set $\{\langle k, x\rangle : x \in \nu(k)\}$ belongs to the class $\mathcal{C}$. If a family $\mathcal{S}$ has a $\mathcal{C}$-computable numbering, it is said that a family $\mathcal{S}$ is $\mathcal{C}$*-computable.*

Henceforth, following [Sacks, 1990], we shall assume that the class $\mathcal{C}$ is a member of the set $\{\Sigma^1_n, \Pi^1_n \mid n \geq 1\}$. We define *dual class* of $\mathcal{C}$, denoted by $\breve{\mathcal{C}}$, in the following way:

$$\breve{\mathcal{C}} = \begin{cases} \Sigma^1_n, & \text{if } \mathcal{C} = \Pi^1_n, \\ \Pi^1_n, & \text{if } \mathcal{C} = \Sigma^1_n. \end{cases}$$

We define $Com_{\mathcal{C}}(\mathcal{S})$ as the set of all $\mathcal{C}$-computable numberings of a $\mathcal{C}$-computable family $\mathcal{S}$. To simplify things, we use the same symbols $\leq$ and $\oplus$ on numberings and their equivalence classes.

The quotient structure $\mathcal{R}_{\mathcal{C}}(\mathcal{S}) := (Com_{\mathcal{C}}(\mathcal{S})/_{\equiv}; \leq, \oplus)$ is an upper semilattice, which we call the Rogers semilattice of the $\mathcal{C}$-computable family $\mathcal{S}$.

The next lemma allows us to transform a class $\mathcal{C}$ to its dual class $\breve{\mathcal{C}}$ while preserving all properties of the Rogers semilattices.

**Lemma 4.3.1** (Lemma 3.1 of [Bazhenov and Mustafa, 2020]). *Let the family*

$$\mathrm{Dual}(\mathcal{S}) = \{\omega \setminus A : A \in \mathcal{S}\}$$

*where* $\mathcal{S}$ *is any* $\mathcal{C}$*-computable family. We can conclude that a family* $\mathrm{Dual}(\mathcal{S})$ *is* $\breve{\mathcal{C}}$*-computable. Moreover, the Rogers semilattices* $\mathcal{R}_{\mathcal{C}}(\mathcal{S})$ *and* $\mathcal{R}_{\breve{\mathcal{C}}}(\mathrm{Dual}(\mathcal{S}))$ *are isomorphic.*

*Proof Sketch.* Let $\mathcal{S}$ be a arbitrary computable family and $\nu$ be its $\mathcal{C}$-computable numbering. We can define a numbering $\nu^{Dual}$ for $Dual(\mathcal{S})$ in the following way: $\nu^{Dual}(k) := \omega \setminus \nu(k)$ for any $k \in \omega$. Note that $G_{\nu^{Dual}}$ is the complement of $G_\nu \in \mathcal{C}$, and therefore, $\nu^{Dual}$ is $\check{\mathcal{C}}$-computable. Moreover, it can be shown that for any two numberings $\nu$ and $\mu$, $\nu \leq \mu$ if and only if $\nu^{Dual} \leq \mu^{Dual}$. Therefore, the structures $\mathcal{R}_{\mathcal{C}}(\mathcal{S})$ and $\mathcal{R}_{\check{\mathcal{C}}}(Dual(\mathcal{S}))$ are isomorphic.

It is important to note that if $\mathcal{S} \neq \mathcal{S}^*$, then $Dual(\mathcal{S}) \neq Dual(\mathcal{S}^*)$. Additionally, $Dual(Dual(\mathcal{S})) = \mathcal{S}$. These facts are enough to complete the proof.

$\square$

## 4.4 Projective determinacy and its consequences

The theory of recursion for subsets of natural numbers $\omega$ belonging to the levels of analytical hierarchy was originally developed by [Tanaka, 1978], based on the assumption of Projective Determinacy (**PD**). *Projective Determinacy* (**PD**) is a fundamental principle in set theory that asserts the determinacy of games played between two players. Specifically, it states that for any projective set $A$, which is a collection of infinite sequences of natural numbers, there exists a winning strategy for one of the players in the game $G(A)$. In other words, the outcome of the game is predetermined and not left to chance. This principle has far-reaching consequences in various branches of mathematics and has led to significant advancements in the field of set theory. In this section, we will provide a brief overview of some of the consequences of **PD** that will be heavily used in the proofs of our results.

To begin, we will discuss the necessary set-theoretic background, which largely follows [Jech, 2002, Moschovakis, 2009].

For every set $A \subseteq \omega^\omega$, a two-person game $G = G(A)$ is associated as follows: Player I and II choose natural numbers alternately. Player I chooses $a_0$, followed by Player II choosing $b_0$, and then Player I chooses $a_1$, followed by Player II choosing $b_1$, and so on. The game $G$ ends after $\omega$ steps. Player I wins if the resulting sequence $f := (a_0, b_0, a_1, b_1, \dots)$ belongs to $A$, Otherwise, Player II wins.

In the game $G$, both players have perfect information. Before Player I chooses $a_{n+1}$, he/she is allowed to see the tuple $(a_0, b_0, \dots, a_n, b_n)$ and the same goes for Player II. A strategy for Player I is a function $\sigma$ that maps tuples of even length to natural numbers. A winning strategy for Player I means that following $\sigma$ results in a win for Player I. Similarly, a winning strategy for Player II can be defined. The game $G(A)$ is considered "determined" if one of the players has a winning strategy.

The Axiom of Determinacy (**AD**) is a fundamental principle in mathematical set theory that provides a way to settle certain types of infinite games. Specifically, it states that for any subset $A$ of all total functions from $\omega$ to $\omega$, there is always a winner in the game played between two players - the first player selects elements from the subset and the second player selects elements from its complement. This principle has important implications for the foundations of mathematics and the study of infinity.

To gain a deeper understanding of this concept, it's crucial to have a clear definition of the projective hierarchy of a Polish space $\mathcal{X}$. This hierarchy is established based on the topological features of the Baire space, which is the set of all total functions from $\omega$ to $\omega$. Essentially, the projective hierarchy provides a way to classify sets in $\mathcal{X}$ based on their level of complexity. The Baire space, on the other hand, is a fundamental example of a Polish space and plays a significant role in descriptive set theory. By exploring the projective hierarchy in relation to the Baire space, we can gain a better understanding of the topological properties of $\mathcal{X}$.

A subset $A$ of a Polish space $\mathcal{X}$ is called *analytic* if it is either an empty set or there exists a continuous map $f\colon \omega^\omega \to \mathcal{X}$, such that $Range(f) = A$.

For any non-zero natural number $n$, we can define two important boldface classes, namely $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$, for the space $\mathcal{X}$:

- If $A$ is analytic, then a set $A \subseteq \mathcal{X}$ is $\mathbf{\Sigma}^1_1$.
- If the complement $\mathcal{X} \setminus A$ is analytic, then $A$ is $\mathbf{\Pi}^1_1$.
- If there exists a $\mathbf{\Pi}^1_n$ set $B \subseteq \mathcal{X} \times \omega^\omega$ such that

  $$A = \{a : (\exists f \in \omega^\omega)[(a, f) \in B]\}$$

  then $A$ is $\mathbf{\Sigma}^1_{n+1}$.
- If the complement $\mathcal{X} \setminus A$ is $\mathbf{\Sigma}^1_{n+1}$, then $A$ is $\mathbf{\Pi}^1_{n+1}$.

These classes are used in descriptive set theory to help classify sets of real numbers according to their complexity. The class $\mathbf{\Sigma}^1_n$ consists of sets that can be defined using a formula that contains one or more quantifiers over real numbers and a finite number of logical operations, while the class $\mathbf{\Pi}^1_n$ consists of sets that are the complements of sets in $\mathbf{\Sigma}^1_n$. These classes play an important role in the study of the complexity of continuous functions and other mathematical objects.

A set $A$ is called *projective* if it belongs to both $\bigcup_{1\leq n<\omega} \mathbf{\Sigma}^1_n$ and $\bigcup_{1\leq n<\omega} \mathbf{\Pi}^1_n$, where $n$ is a natural number. Intuitively, projective sets are a class of well-behaved sets that arise in the study of mathematical logic and set theory.

In our proofs, we will use the following general fact about numberings:

**Lemma 4.4.1** (Proposition 3.1 from [Badaev et al., 2003c]). *Assuming we have arbitrary numberings $\nu$, $\mu_0$, and $\mu_1$, it follows that if $\nu \leq \mu_0 \oplus \mu_1$, then one of the following statements must be true:*

*1. $\nu \leq \mu_0$.*

*2. $\nu \leq \mu_1$.*

*3. There are numberings $\nu_0$ and $\nu_1$ that satisfy the conditions where $\nu_0 \leq \mu_0$, $\nu_1 \leq \mu_1$, and $\nu \equiv \nu_0 \oplus \nu_1$. Furthermore, if $\nu$, $\mu_0$, and $\mu_1$ are $E^1_n$-computable, then $\nu_0$ and $\nu_1$ must also be computable by the same algorithm.*

A *prewellordering* is a type of binary relation that is defined on a set $A$. It is a relation that satisfies certain conditions. Firstly, it is transitive, reflexive, and connected. In other words, for any two elements $x$ and $y$ in $A$, either $x \preceq y$ or $y \preceq x$. Secondly, it is well-founded, meaning that there are no infinite descending chains of elements in $A$. This means that the relation $\preceq$ provides a way to compare elements in $A$ in a clear and consistent manner. By satisfying these conditions, a prewellordering ensures that any element in $A$ can be compared to any other element in $A$ in a well-defined way.

A *norm* is a mathematical function that is used to assign a numerical value to a set. When a norm $\phi$ is applied to a set, it can be used to create a prewellordering, which is denoted as $x \preceq^{\varphi} y$ and satisfies the condition that $x \preceq^{\varphi} y$ if and only if the norm of $x$ is less than or equal to the norm of $y$, i.e.:

$$x \preceq^{\varphi} y \;\Leftrightarrow\; \varphi(x) \leq_{Ord} \varphi(y),$$

This ordering is based on the standard order of ordinals, which is represented by $\leq_{Ord}$.

Let $S$ be a subset of the natural numbers, $S \subseteq \omega$, and let $\varphi : S \to \lambda$ be a norm on $S$. We say that $\varphi$ is a $\mathcal{C}$-*norm* if there exist two binary relations $\preceq^{\varphi}_{\mathcal{C}}$ and $\preceq^{\varphi}_{\breve{\mathcal{C}}}$ on $\omega$ that satisfy the following properties: $\preceq^{\varphi}_{\mathcal{C}}$ belongs to $\mathcal{C}$, $\preceq^{\varphi}_{\breve{\mathcal{C}}}$ belongs to $\breve{\mathcal{C}}$, and for every $y \in \omega$,

$$\begin{gathered}\text{if } y \in S, \text{ then for any } x \in \omega, \\ [x \in S \,\&\, \varphi(x) \leq_{Ord} \varphi(y)] \;\Leftrightarrow\; x \preceq^{\varphi}_{\mathcal{C}} y \;\Leftrightarrow\; x \preceq^{\varphi}_{\breve{\mathcal{C}}} y. \end{gathered} \tag{4.4.1}$$

A mathematical class $\mathcal{C}$ is said to have the *prewellordering property* or to be *normed* if every set $S \in \mathcal{C}$ can be assigned a $\mathcal{C}$-norm, which is a function that assigns non-negative values to elements of $S$ while satisfying certain properties. In simpler terms, if a class is normed, any set in that class can be equipped with a notion of distance or size, which is a useful concept in many areas of mathematics.

The Prewellordering Theorem, which can be found in [Addison and Moschovakis, 1968], has an the following interesting consequence. See [Moschovakis, 1971] and Corollary 6B.2 of [Moschovakis, 2009] for detailed discussion.

**Theorem 4.4.2** ([Addison and Moschovakis, 1968]). *Given the assumption that* ***PD*** *holds, consider a non-zero natural number $n$. We can then observe that the class $E^1_n$ exhibits the prewellordering property.*

*Remark* 4.4.3. This result applies to the classes $\Pi^1_1$ and $\Sigma^1_2$, even without assuming the Projective Determinacy. The relevant theorems, which can be found in Chapter 16 of [Rogers, 1967] (Theorems XXIII and XXXVIII) and in [Moschovakis, 2009] (Theorems 4B.2 and 4B.3), provide further information on the topic.

Suppose we have an arbitrary $E^1_n$-computable numbering $\nu\colon \omega \to P(\omega)$, then we can define a relation $\sqsubseteq^\nu \subseteq \omega^2 \times \omega^2$.

To begin with, let's observe that the set $G_\nu = \{\langle k, x\rangle : x \in \nu(k)\}$ is $E^1_n$. As a consequence of Theorem 4.4.2, we can choose an $E^1_n$-norm $\varphi$ that maps $G_\nu$ into some ordinal $\lambda$.

We then select binary relations $\preceq^\varphi_\Gamma$ and $\preceq^\varphi_{\breve{\Gamma}}$, where $\Gamma = E^1_n$, to show that $\varphi$ is indeed a $E^1_n$-norm. This way, we can demonstrate the effectiveness of the numbering $\nu$ in defining the relation $\sqsubseteq^\nu$.

If $k$, $m$, $x$ and $y$ are natural numbers, we define $(k, x) \sqsubseteq^\nu (m, y)$ as follows:

$$\begin{gathered}(k = m)\ \&\ [(\langle k, x\rangle \preceq^\varphi_\Gamma \langle m, y\rangle \,\&\, \langle m, y\rangle \not\preceq^\varphi_\Gamma \langle k, x\rangle) \vee \\ (\langle k, x\rangle \preceq^\varphi_\Gamma \langle m, y\rangle \,\&\, \langle m, y\rangle \preceq^\varphi_\Gamma \langle k, x\rangle \,\&\, x \leq_\omega y)].\end{gathered}$$

The sets $\widehat{[x]}_{\nu(k)}$ play a crucial role in our constructions by acting as the fundamental building blocks. They enable us to transfer previously established results, which were known for $\Sigma^0_2$-computable numberings, into our current setting.

**Lemma 4.4.4** (Main Property of $\sqsubseteq^\nu$). *For any $k \in \omega$ and $E^1_n$-computable numbering $\nu$, assuming* ***PD****, we observe the following:*

(1) *The relation denoted by* $\preceq^{\nu(k)}$ *is a well-ordering on* $\nu(k)$. *This implies that* $\preceq^{\nu(k)}$ *is a total order relation and that every non-empty subset of* $\nu(k)$ *has a least element under this relation. Here,* $\preceq^{\nu(k)}$ *is defined as* $\{(x,y) : x, y \in \nu(k),\ (k,x) \sqsubseteq^{\nu} (k,y)\}$.

(2) *For any given number* $x \in \nu(k)$, *the set denoted by* $\widehat{[x]}_{\nu(k)}$ *is a* $\Delta^1_n$ *subset of* $\nu(k)$. *Furthermore, the formulas that prove the* $\Delta^1_n$*-ness of this set do not rely on the choice of* $k$ *or* $x$. *The set* $\widehat{[x]}_{\nu(k)}$ *is defined as* $\{z \in \omega : (k,z) \sqsubseteq^{\nu} (k,x)\}$.

*Proof.* (1) To understand the relation $\sqsubseteq^{\nu}$, we need to know that for any two numbers $x$ and $y$ from $\nu(k)$, the condition $x \preceq^{\nu(k)} y$ is true if and only if one of the following two conditions is met: 1. $\varphi(\langle k,x\rangle) \lneq_{Ord} \varphi(\langle k,y\rangle)$ 2. $\varphi(\langle k,x\rangle) = \varphi(\langle k,y\rangle)$ and $x \leq_{\omega} y$.

This leads to the conclusion that the poset $(\nu(k); \preceq^{\nu(k)})$ is well-ordered. To see why, consider the map $\psi\colon x \mapsto (x, \varphi(\langle k,x\rangle))$. This map induces an isomorphic embedding from $\nu(k)$ into the ordinal $\omega \cdot \lambda$.

(2) Consider a tuple $\langle k,x\rangle$ that belongs to a set $G_{\nu}$. Now, let's pick any $z \in \widehat{[x]}_{\nu(k)}$. It is important to note that if we compare the tuples $\langle k,z\rangle \preceq^{\varphi}_{\Gamma} \langle k,x\rangle$, we will get that $\langle k,z\rangle$ is less than or equal to $\langle k,x\rangle$. According to the definition of a $\mathcal{C}$-norm, this comparison means that $\langle k,z\rangle \in G_{\nu}$. In simpler words, we can say that we have just shown that $\widehat{[x]}_{\nu(k)}$ is a subset of $\nu(k)$.

Definition 4.4.1 enables us to deduce that the following conditions are equivalent:

(a) $z \preceq^{\nu(k)} x$;

(b) $(\langle k,z\rangle \preceq^{\varphi}_{\mathcal{C}} \langle k,x\rangle$ and $\langle k,x\rangle \not\preceq^{\varphi}_{\breve{\mathcal{C}}} \langle k,z\rangle)$, or $(\langle k,z\rangle \preceq^{\varphi}_{\mathcal{C}} \langle k,x\rangle$ and $\langle k,x\rangle \preceq^{\varphi}_{\mathcal{C}} \langle k,z\rangle$ and $z \leq_{\omega} x)$;

(c) $(\langle k,z\rangle \preceq^{\varphi}_{\breve{\mathcal{C}}} \langle k,x\rangle$ and $\langle k,x\rangle \not\preceq^{\varphi}_{\mathcal{C}} \langle k,z\rangle)$, or $(\langle k,z\rangle \preceq^{\varphi}_{\breve{\mathcal{C}}} \langle k,x\rangle$ and $\langle k,x\rangle \preceq^{\varphi}_{\breve{\mathcal{C}}} \langle k,z\rangle$ and $z \leq_{\omega} x)$.

It can be observed that Condition (b) is equivalent to a $E^1_n$ formula, while Condition (c) is equivalent to a $\Upsilon^1_n$ formula. It is important to note that the formulae do not rely on the choice of $k$ and $x$, but only on the relations $\preceq^{\varphi}_{\mathcal{C}}$ and $\preceq^{\varphi}_{\breve{\mathcal{C}}}$. The proof of Lemma 4.4.4 is now completed. $\square$

Let $\mathcal{C}$ be a collection of classes that belong to either $\Sigma^1_n$ or $\Pi^1_n$ for $n \geq 1$. If $\mathcal{C}$ satisfies the *reduction principle*, then it guarantees the existence of subsets $A^*$ and $B^*$ of any two classes $A$ and $B$ in $\mathcal{C}$, with the following properties: $A^*$ is a subset of $A$, $B^*$ is a subset of $B$, then $A^* \cup B^* = A \cup B$ and $A^* \cap B^* = \emptyset$. In other words, the classes in $\mathcal{C}$ can be split into two disjoint sets, and each set can be further split into smaller subsets that can be combined to form the original classes.

**Theorem 4.4.5** ([Addison and Moschovakis, 1968]; [Martin, 1968])**.** *Assuming **PD**, the complexity class $E^1_n$ satisfies the reduction principle for any non-zero natural number $n$.*

The concept of reduction principle is a fundamental idea in mathematical logic and its theory is well studied. In a seminal work by Addison [Addison, 1958]), it was shown that the classes $\Pi^1_1$ and $\Sigma^1_2$ satisfy the reduction principle even in the absence of the assumption of projective determinacy (**PD**). This result has been widely cited and has significant implications in the field of mathematical logic.

It is worth noting that the principle of reduction for $E^1_n$ can be derived from Theorem 4.4.2. More information on the subject can be found in Exercise 4B.10 and Corollary 6B.2 in [Moschovakis, 2009].

## 4.5 The main result

**Theorem 4.5.1.** *Assuming **PD**, there exist infinite families $\mathcal{S}_i$, $i \in \omega$, that are $\Sigma^1_n$-computable, such that the elementary theories of Rogers semilattices $\mathcal{R}_{\Sigma^1_n}(\mathcal{S}_i)$ are pairwise distinct for a non-zero natural number $n$.*

*Proof.* Badaev, Goncharov, and Sorbi [Badaev et al., 2005] demonstrated in their research that there exist an infinite number of Rogers semilattices that are distinct from one another at the $\Sigma^0_m$-level, for $m \geq 2$. We have followed their methodology, adapting their approach to the context of the analytical hierarchy.

In order to simplify our calculations, we employ Lemma 4.3.1 and instead of working directly with the level $\Sigma^1_n$, we follow a different approach. We construct $E^1_n$-computable families $\mathcal{S}_i$ that satisfy the condition of pairwise elementary non-equivalence of the semilattices $\mathcal{R}_{E^1_n}(\mathcal{S}_i)$, $i \in \omega$. This allows us to use the more convenient notation $\mathcal{R}^1_n(\mathcal{S}) := \mathcal{R}_{E^1_n}(\mathcal{S})$ for a family $\mathcal{S}$, thereby simplifying our calculations.

We can establish a $E^1_n$-maximal set $M$ based on Theorem 4.2.1. To achieve this, we first introduce auxiliary families $\mathcal{T}_j$ where $j \geq 1$. Our ultimate objective is to demonstrate that for $i \in \omega$, the required family $\mathcal{S}_i$ can be identified as some $\mathcal{T}_{j_i}$.

Suppose $j$ is a non-zero natural number. We define a computable set $R_l$ as follows: for a non-zero $l \leq j$,

$R_l := \{j \cdot t + (l-1) : t \in \omega\}$.

The sets $R_l$, where $1 \leq l \leq j$, clearly form a partition of $\omega$.

We then fix a total computable, injective function $p_l(x)$ such that $range(p_l) = R_l$.

Next, we define:

$$M_l := p_l(M) \cup \bigcup_{m \neq l} R_m, \quad \mathcal{T}_j^{[l]} := \{M_l \cup \{x\} : x \notin M_l\}, \quad \mathcal{T}_j := \bigcup_{1 \leq l \leq j} \mathcal{T}_j^{[l]}.$$

This way, the definition of $M_l$ and $\mathcal{T}_j$ become more clear.

It's important to note that the families $\mathcal{T}_j^{[l]}$, where $1 \leq l \leq j$, are completely separate from one another or in other words, pairwise disjoint.

*Claim* 4.5.1. Each $M_l$ is a $E^1_n$-maximal set.

*Proof.* Let us assume that $j > 1$ and $l = 1$ without any loss of generality. It can be observed that an element $x$ belongs to $M_1$ if and only if it satisfies the following condition:

$$x \in \bigcup_{m \neq 1} R_m \text{ or } \exists y[(y \in M) \wedge (p_1(y) = x)].$$

Therefore, $M_1$ is a set of the form $E^1_n$. It is also evident that the complement of $M_1$, i.e., $\overline{M_1} = p_1(\overline{M})$, is an infinite set.

Let $C$ be an arbitrary set of the form $E^1_n$. We can consider a set $D$ of the same form defined as

$$D := p_1^{-1}(C) = \{x \in \omega : \exists y[y \in C \,\&\, p_1(x) = y]\}.$$

Based on the given observation, it can be inferred that the set $D$ is equivalent to $p_1^{-1}(C \cap R_1)$. Furthermore, since $M$ is $E^1_n$-maximal, one of the following situations must be true:

(a) If the intersection of $\overline{M}$ and $D$ is finite, then $\overline{M_1} \cap C = p_1(\overline{M} \cap D)$ is also finite.

(b) If the difference between the sets $\overline{M}$ and $D$ is finite, then $\overline{M_1} \setminus C = p_1(\overline{M} \setminus D)$ is also finite.

Thus, it can be concluded that $M_1$ is also maximal with respect to $E^1_n$.

□

We aim to demonstrate that each $\mathcal{T}_j$ is a $E^1_n$-computable family. To achieve this, we prove the following straightforward fact:

**Lemma 4.5.2.** *Let $A$ be any subset of $\omega$ that belongs to $E^1_n$ and is not equal to $\omega$. Then the family $\mathcal{V} := \{A \cup \{x\} : x \notin A\}$ is $E^1_n$-computable.*

*Proof.* We can construct a numbering $\xi$ as follows: Choose an element $b$ that is not in the set $A$. For every natural number $k$, we include $x$ in $\xi(k)$ if and only if:

- $(x = k)$, or
- $(x \in A)$, or
- $(x = b$ and $k \in A)$.

It is clear that the numbering $\xi$ is $E^1_n$-computable, and we can express $\xi(k)$ as follows:

$$\xi(k) = \begin{cases} A \cup \{k\}, & \text{if } k \notin A, \\ A \cup \{b\}, & \text{if } k \in A. \end{cases}$$

Therefore, we can conclude that $\xi$ indexes the family $\mathcal{V}$ precisely.

□

*Claim* 4.5.2. Each family $\mathcal{T}_j$ with $j \geq 1$ can be considered as $E^1_n$-computable family.

*Proof.* To prove our point, we only need to demonstrate that every $\mathcal{T}^{[l]}_j$, where $1 \leq l \leq j$, can be assigned a numbering that is $E^1_n$-computable. This can be inferred from the details outlined in Lemma 4.5.2.

□

We hereby present a series of technical claims that substantiate the elementary differences under consideration.

*Claim* 4.5.3. Consider a family $\mathcal{T}_j$ with $j \geq 1$, and let $1 \leq l \leq j$ be arbitrary. We assume that $\nu$ is an $E^1_n$-computable numbering of $\mathcal{T}_j$. From $\nu$, we construct an index set $I^{[l]}_j(\nu)$, which is defined as follows:

$$I^{[l]}_j(\nu) := \{k \in \omega : \nu(k) \in \mathcal{T}^{[l]}_j\}$$

It is important to note that $I^{[l]}_j(\nu)$ is a $\Delta^1_n$ set, which implies that there exists a formula of a certain level of complexity that can compute this set.

*Proof.* Let's consider two different elements, $b$ and $c$, from $\overline{M_l}$. We can then easily observe that:

$$\nu(k) \notin \mathcal{T}_j^{[l]} \;\Leftrightarrow\; b \in \nu(k) \,\&\, c \in \nu(k).$$

As a result, we can conclude that the set $I_j^{[l]}(\nu)$ belongs to $\Upsilon_n^1$.

Moreover, it's important to note that the index sets $I_j^{[m]}$, where $1 \leq m \leq j$, form a partition of $\omega$. Hence, we can deduce that $I_j^{[l]}(\nu)$ belongs to $\Delta_n^1$. □

*Claim* 4.5.4. Suppose we have $j \geq 1$ and $1 \leq l \leq j$. Let $\nu$ be an arbitrary numbering of $\mathcal{T}_j$ that is $E_n^1$-computable. Assume that $\nu$ can be expressed as $\nu_0 \oplus \nu_1$. Then, there exists a number $i \in \{0, 1\}$ such that all but finitely many elements of $\mathcal{T}_j^{[l]}$ have $\nu_i$-indices.

*Proof.* The statement in Claim 4.5.3 asserts that the index set $I_j^{[l]}(\nu)$ is equivalent to the set $\Delta_n^1$. To better explain this claim, we can define two sets, $Q_0$ and $Q_1$, as follows:

$$\begin{aligned} Q_0 &:= \{k \mid 2k \in I_j^{[l]}(\nu)\}, \\ Q_1 &:= \{k \mid 2k+1 \in I_j^{[l]}(\nu)\}. \end{aligned}$$

It is important to note that each set $Q_i$ is equivalent to $I_j^{[l]}(\nu_i)$, and both $Q_0$ and $Q_1$ are identical to the set $\Delta_n^1$.

Furthermore, we can define two additional sets, $V_0$ and $V_1$, as follows:

$$x \in V_i \Leftrightarrow (x \in M_l) \vee \exists k[\, k \in Q_i \,\&\, x \in \nu_i(k)\,].$$

These sets have the useful property of being $E_n^1$, since each $\nu_i$ is a $E_n^1$-computable numbering. It is also worth noting that both $V_0$ and $V_1$ contain the set $M_l$.

Given that each set in $\mathcal{T}_j^{[l]}$ possesses a $(\nu_0 \oplus \nu_1)$-index, it follows that $V_0 \cup V_1 = \omega$. Additionally, since $M_l$ is $E_n^1$-maximal, there must exist at least one $V_i$ such that $V_i =^* \omega$. Consequently, only a limited number of sets $M_l \cup \{x\}$ in $\mathcal{T}_j^{[l]}$ lack $\nu_i$-indices, namely, those with $x \in \omega \setminus V_i$.

□

The fundamental concept behind the desired elementary differences is to work with *minimal pairs*.

Let us consider a family of sets, denoted by $\mathcal{V}$. We assume that this family is $E_n^1$-computable. Now, suppose we have two numberings of $\mathcal{V}$, denoted by $\nu_0$ and $\nu_1$, that are also $E_n^1$-computable.

**Definition 4.5.3** ([Badaev et al., 2005])**.** If two numberings $\nu_0$ and $\nu_1$ of $E^1_n$-computable family $\mathcal{V}$ induce a minimal pair inside $\mathcal{R}^1_n(\mathcal{V})$, it means that there is no other computable numbering $\mu$ of $\mathcal{V}$ that is $\mu \leq \nu_0$ and $\mu \leq \nu_1$. In other words, $\nu_0$ and $\nu_1$ are the smallest numberings we can find.

The lemma presented below provides a condition that is sufficient for discovering two distinct $E^1_n$-computable numberings of $\mathcal{T}_j$. These numberings *do not induce* a minimal pair, which is a pair of sequences that are distinct but cannot be distinguished by any computable function. By satisfying the conditions of this lemma, we can guarantee the existence of two computable numberings that will avoid such a scenario.

To clarify, we use the term "minimal pair" to describe two binary strings that have the smallest possible difference, which is to say, they differ in only a single bit.

Moving forward, when we talk about a binary string $\sigma$ that has a non-zero length $m$, we will refer to it as a tuple $(\sigma(1), \sigma(2), \ldots, \sigma(m))$. In addition, we use $|\sigma|$ to indicate the length of $\sigma$.

**Lemma 4.5.4.** *Suppose $\alpha^{[0]}_1$, $\alpha^{[1]}_1$, $\alpha^{[0]}_2$, $\alpha^{[1]}_2$, $\ldots$, $\alpha^{[0]}_{m+1}$, $\alpha^{[1]}_{m+1}$ be $E^1_n$-computable numberings of the family $\mathcal{T}_j$ and $j \geq 1$ and $m \geq j$. If $\alpha^{[0]}_1 \oplus \alpha^{[1]}_1 \equiv \alpha^{[0]}_2 \oplus \alpha^{[1]}_2 \equiv \cdots \equiv \alpha^{[0]}_{m+1} \oplus \alpha^{[1]}_{m+1}$, then there exist a binary string $\sigma$ and a $E^1_n$-computable numbering $\delta$ of $\mathcal{T}_j$ such that:*

*- the length of $\sigma$ is $m$.*

*- $\delta \leq \alpha^{[0]}_{m+1}$.*

*- $\delta \leq \alpha^{[\sigma(1)]}_1 \oplus \alpha^{[\sigma(2)]}_2 \oplus \cdots \oplus \alpha^{[\sigma(m)]}_m$.*

*Proof.* The following point is worth noting: If the numbering $\alpha^{[0]}_{m+1}$ can be reduced to any of the $\alpha^{[\rho]}_i$ with $1 \leq i \leq m$ and $\rho \in \{0, 1\}$, we can simply pick $\delta$ as $\alpha^{[0]}_{m+1}$ and the proof is complete. It is, therefore, safe to assume that $\alpha^{[0]}_{m+1}$ is not reducible to any of these $\alpha^{[\rho]}_i$, without losing the generality of the argument.

In order to find the desired objects, we need to follow a set of well-defined steps. Firstly, we need to select a value $\sigma(i)$ from the set $\{0, 1\}$ for every non-zero number $i$, where $i \leq j$. This selection must be followed by choosing a numbering $\delta^{[\sigma(i)]}_i$ for some subfamily of $\mathcal{T}_j$. It is essential to ensure that $\delta^{[\sigma(i)]}_i \leq \alpha^{[\sigma(i)]}_i$ and $\delta^{[\sigma(i)]}_i \leq \alpha^{[0]}_{m+1}$. Additionally, we must ensure that the family

$$\{A \in \mathcal{T}^{[i]}_j : A \text{ does not have a } \delta^{[\sigma(i)]}_i\text{-index}\}$$

has a finite number of elements.

Once we have established this, we can proceed to search for the desired objects. We begin by using Lemma 4.4.1 to show that $\alpha^{[0]}_{m+1} \leq \alpha^{[0]}_i \oplus \alpha^{[1]}_i$, which means that $\alpha^{[0]}_{m+1} = \delta^{[0]}_i \oplus \delta^{[1]}_i$, where $\delta^{[\rho]}_i \leq \alpha^{[\rho]}_i$. We then use Claim 4.5.4 to find at least one $\rho_i \in \{0,1\}$ such that all but finitely many elements from $\mathcal{T}^{[i]}_j$ have $\delta^{[\rho_i]}_i$-indices. We choose one such $\rho_i$, and define $\sigma(i) := \rho_i$.

The selection of $\delta^{[\sigma(i)]}_i$, where $1 \leq i \leq j$, indicates that

$$\delta^* := \delta^{[\sigma(0)]}_0 \oplus \delta^{[\sigma(1)]}_1 \oplus \cdots \oplus \delta^{[\sigma(j)]}_j \leq \alpha^{[0]}_{m+1},$$

$$\delta^* \leq \alpha^{[\sigma(0)]}_0 \oplus \alpha^{[\sigma(1)]}_1 \oplus \cdots \oplus \alpha^{[\sigma(j)]}_j.$$

If all the sets in the family $\mathcal{T}_j$ are indexed by $\delta^*$, then we simply set $\delta$ equal to $\delta^*$. Otherwise, if there are only finitely many sets $B_0, B_1, \ldots, B_r$ in $\mathcal{T}_j$ that do not have $\delta^*$-indices, we set $\delta'(k)$ as follows: if $k \leq r$, then $\delta'(k) = B_k$, otherwise $\delta'(k) = B_0$. Then, we set $\delta := \delta^* \oplus \delta'$.

It is easy to see that $\delta$ satisfies both $\delta \leq \alpha^{[0]}_{m+1}$ and $\delta \leq \alpha^{[\sigma(0)]}_0 \oplus \alpha^{[\sigma(1)]}_1 \oplus \cdots \oplus \alpha^{[\sigma(j)]}_j$. Therefore, for any $i$ such that $j < i \leq m$, we can arbitrarily choose the value of $\sigma(i)$. This completes the proof of Lemma 4.5.4.

$\square$

Now, we will demonstrate the process of obtaining minimal pairs within the context of $\mathcal{R}^1_n(\mathcal{T}^{[l]}_j)$. It is important to note that this applies specifically to cases where $j$ is greater than or equal to 1, and $l$ is between 1 and $j$. By following these steps, we can efficiently identify and distinguish minimal pairs in this particular setting.

Start by selecting two distinct numbers, $a^{[0]}$ and $a^{[1]}$, from $\overline{M}$. It's important to note that $p_l$ is a computable bijection from $\omega$ onto $R_l$. For $\rho \in \{0,1\}$ and $k \in \omega$, we define:

$$\gamma^{[\rho]}_l(k) := \begin{cases} M_l \cup \{p_l(a^{[\rho]})\}, & \text{if } k \in M, \\ M_l \cup \{p_l(k)\}, & \text{if } k \notin M. \end{cases}$$

We can use a similar argument as the one presented in Lemma 4.5.2 to demonstrate that $\gamma^{[\rho]}_l$ serves as a $E^1_n$-computable numbering for the family $\mathcal{T}^{[l]}_j$. This means that the elements in $\mathcal{T}^{[l]}_j$ can be effectively enumerated using $\gamma^{[\rho]}_l$, which is an important result in computational theory.

*Claim* 4.5.5. The pair of numberings $\gamma^{[0]}_l$ and $\gamma^{[1]}_l$ are fundamental as they form a minimal pair, which is the smallest possible pair of elements that are distinct within the set $\mathcal{R}^1_n(\mathcal{T}^{[l]}_j)$.

*Proof.* Assuming the existence of a numbering $\xi$ of $\mathcal{T}_j^{[l]}$ that satisfies both $\xi \leq \gamma_l^{[0]}$ and $\xi \leq \gamma_l^{[1]}$, we arrive at a contradiction. For each $\rho \in \{0,1\}$, we can define a computable function $f_\rho$ that reduces $\xi$ to $\gamma_l^{[\rho]}$.

For any given number $k$, the following statement is true:

1) If $f_0(k) = f_1(k)$, then $f_0(k)$ does not belong to $M$. To see why, let's assume that $f_0(k)$ does belong to $M$. This implies that:

$$M_l \cup \{p_l(a^{[0]})\} = \gamma_l^{[0]}(f_0(k)) = \xi(k) = \gamma_l^{[1]}(f_1(k)) = M_l \cup \{p_l(a^{[1]})\}$$

However, this contradicts with the fact that $a^{[0]} \neq a^{[1]}$.

2) If $\xi(k) \neq M_l \cup \{p_l(a^{[0]})\}$ and $\xi(k) \neq M_l \cup \{p_l(a^{[1]})\}$, then we can deduce that $f_0(k)$ is equal to $f_1(k)$. Both $f_0(k) \notin M$ and $f_1(k) \notin M$, which supports this conclusion. We can express this as:

$$M_l \cup \{p_l(f_0(k))\} = \gamma_l^{[0]}(f_0(k)) = \xi(k) = \gamma_l^{[1]}(f_1(k)) = M_l \cup \{p_l(f_1(k))\}.$$

Since we know that function $p_l$ is injective, we can confidently state that $f_0(k) = f_1(k)$.

Combining these two facts, we can conclude that the set $W$, defined as follows:

$$W := \{y \in \omega : \exists k(f_0(k) = f_1(k) = y)\}$$

is an infinite computably enumerable subset of $\overline{M}$. This, however, contradicts the $E_n^1$-maximality of $M$. Hence, it is clear that $\gamma_l^{[0]}$ and $\gamma_l^{[1]}$ must induce a minimal pair.

□

As we approach the end of the proof, it's important to highlight the next lemma, which is the last critical ingredient. This lemma will offer a sufficient condition for having *a lot of* of minimal pairs within $\mathcal{R}_n^1(\mathcal{T}_j)$. To ensure completeness, we will first revisit a simple combinatorial fact before presenting the lemma.

*Claim* 4.5.6. For any $N \geq 1$ and $k \geq 0$, let $j \geq 2^{N+k}$. Suppose we have a set $J := \{1, 2, \ldots, j\}$ containing $j$ elements, and then we can find $N$ subsets of $J$, denoted by $F_1, F_2, \ldots, F_N$, such that for any binary string of length $N$, denoted by $\sigma$, the intersection of the subsets $F_1^{\sigma(1)}, F_2^{\sigma(2)}, \ldots, F_N^{\sigma(N)}$ contains at least $2^k$ elements, i.e.

$$card(F_1^{\sigma(1)} \cap F_2^{\sigma(2)} \cap \cdots \cap F_N^{\sigma(N)}) \geq 2^k.$$

We use the notation $F^1 := F$ and $F^0 := J \setminus F$ for subsets $F$ of $J$.

*Proof Sketch.* To prove the statement, we only need to provide evidence for $j = 2^{N+k}$. After doing so, we can equate $J$ with $J^*$, which consists of all binary strings of length $N + k$. For any non-zero $i \leq N$, we define $F_i$ as:

$$F_i := \{\tau \in J^* : \tau(i) = 1\}.$$

□

**Lemma 4.5.5.** *The given statement asserts that for any positive integer value of $m \geq 1$ and any $j$ that is $j \geq 2^{2^m+m+1}$, there exist computable numberings of the family $\mathcal{T}_j$ which are denoted by $\beta_1^{[0]}, \beta_1^{[1]}, \beta_2^{[0]}, \beta_2^{[1]}, \dots, \beta_{2^m}^{[0]}, \beta_{2^m}^{[1]}$. These numberings satisfy the following conditions:*

*(1) $\beta_1^{[0]} \oplus \beta_1^{[1]} \equiv \beta_2^{[0]} \oplus \beta_2^{[1]} \equiv \cdots \equiv \beta_{2^m}^{[0]} \oplus \beta_{2^m}^{[1]}$. This means that the pairings of each $\beta_i^{[0]}$ and $\beta_i^{[1]}$ are equal to each other.*

*(2) The numberings $\beta_i^{[0]}$ and $\beta_i^{[1]}$ create a minimal pair inside $\mathcal{R}_n^1(\mathcal{T}_j)$ for any non-zero $i \leq 2^m$.*

*(3) Suppose we have a binary string $\sigma$ with a length $|\sigma| = t$ and a set $I = \{i_1 <_\omega i_2 <_\omega \cdots <_\omega i_t\} \subset \{1, 2, \dots, 2^m\}$ for any non-zero number $t \leq m$. Then the following numberings induce a minimal pair inside a family $\mathcal{R}_n^1(\mathcal{T}_j)$ for any $\rho \in \{0, 1\}$ and any $i \in \{1, 2, \dots, 2^m\} \setminus I$:*

$$\beta_i^{[\rho]} \text{ and } \beta_{i_1}^{[\sigma(1)]} \oplus \beta_{i_2}^{[\sigma(2)]} \oplus \cdots \oplus \beta_{i_t}^{[\sigma(t)]}$$

*Proof.* We start by defining the set $J$ as $J := \{1, 2, \dots, j\}$. According to Claim 4.5.6, it is possible to select some subsets $F_1, F_2, \dots, F_{2^m}$ of the set $J$ such that for any binary string $|\sigma| = 2^m$, we can ensure that:

$$card(F_1^{\sigma(1)} \cap F_2^{\sigma(2)} \cap \cdots \cap F_{2^m}^{\sigma(2^m)}) \geq 2^{m+1}.$$

Next, we define a numbering $\beta_i^{[\rho]}$ for every non-zero $i \leq 2^m$ and every $\rho \in \{0, 1\}$ as follows:

$$\beta_i^{[\rho]} := \Big(\bigoplus_{l \in F_i^0} \gamma_l^{[1-\rho]}\Big) \oplus \Big(\bigoplus_{l \in F_i^1} \gamma_l^{[\rho]}\Big), \tag{4.5.1}$$

where the numberings $\gamma_l^{[0]}$ and $\gamma_l^{[1]}$ are the same as those in Claim 4.5.5. We will now proceed to demonstrate that the numberings $\beta_i^{[\rho]}$ satisfy the lemma.

(1) For any value of $i$ that is not equal to zero and is less than or equal to $2^m$, it is evident that:

$$\beta_i^{[0]} \oplus \beta_i^{[1]} \equiv \bigoplus_{1\leq l\leq j} (\gamma_l^{[0]} \oplus \gamma_l^{[1]}).$$

(2) Suppose that there exists a numbering $\xi$ of the family $\mathcal{T}_j$ such that $\xi \leq \beta_i^{[0]}$ and $\xi \leq \beta_i^{[1]}$. This assumption, however, leads to a contradiction. To make the situation less complex, we can assume that the number 1 belongs to $F_i$. As a result, we can conclude that $\gamma_1^{[0]} \leq \beta_i^{[0]}$ and $\gamma_1^{[1]} \leq \beta_i^{[1]}$.

It is important to note that each family $\mathcal{T}_j^{[l]}$, where $1 \leq l \leq j$, is completely independent of the others. Consequently, it is impossible for $\xi$ to be less than or equal to $\alpha_l^{[\rho]}$ for all values of $l$ and $\rho$. Since $\xi \leq \beta_i^{[1]}$, we can apply Lemma 4.4.1 to deduce that there exist index values $\xi_l$, where $1 \leq l \leq j$, that precisely indexes the family $\mathcal{T}_j^{[l]}$. This can be expressed as:

$$\xi \equiv \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_j,$$

where each $\xi_l$ can be reduced to an appropriate $\gamma_l^{[\rho]}$ taken from the decomposition of $\beta_i^{[1]}$ (by equation (4.5.1)). Specifically, we have $\xi_1 \leq \gamma_1^{[1]}$.

Considering the relation between $\xi$ and $\beta_i^{[0]}$, we can deduce that $\xi_1$ can be reduced to $\gamma_1^{[0]}$. As a result, it becomes clear that the numberings $\gamma_1^{[0]}$ and $\gamma_1^{[1]}$ do not constitute a minimal pair. This fact contradicts the Claim 4.5.5.

(3) Suppose that there exists a numbering denoted by $\xi$ for $\mathcal{T}_j$, which satisfies the conditions $\xi \leq \beta_i^{[\rho]}$ and $\xi \leq \beta_{i_1}^{[\sigma(1)]} \oplus \beta_{i_2}^{[\sigma(2)]} \oplus \cdots \oplus \beta_{i_t}^{[\sigma(t)]}$. By following the same steps as in the proof of (2), we can break down $\xi$ into $\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_j$, where every $\xi_l$ indexes $\mathcal{T}_j^{[l]}$. Additionally, we can simplify $\xi_l$ to an appropriate $\gamma_l^{[\varepsilon]}$, which is obtained from (4.5.1) for the numbering $\beta_i^{[\rho]}$.

Suppose that $t+1 \leq m+1 \leq 2^m$. In such a scenario, there is a specific number, denoted as $l^*$, which lies in the intersection of several sets, namely

$$l^* \in F_i^{\rho} \cap F_{i_1}^{1-\sigma(1)} \cap F_{i_2}^{1-\sigma(2)} \cap \cdots \cap F_{i_t}^{1-\sigma(t)}.$$

It is worth noting that $l^*$ belongs to $F_i^{\rho}$, which implies that $\gamma_{l^*}^{[1]}$ appears in the decomposition of $\beta_i^{[\rho]}$ provided by equation (4.5.1), while $\gamma_{l^*}^{[0]}$ does not appear. Consequently, we can deduce that $\xi_{l^*}$ can be reduced to $\gamma_{l^*}^{[1]}$.

We find that only $\gamma_{l^*}^{[0]}$ appears in the decomposition, not $\gamma_{l^*}^{[1]}$, in the corresponding decomposition of $\beta_{i_p}^{[\sigma(p)]}$ for non-zero values of $p$ such that $p \leq t$. By reducing $\xi$ to

$\beta_{i_1}^{[\sigma(1)]} \oplus \cdots \oplus \beta_{i_t}^{[\sigma(t)]}$, we can determine that $\xi_{l^*}$ can be reduced to $\gamma\overset{\Theta}{[}0]_{l^*}$. This means that $\gamma_{l^*}^{[0]}$ and $\gamma_{l^*}^{[1]}$ do not form a minimal pair, which goes against the claim made in Claim 4.5.5. As a result, we can confirm the validity of Lemma 4.5.5.

□

To complete the proof, it is necessary to introduce a function known as $h(x)$ that can be computed. This function operates as follows:

$$h(0) := 1, \quad h(e+1) := 2^{2^{h(e)}+h(e)+1}.$$

Our next step is to demonstrate that the Rogers $E_n^1$-semilattices of the families $\mathcal{S}_i := \mathcal{T}_{h(i)}$ differ in their elementary theories when compared to one another.

Assume $i$ is a value smaller than $e$. As per the statement provided in Lemma 4.5.4, the structure $\mathcal{R}_n^1(\mathcal{S}_i)$ possesses a particular characteristic. Given any $\alpha_1^{[0]}$, $\alpha_1^{[1]}$, $\alpha_2^{[0]}$, $\alpha_2^{[1]}$, …, $\alpha_{h(i)+1}^{[0]}$, $\alpha_{h(i)+1}^{[1]}$, such that

$$\alpha_1^{[0]} \oplus \alpha_1^{[1]} \equiv \alpha_2^{[0]} \oplus \alpha_2^{[1]} \equiv \cdots \equiv \alpha_{h(i)+1}^{[0]} \oplus \alpha_{h(i)+1}^{[1]},$$

we can obtain a binary string $\sigma$. The length of this string must be equal to $h(i)$. Furthermore, the ordering of $\alpha_{h(i)+1}^{[0]}$ and $\alpha_1^{[\sigma(1)]} \oplus \alpha_2^{[\sigma(2)]} \oplus \cdots \oplus \alpha_{h(i)}^{[\sigma(h(i))]}$ should not result in a minimal pair.

Based on Lemma 4.5.5, it can be concluded that the property does not hold within $\mathcal{R}_n^1(\mathcal{S}_e)$. This is because $h(e) \geq h(i+1) = 2^{2^{h(i)}+h(i)+1}$. As a result, Theorem 4.5.1 has been proven.

□

Remark 4.2.2 leads to the following conclusion:

**Corollary 4.5.6.** *Without assuming **PD**, Theorem 4.5.1 still holds true for classes* $\Sigma_1^1$ *and* $\Sigma_2^1$.

## 4.6 Further discussion

Upon careful examination of the preceding sections, we are interested in exploring the potential results that can be obtained by applying alternative set-theoretic assumptions, instead of **PD**, on Rogers semilattices of $\mathcal{C}$-computable families if $\mathcal{C}$ is the class of the analytical hierarchy.

We will provide a case study for the problem and assume the Axiom of Constructibility ($V = L$). We will also list some of the outcomes that can be achieved with this assumption.

For clarity, the Axiom of Constructibility states that every set is constructible, and one can find the formal statement of the axiom in Chapter 13 of [Jech, 2002]. Note that the prewellordering property is the critical feature of class $E_n^1$, which was extensively used in the previous sections.

**Theorem 4.6.1** (refer to Exercises 5A.3 and 4B.10 of [Moschovakis, 2009])**.** *Assuming $V = L$, we can conclude that for $n \geq 3$, the class $\Sigma_n^1$ has the prewellordering property. Consequently, $\Sigma_n^1$ satisfies the reduction principle, which is another important property of this class.*

**Corollary 4.6.2** ($\mathbf{V = L}$)**.** *For a finite collection of sets $\mathcal{S}$ that belong to $\Sigma_n^1$ and for any $n \geq 3$:*

*1. The Rogers semilattice $\mathcal{R}_{\Sigma_n^1}(\mathcal{S})$ has the greatest element only if $\mathcal{S}$ contains a least element under the relation of set inclusion $\subseteq$.*

*2. If $\mathcal{R}_{\Sigma_n^1}(\mathcal{S})$ does not have greatest element, then each element in $\mathcal{R}_{\Sigma_n^1}(\mathcal{S})$ has a minimal cover.*

We have noticed a straightforward observation:

*Remark* 4.6.3. Assuming that $\mathcal{S}$ is a finite family of $\Sigma_3^1$ sets, there are two scenarios to consider:

(a) If we assume **PD**, then it follows that $\mathcal{R}_{\Sigma_3^1}(\mathcal{S})$ has a greatest element if and only if $\mathcal{S}$ contains a *greatest* element under the subset relation $\subseteq$.

(b) If we assume ($V = L$), then $\mathcal{R}_{\Sigma_3^1}(\mathcal{S})$ has a greatest element if and only if $\mathcal{S}$ contains a *least* element under $\subseteq$.

In conclusion, it is important to note that all of our proofs were based solely on properties that are inherent to Spector pointclasses, which can be found in Section 4C of [Moschovakis, 2009]. Therefore, we can confidently state that the conclusions drawn from our analysis are firmly grounded in well-established principles and concepts. Therefore, we formulate:

**Problem 4.6.4.** Develop the theory of Rogers semilattices for Spector pointclasses.

# Chapter 5

# Conclusion

Since the late 1960s, research in the theory of numberings has primarily focused on Rogers semilattices. The research problems introduced by Yu. Ershov [Ershov, 1967, Ershov, 1968, Ershov, 1977b] for Rogers semilattices of the families of sets from arithmetical, analytical, and Ershov hierarchies have been thoroughly investigated and mostly solved. However, the classical studies of Rogers semilattices of punctual numberings, polynomial-time numberings, as well as limitwise monotonic numberings have become a new area of research.

This work presents several results that reflect the algebraic and elementary properties of the Rogers semilattices for different families of sets. Specifically, we have considered limitwise monotonic numberings for a family of limitwise monotonic sets. We have investigated the properties of Rogers semilattices of limitwise numberings and obtained the following results:

(1) For any $\Sigma^0_2$-computable family $\mathcal{S}$, there exists a l.m. family $\mathcal{S}'$ such that the semilattices $\mathcal{R}_{lm}(\mathcal{S}')$ and $\mathcal{R}^0_2(\mathcal{S})$ are isomorphic (Theorem 3.3.2). In particular, this implies that there are infinitely many pairwise non-elementarily-equivalent Rogers semilattices for l.m. families (Corollary 3.3.5).

(2) In addition, if an infinite l.m. family $\mathcal{S}$ contains only infinite sets, then the Rogers semilattices $\mathcal{R}_{lm}(\mathcal{S})$ and $\mathcal{R}^0_2(\mathcal{S})$ are *equal* (Theorem 3.3.3).

(3) There are infinitely many pairwise non-isomorphic Rogers semilattices for $\Sigma^0_1$-computable families which can be realized as Rogers semilattices for l.m. families.(Proposition 3.4.2 and Observation 3.4.1). In particular, the semilattice of c.e. $m$-degrees $\mathbf{R}_m$ is isomorphic to the Rogers semilattice of a l.m. family.

(4) if a l.m. family $\mathcal{S}$ contains more than one element, then the poset $\mathcal{R}_{lm}(\mathcal{S})$ is infinite, and it is not a lattice (Theorem 3.5.1).

(5) the index set of limitwise monotonic numberings (in the class of all $\Sigma^0_2$-computable numberings) is $\Sigma^0_4$-complete (Theorem 3.6.1)

The Rogers semilattice was studied using a different approach in the analytical hierarchy. When working in this hierarchy, additional set-theoretic axioms are required to obtain significant results for the families of sets at levels beyond level 3. It was shown in [Bazhenov and Mustafa, 2020, Bazhenov et al., 2020, Bazhenov et al., 2022] that assuming the axiom of analytical determinacy leads to many well-known results for numberings in the arithmetical hierarchy being transferred to the analytical hierarchy. In this work, we continued the investigation of [Bazhenov and Mustafa, 2020] and proved that there are infinitely many Rogers semilattices at each level of the analytical hierarchy that are not equivalent to each other. This result is an adaptation of the work by S.A. Badaev, S.S. Goncharov, and A. Sorbi [Badaev et al., 2005].

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