

Modelling and Analysis of Magnetic Force Between Two Permanent Magnets

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Abstract

This capstone thesis studies magnetic force between two permanent magnets. The analytical formula that models the force between two magnets is derived where Gauss Hypergeometric and Gamma functions will be used in the modelling of magnetic force. Also, we derive its asymptotics for small distances. There is an experimental part when we use magnets of different sizes separated by different distances. Very small distances that cannot be achieved in experiments will be simulated using the open-source Python package Magpylib or the commercial FEM package COMSOL. The analytical and numerical results will be validated by experiments.

Keywords: cylindrical magnets, magnetic force, hypergeometric function, COMSOL, Magpylib

1. Introduction

Magnetic interactions lie in the basis of different micro and macrodevices- including magnetic clamps, sensors and resonance systems, where precise knowledge of the force of magnetic attraction as a function of the distance between magnets is required. In modeling arrays of magnetic elements, elementary dipole approximation is widely used, but this is only correct for sufficiently large distances between elements, in the near-contact case, the force becomes significantly nonlinear, and for a more precise prediction, it is necessary to take into account the geometry and shape effects. This transition from dipole approximations to geometrically accurate models is emphasized in works on the interaction between cylindrical magnets: the dipole approach is not always acceptable, and for small distances a more advanced analytical description is required.

In the literature, there are several approaches for field and force calculation of cylindrical magnets. On

the one hand, there are analytical and semi-analytical models for coaxial cylindrical/disk magnets which are based on magnetostatic energy and special functions.

For example, one of the most cited analytical literature is the works by Agashe&Arnold (2008, 2009) [1] [2]. The authors derive analytical expressions for the forces between cuboidal and cylindrical magnets using Kelvin's formula, magnetic fields from an equivalent surface current model, and Biot-Savart's law.

Derby & Olbert (2010) [3] obtain expressions for the components of the field of a finite solenoid, reducing the integrals to elliptic special functions and proposing stable methods for their calculation. The importance of this approach is that it provides a fast and accurate expressions for magnetic field components, B_z and B_r , after these components are known the force between the disks can be obtained using the energy method, $\mathcal{F} = -dE/dD$, through evaluation of the Maxwell stress tensor, or by calculating the interaction of surface magnetic charge distributions.

The work by Ravaut et al. (2010) [4] proposes the set of formulas for direct calculation of force: they present a unified analytical approach for coaxial cylindrical sources (magnets/coils), derive expressions for B , $\mathcal{F}_z(d)$ and for axial stiffness $Kz = d\mathcal{F}/dZ$ using elliptic integrals, and verify the accuracy by comparison with FEM/filament method. For our capstone project, it is convenient to use it as an analytical basis for checking the integral representation of force and comparison with the numerical validation.

Vokoun et al. (2009, 2011) [5] represent the key work in the concept of axial forces between cylindrical magnets. They derive the force using magnetostatic energy and its derivative with respect to the distance, and the energy itself is expressed as the integrals in the Bessel basis (Hankel/Fourier-Bessel representations). This approach is essential for our problem since it works in axisymmetry and remains applicable at small distances keeping the assumption of uniform magnetization. The papers also contain the discussion of experimental verification and limits of dipole approximation.

In applied problems, in particular, in microscale, it is important to answer questions of accuracy of "closed" formulas over a wide range. Several papers show that some closed form expressions might lose accuracy precisely with small gaps.

The goal of this capstone research is to derive and analyse the expression for force between two cylindrical permanent magnets at small distances. These magnets can be represented as equations of oscillator motion. In our research we are considering two coaxial cylindrical magnets with the same radius (R) and from the same material with magnetization (M). The interacting force between magnets is identified by the magnetostatic energy system. Due to the fact that the magnetic field in our case is static, the system is conservative, which means that mechanical force is linked with potential energy. Generally, force equals negative gradient of

energy. Therefore, force is directed to decrease energy. The key mathematical accent of our work is that we transform initial integral of force to integral of $(I(y) = \int_0^\infty y \frac{J_1^2(q)}{q} e^{-yq} dq)$ and express it with hypergeometric functions. After that we will use asymptotic expansion for later practical use. Finally, we will validate results of force with numerical calculations on COMSOL for chosen parameters of NdFeB-magnets.

The paper is organized as follows: (1) derivation of force integral and reduction to $I(y)$; (2) getting hypergeometric representation and checking equality of forms (Maple and WolframAlpha); (3) asymptotic decomposition for small and large y and discussion on application; (4) numeric validation (COMSOL) and results for later dynamic modelling of magMEMS-oscillators.

2. Derivation of the axial attraction force between two coaxial cylindrical magnets

We consider two identical and uniformly magnetized cylindrical magnets. They have common symmetry axis, which is the z axis. The magnets have equal radius (R) and magnetization saturation (M). We assume that magnets are made of same material and M is constant as well as uniformly distributed inside of the magnets. We denote t_1 and t_2 as thickness or in other words as height of the magnets which are cylindrical. In order to keep dimensionless, the aspect ratios are defined.

$$\tau_i = \frac{t_i}{2R}, \quad i = 1, 2,$$

The distance between the centers of the magnets is denoted as Z . Similarly, the dimensionless ratio of it is given by

$$\zeta = \frac{Z}{R}.$$

This model is axisymmetric. They are placed perfectly above each other. Therefore, all lateral forces cancel. The net force is only axial as the force acts only along the Z -axis.

$$\mathcal{F} = \mathcal{F}_z \hat{\mathbf{z}}.$$

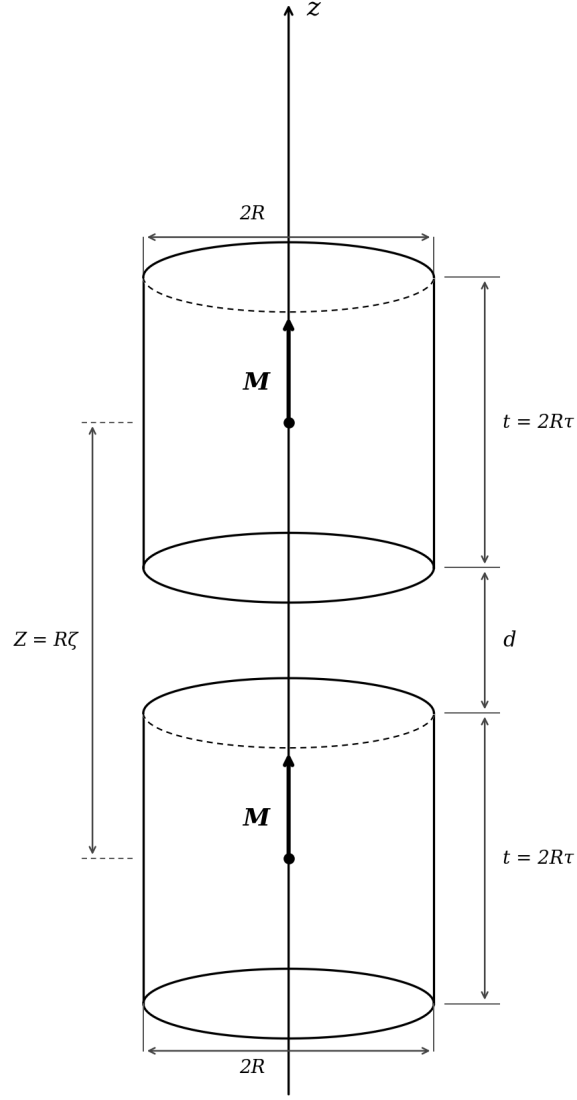


Figure 1: Two cylindrical magnets with magnetization M , radius R , thickness t separated by distance d

2.1. The conservative force resulting from magnetostatic interaction energy

In this work, the force in terms of energy is defined as the rate of change of the energy over the distance, mathematically it corresponds to the negative gradient of magnetostatic interaction energy of the two-magnet

system

$$\mathcal{F} = -\nabla E, \quad (1)$$

where the energy E is an interaction energy in the coaxial configuration and is given as [5]

$$E(Z) = -2\pi\mu_0 M^2 R^3 J_d(\tau_1, \tau_2, \zeta) + \text{const}. \quad (2)$$

In the case of magnets, the energy and the force depend on the separation between magnets Z , the overall expression of the force can be reduced to the following

$$\mathcal{F}_z = -\frac{\partial E}{\partial Z}. \quad (3)$$

The sign is important because for magnets, the attraction force which works to decrease the separation between magnets, the sign is negative, for the repulsion force it is positive.

2.2. Magnetization model

For a uniformly magnetized body, where the vector of magnetization is distributed equally in every point of the volume and with the same direction over the volume, a convenient representation for magnetostatic calculations is the bound magnetic surface charge density

$$\sigma_m = \mathbf{M} \cdot \mathbf{n}, \quad (4)$$

$\mathbf{n}$ is the outward unit normal. In our model an axially magnetized cylinder has magnetization expression in vector term as $\mathbf{M} = M\hat{\mathbf{z}}$:

- On the side surfaces n and $\hat{z}$ are perpendicular to each other, $\mathbf{n} \perp \hat{\mathbf{z}}$, hence magnetic surface charge density $\sigma_m = 0$.
- On the two end faces, $\mathbf{n} = \pm\hat{\mathbf{z}}$, hence $\sigma_m = \pm M$. When n and $\hat{z}$ have the same direction M is charged positively and when they have opposite directions M has negative sign.

Therefore, we can consider the cylindrical magnets as two uniformly charged disks with radius R , and they have opposite signs.

2.3. Bessel and Hankel integrals

In the basis of any problem related to fields there is potential of singular point. According to the laws of electrostatics and magnetostatics the potential created by point at any other point in the plane is inversely

proportional to the distance between the points. Let r be point of observation and r' is point where the source of the charge is located. For cylinders, it is more convenient to use cylindrical coordinates (ρ, ϕ, z) .

Mathematically to model the potential of one point we use following expansion (after integrating out the azimuthal angle) [5]

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \int_0^\infty J_0(k\rho) J_0(k\rho') e^{-k|z-z'|} dk, \quad (5)$$

where (ρ, z) and (ρ', z') are cylindrical coordinates and J_0 is the Bessel function of the first kind. The cylinders are uniformly magnetized and axially symmetric, magnetization at the points do not depend on azimuthal angle.

In order to model the whole field of the disk we need to sum (integrate) the potentials of all the points on it. The integration element of disk area in cylindrical coordinates: $dS' = \rho' d\rho' d\phi'$.

$$\text{The disk potential} \propto \int_0^R (\text{point potential}) \cdot \rho' d\rho'$$

According to the Bessel standard identity

$$\int x J_0(x) dx = x J_1(x).$$

As we are considering the disks, the z coordinate is the same for all points on the disk, which makes the $z - z' = 0$. Summarizing all the above expressions we got the following for the one disk ($\rho = 0, \rho' = R$):

$$\int_0^R \rho' J_0(k\rho') d\rho' = \frac{R J_1(kR)}{k}, \quad (6)$$

J_1 is the Bessel function of the first kind of order one. The result shows that the contribution of one disk is $\frac{R J_1(kR)}{k}$, and for the interaction of two disks we got the product

$$\left(\frac{R J_1(kR)}{k} \right)^2 = R^2 \frac{J_1^2(kR)}{k^2}. \quad (7)$$

The dimensionless wavenumber

$$q = kR, \quad (8)$$

which leads to $R^4 \frac{J_1^2(q)}{q^2}$. It will be used in further integrations.

In the Hankel representation, the exponential factors are of the form $e^{-k\Delta z} = e^{-q(\Delta z/R)}$. Considering the superposition of two ends with opposite 'charges' and thickness $t_i = 2R\tau_i$, it results in difference of exponentials.

$$\sinh(q\tau_i) = \frac{e^{q\tau_i} - e^{-q\tau_i}}{2}, \quad (9)$$

Therefore, thickness dependence of cylinders is denoted by $\sinh(q\tau_1) \sinh(q\tau_2)$.

Now we move on to the interaction between two magnets. We need to combine potential of interaction of two disks (7), thickness dependence (9) and exponential for separation of magnets. The obtained coupling integral is provided below

$$J_d(\tau_1, \tau_2, \zeta) = 2 \int_0^\infty \frac{J_1^2(q)}{q^2} \sinh(q\tau_1) \sinh(q\tau_2) e^{-q\zeta} dq. \quad (10)$$

where $e^{-q\zeta}$ stands for decay with separation.

2.4. Magnetic force in terms of Bessel/Hankel integrals

The interaction energy in the coaxial configuration is [5]

$$E(Z) = -2\pi\mu_0 M^2 R^3 J_d(\tau_1, \tau_2, \zeta) + \text{const.} \quad (11)$$

Using the conservative-force relation (3) leads to

$$\mathcal{F}_z = -\frac{\partial E}{\partial Z} = 2\pi\mu_0 M^2 R^3 \frac{\partial J_d}{\partial Z}. \quad (12)$$

Step 1:. According to the chain rule and since $\zeta = Z/R$,

$$\frac{\partial}{\partial Z} = \frac{1}{R} \frac{\partial}{\partial \zeta}, \quad \Rightarrow \quad \frac{\partial J_d}{\partial Z} = \frac{1}{R} \frac{\partial J_d}{\partial \zeta}. \quad (13)$$

Step 2: . From (10), only factor that depends on ζ is $e^{-q\zeta}$; hence,

$$\frac{\partial}{\partial \zeta} (e^{-q\zeta}) = -q e^{-q\zeta}. \quad (14)$$

Plugging into the $\frac{\partial J_d}{\partial \zeta}$ leads to

$$\begin{aligned} \frac{\partial J_d}{\partial \zeta} &= 2 \int_0^\infty \frac{J_1^2(q)}{q^2} \sinh(q\tau_1) \sinh(q\tau_2) (-q e^{-q\zeta}) dq \\ &= -2 \int_0^\infty \frac{J_1^2(q)}{q} \sinh(q\tau_1) \sinh(q\tau_2) e^{-q\zeta} dq. \end{aligned} \quad (15)$$

Step 3:. After combining results from (13) and (15) and converting back to $\frac{\partial J_d}{\partial Z}$ leads to,

$$\frac{\partial J_d}{\partial Z} = -\frac{2}{R} \int_0^\infty \frac{J_1^2(q)}{q} \sinh(q\tau_1) \sinh(q\tau_2) e^{-q\zeta} dq. \quad (16)$$

Substituting this back into the force equation (12) results

$$\begin{aligned} \mathcal{F}_z &= 2\pi\mu_0 M^2 R^3 \left[-\frac{2}{R} \int_0^\infty \frac{J_1^2(q)}{q} \sinh(q\tau_1) \sinh(q\tau_2) e^{-q\zeta} dq \right] \\ &= -4\pi\mu_0 M^2 R^2 \int_0^\infty \frac{J_1^2(q)}{q} \sinh(q\tau_1) \sinh(q\tau_2) e^{-q\zeta} dq. \end{aligned} \quad (17)$$

Since $\tau_1, \tau_2 > 0$ and $\zeta > 0$, the integral will be positive. Consequently $F_z < 0$, which corresponds to the attraction of magnets.

$$\mathcal{F}_z = -4\pi\mu_0 M^2 R^2 \int_0^{+\infty} \frac{J_1^2(q)}{q} \sinh(q\tau_1) \sinh(q\tau_2) e^{-q\zeta} dq \quad (18)$$

Because cylinders have same thickness, radius, height, and other parameters, $\tau_1 = \tau_2 = \tau$ and placed symmetrically, it leads to

$$\sinh^2(\tau q) = \frac{1}{4}(e^{2\tau q} - 2 + e^{-2\tau q})$$

Introducing the parameter $K = -4\pi\mu_0 M^2 R^2$, we have, (17) will become:

$$\begin{aligned} \mathcal{F}[\zeta; \tau; K] = & \frac{1}{4}K \cdot \int_0^{\infty} \frac{J_1^2(q)}{q} \cdot e^{-(\zeta-2\tau)q} dq \\ & - \frac{1}{2}K \cdot \int_0^{\infty} \frac{J_1^2(q)}{q} \cdot e^{-\zeta q} dq \\ & + \frac{1}{4}K \cdot \int_0^{\infty} \frac{J_1^2(q)}{q} \cdot e^{-(\zeta+2\tau)q} dq \end{aligned} \quad (19)$$

Hence, there is a need to analyze the integral of type:

$$I(y) = \int_0^{\infty} \frac{J_1^2(q)}{q} \cdot e^{-yq} dq \quad (20)$$

According to the Maple software this integral can be expressed as hypergeometric function:

$$I(y) = \int_0^{\infty} \frac{1}{q} e^{-yq} (J_1(q))^2 dq = \frac{1}{4y^2} {}_3F_2\left(1, \frac{3}{2}, \frac{3}{2}; 2, 3; -\frac{4}{y^2}\right) \quad (21)$$

According to the Mathematica Wolfram, the above hypergeometric function can be expressed is simplified the following way:

$$\frac{1}{4y^2} {}_3F_2\left(1, \frac{3}{2}, \frac{3}{2}; 2, 3; -\frac{4}{y^2}\right) = \frac{1}{2} - \frac{1}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; -\frac{4}{y^2}\right) \quad (22)$$

The details of Maple and Mathematics Wolfram software are provided in Appendix A. Hence, the force above can be written as:

$$\begin{aligned} \mathcal{F}(\zeta; \tau; K) = & \frac{1}{4}K \left(\frac{1}{2} - \frac{1}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; -\frac{4}{(\zeta-2\tau)^2}\right) \right) \\ & - \frac{1}{2}K \left(\frac{1}{2} - \frac{1}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; -\frac{4}{\zeta^2}\right) \right) \\ & + \frac{1}{4}K \left(\frac{1}{2} - \frac{1}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; -\frac{4}{(\zeta+2\tau)^2}\right) \right) \end{aligned} \quad (23)$$

First of all, there is a need to compare the Hypergeometric function provided by Maple software to the simplified version of Mathematica Wolfram:

$$\begin{aligned} I(y) &= \frac{1}{4y^2} {}_3F_2\left(1, \frac{3}{2}, \frac{3}{2}; 2, 3; -\frac{4}{y^2}\right) \\ &= \frac{1}{2} - \frac{1}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; -\frac{4}{y^2}\right) \end{aligned} \quad (24)$$

Indeed, the Figure 2 shows that they are identical:

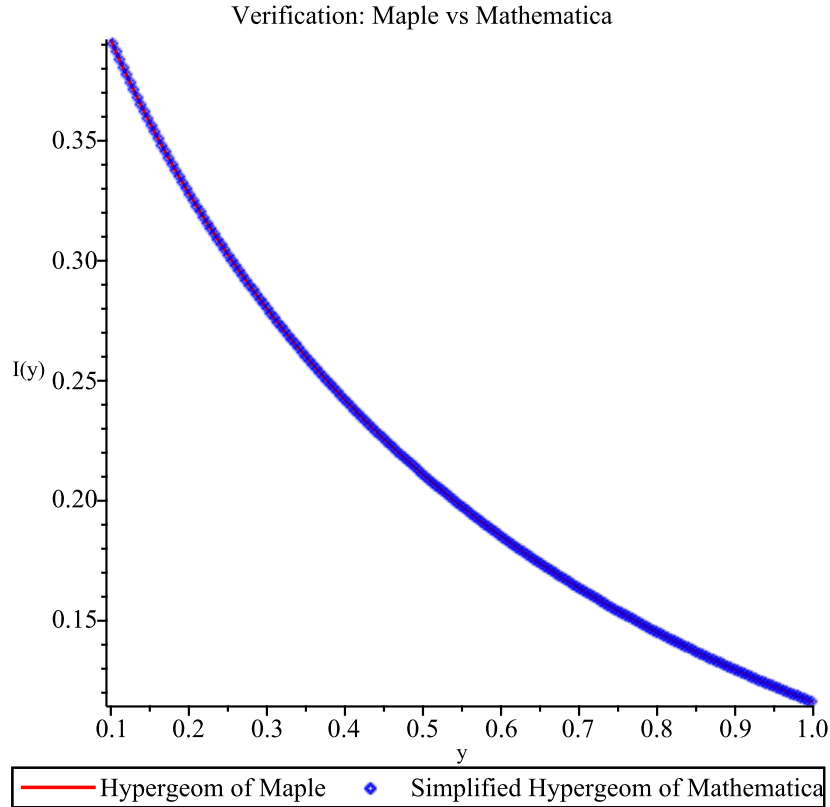


Figure 2: Maple vs Mathematica Wolfram

3. Hypergeometric function asymptotic series

We need to analyze the hypergeometric function:

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; -\frac{4}{y^2}\right)$$

From Abramowitz and Stegun book Hypergeometric function defined as: [6]:

$${}_2F_1(a, a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}(-z)^{-a} \times \left[\sum_{n=0}^{\infty} \frac{(a)_n(1-c+a)_n}{(n!)^2} z^{-n} (\ln(-z) + 2\psi(n+1) - \psi(a+n) - \psi(c-a-n)) \right] \quad (25)$$

where

$$|\arg(-z)| < \pi, \quad |z| > 1, \quad (c-a) \neq 0, \pm 1, \pm 2, \dots$$

Taking only the first term of the asymptotic series, setting $n=0$, we obtain the following result for the sum

$$\frac{\Gamma(2)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} \left(\frac{4}{y^2}\right)^{-\frac{1}{2}} \left[\ln\left(\frac{4}{y^2}\right) + 2\psi(1) - \psi\left(\frac{1}{2}\right) - \psi\left(\frac{3}{2}\right) \right]$$

where ψ is digamma function. The explanations of gamma and digamma functions are provided in Appendix A. Because of following result

$$\sum_{n=0}^0 \frac{(1/2)_0(1-2+1/2)_0}{(0!)^2} \left(-\frac{4}{y^2}\right)^0 = 1$$

as Pochhammer's symbol $(a)_0 = 1$, $(a)_1 = a$, $(a)_2 = a(a+1)$. For half integers ψ can be defined as:

$$\psi\left(\frac{1}{2} + n\right) = -\gamma - 2\ln(2) + \sum_{k=1}^n \frac{2}{2k-1}$$

So, we have:

$$\begin{aligned} \psi\left(\frac{1}{2}\right) &= -\gamma - 2\ln(2) \\ \psi\left(\frac{3}{2}\right) &= -\gamma - 2\ln(2) + 2 \\ \psi\left(\frac{5}{2}\right) &= -\gamma - 2\ln(2) + \frac{8}{3} \\ \psi\left(-\frac{1}{2}\right) &= -\gamma - 2\ln(2) + 2 \end{aligned}$$

and also $\psi(1) = -\gamma$, $\psi(2) = -\gamma + 1$. Additionally, $\Gamma(2) = 1$, $\Gamma(0.5) = \sqrt{\pi}$ and $\Gamma(1.5) = \frac{1}{2}\sqrt{\pi}$. Similarly, solving for $n=1$ and $n=2$. The summing term of (25) for $n=1$:

$$\begin{aligned} & \frac{(\frac{1}{2})_1(1-2+\frac{1}{2})_1}{(1!)^2} \left(-\frac{4}{y^2}\right)^{-1} \left[\ln\left(-\left(-\frac{4}{y^2}\right)\right) + 2\psi(1+1) - \psi\left(\frac{1}{2}+1\right) - \psi\left(2-\frac{1}{2}-1\right) \right] \\ &= \frac{(1/2)(-1/2)}{(1!)^2} \left(-\frac{y^2}{4}\right) \left[\ln\left(\frac{4}{y^2}\right) + 2\psi(2) - \psi\left(\frac{3}{2}\right) - \psi\left(\frac{1}{2}\right) \right] \\ &= \frac{y^2}{16} \left[\ln\left(\frac{4}{y^2}\right) + 4\ln 2 \right]. \end{aligned}$$

The summing term of (25) for $n=2$

$$\begin{aligned}
& \frac{(\frac{1}{2})_2 (1-2+\frac{1}{2})_2}{(2!)^2} \left(-\frac{4}{y^2}\right)^{-2} \left[\ln\left(-\left(-\frac{4}{y^2}\right)\right) + 2\psi(2+1) - \psi\left(\frac{1}{2}+2\right) - \psi\left(2-\frac{1}{2}-2\right) \right] \\
&= \frac{(3/4)(-1/4)}{16} \left(\frac{y^4}{16}\right) \left[\ln\left(\frac{4}{y^2}\right) + 2\psi(3) - \psi\left(\frac{5}{2}\right) - \psi\left(-\frac{1}{2}\right) \right] \\
&= -\frac{3y^4}{4096} \left[\ln\left(\frac{4}{y^2}\right) + 4\ln 2 - \frac{5}{3} \right].
\end{aligned}$$

Finally, after plugging in all values and simplification it will be equal to:

$$\begin{aligned}
{}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 2, -\frac{4}{y^2}\right) &= \frac{y}{\pi} \left((6\ln(2) - 2\ln(y) - 2) + \frac{y^2}{16} (6\ln(2) - 2\ln(y)) \right. \\
&\quad \left. - \frac{3y^4}{1024} (6\ln(2) - 2\ln(y) - \frac{5}{3}) + O(y^6) \right).
\end{aligned}$$

for $y < 2$. For large y , $|z| \ll 1$, because $z = -4/y^2$, the ordinary Maclaurin series can be used to expand the hypergeometric function

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)2!}z^2 + \dots$$

In our case, $a = b = \frac{1}{2}$, $c = 2$, $z = -\frac{4}{y^2}$. Therefore, final expression for hypergeometric function for $y > 2$ is:

$$\begin{aligned}
{}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 2, -\frac{4}{y^2}\right) &= 1 + \frac{(1/2)^2}{2} \cdot \left(-\frac{4}{y^2}\right) + \frac{(1/2)(3/2)(1/2)(3/2)}{2 \cdot 3 \cdot 2} \cdot \left(-\frac{4}{y^2}\right)^2 \\
&+ \dots = 1 - \frac{1}{2y^2} + \frac{3}{4y^4} + O(y^{-6}).
\end{aligned}$$

for $y > 2$.

4. Results and Discussion

4.1. COMSOL simulations

In order to verify our analytical expression of force accuracy, there is a need to do simulations to compare results. In our case magnets are identical, so the parameters for simulations are $R = 1$ cm,

$$t_1 = t_2 = 2R\tau = 2 * 1 * 0.5 = 1 \text{ cm}$$

and magnetic flux density $B = 1.35\text{T}$ (N45 (Sintered NdFeB)). Therefore, $\tau = \frac{t}{2R} = 0.5$. By COMSOL software following M magnetization is obtained:

$$M = \frac{B}{\mu_0} = \frac{1.35}{4\pi \cdot 10^{-7}} \quad (26)$$

Hence, if we plug in all the above parameters into our simplified force equation the following formula 23 is derived:

$$\begin{aligned}\mathcal{F}(\zeta; \tau; K) = & \frac{\pi \cdot 1.35^2 \cdot 0.01^2}{\mu_0} \left(\frac{1}{2} - \frac{1}{2} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 2; -\frac{4}{(\zeta - 2\tau)^2} \right) \right) \\ & - \frac{2\pi \cdot 1.35^2 \cdot 0.01^2}{\mu_0} \left(\frac{1}{2} - \frac{1}{2} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 2; -\frac{4}{\zeta^2} \right) \right) \\ & + \frac{\pi \cdot 1.35^2 \cdot 0.01^2}{\mu_0} \left(\frac{1}{2} - \frac{1}{2} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 2; -\frac{4}{(\zeta + 2\tau)^2} \right) \right)\end{aligned}$$

Now the only unknown parameter is ζ . As example we can set the distance (d) between two facing disks of the magnets to 0.002 – 0.01 cm. Consequently, ζ will be equal to from 1.002 to 1.01. Because

$$\zeta = \frac{Z}{R} = \frac{0.5t_1 + 0.5t_2 + d}{R} = \frac{t + d}{R} = 1 + d$$

In COMSOL using the magnets of the above parameters and placing them symmetrically we got the following points. They were placed on graph with analytical results from hypergeometric force function.

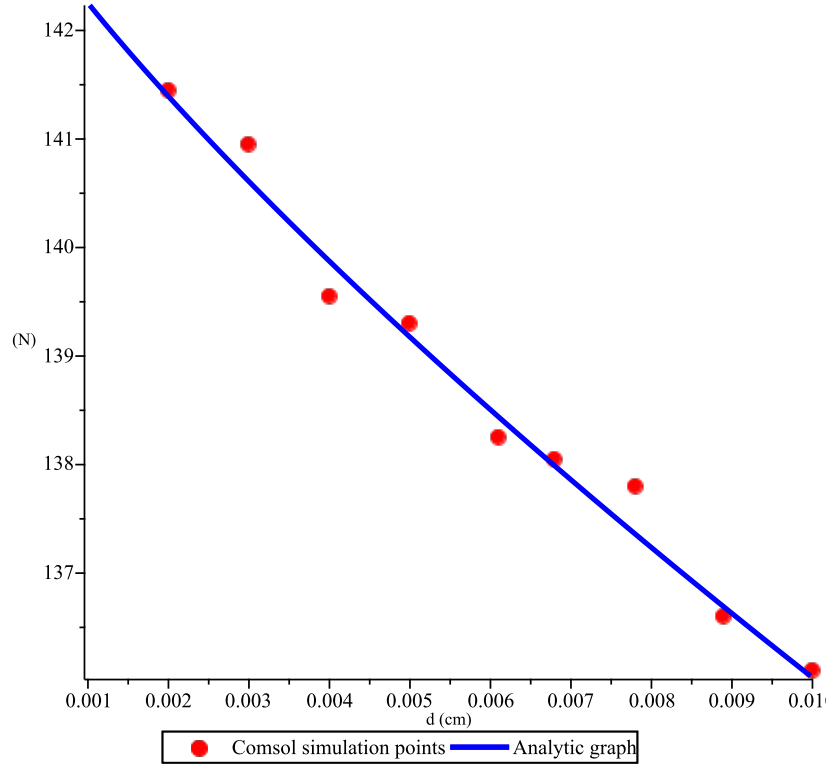


Figure 3: COMSOL vs Hypergeometric function

From Figure 3, we can observe that they follow the same trend. Therefore, our analytical equation can be used to model the magnetic force between two identical cylindrical magnets placed axially symmetrical.

4.2. Experimental results

In order to check correctness of our approximation of hypergeometric functions, we conducted experiments to compare with real-life experimental results. We used different magnets with different heights and radii. In order to measure force between them we used two types of scales. The first one is a scale that was bought to get exact measure up to milligrams which is provided on Figure 4. The second scale was built with the help of Arduino on Figure 5. We used measurements from the second scale to ensure correctness of the first scales. Because Arduino scale does not have components that might be affected by magnets. Therefore, its measurements are accurate.

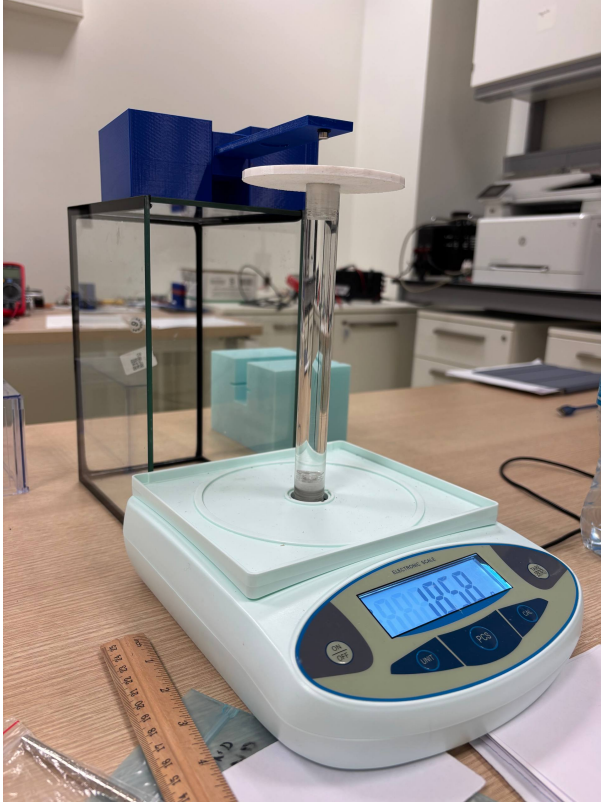


Figure 4: Experimental setup with digital scale.

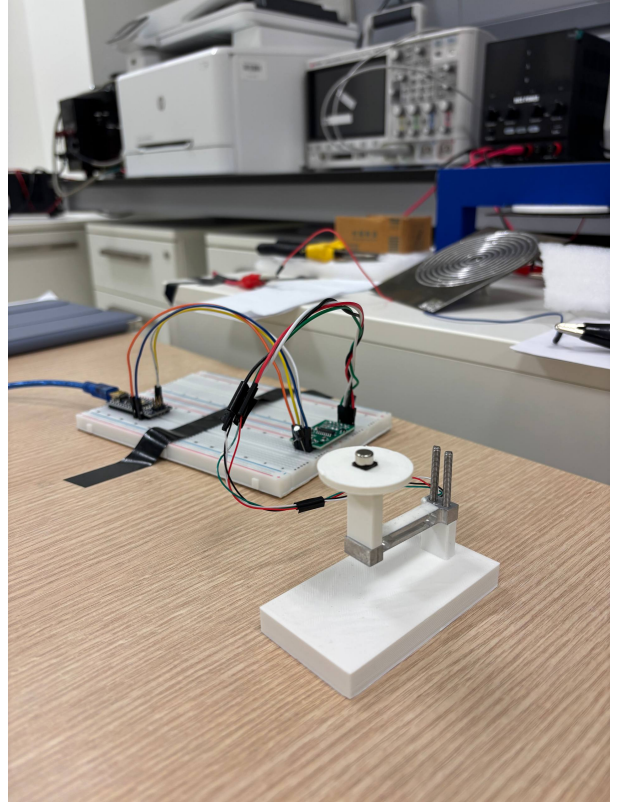


Figure 5: Arduino-based measurement setup.

The force is calculated by formula

$$Force = m * g$$

where m is obtained from scales, $g = 9.81m/s^2$ gravitational acceleration. The first magnets with radius $R = 4.65mm$ and thickness $t = 1.7mm$ have following results.

Table 1: Comparison of experimental, analytical, and Magpy magnetic force values for magnets with R=4.65mm and t=1.7mm

Distance, mm	Experimental force, N (Arduino)	Experimental force, N (Ordinary)	Exact hypergeometric force, N	Magpy force, N
5	0.20177208	0.20152683	0.202040873	0.201486
10	0.04801995	0.04762755	0.046079913	0.046008
15	0.01453842	0.01446975	0.014602674	0.014585
20	0.00576828	0.00585657	0.005802239	0.005797
25	0.00272718	0.00264870	0.002703907	0.002702
30	0.00154017	0.00146169	0.001414469	0.001414
35	0.00088290	0.00083385	0.000806894	0.000807
40	0.000521892	0.00051993	0.000492148	0.000492
45	0.000317844	0.00031392	0.000316525	0.000316
50	0.00009810	0.00006867	0.000212514	0.000212

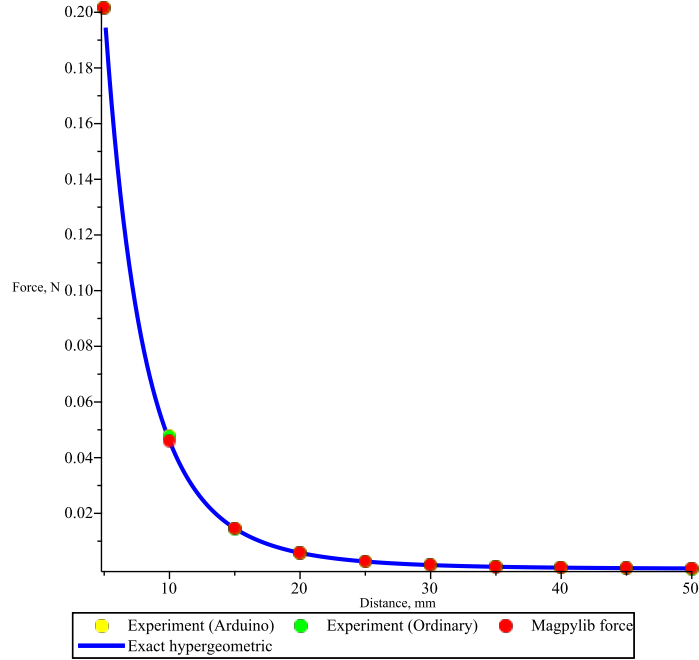


Figure 6: Comparison of analytical results with experimental points and Magpy simulations for magnets 1

From Figure 6 it can be seen that our force derivation based on approximation of hypergeometric function has similar values as experimental results for both scales as well as Magpylib simulations.

Similarly experiment was repeated for magnets with radius $R = 2.45mm$ and thickness $t = 3.22mm$.

Table 2: Comparison of experimental, analytical, and Magpylib magnetic force values for magnets with $R=2.45mm$ and $t=3.22mm$

Distance, mm	Experimental force, N (Arduino)	Experimental force, N (Ordinary)	Exact hypergeometric force, N	Magpy force, N
5	0.29368197	0.3019518	0.300520714	0.306581
10	0.06937632	0.07064181	0.056094838	0.05661
15	0.01797192	0.01822698	0.01667764	0.016762
20	0.00749484	0.00750465	0.006516708	0.006537
25	0.00334521	0.00368856	0.003034374	0.003041
30	0.00175599	0.00180504	0.001594708	0.001597
35	0.00095157	0.00102024	0.000915466	0.000917
40	0.00066708	0.00069651	0.000562039	0.000563
45	0.0003924	0.00041202	0.000363744	0.000364
50	0.00031392	0.00030411	0.000245631	0.000246

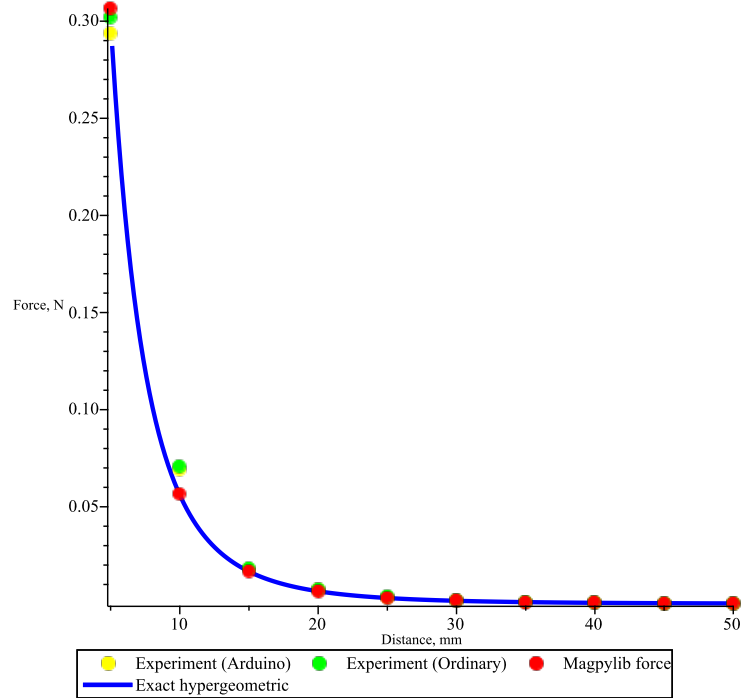


Figure 7: Comparison of analytical results with experimental points and Magpylib simulations for magnets 2

Figure 7 represent similar results. Therefore, approximation of hypergeometric function in force follows the trend as experimental results. Similarly, the Magpylib simulations provide same results. The small difference in small distance might result from error in measuring distance by hand in experimental part.

5. Conclusion

This work investigated the magnetic force between two coaxial cylindrical permanent magnets. The analytical formula of the force based on the magnetostatic energy was obtained using special functions, including hypergeometric functions. This enables us to describe the magnetic force behaviour at small distances. The results show that the magnetic force increases nonlinearly when we decrease the distance between the magnets (limitation of dipolar approximation). Experimental illustrations and validations, both COMSOL, Magpylib simulations and real life experiments, are also presented. They follow the same trend of the dependence of the force on the distance. Overall, the obtained results confirm that the suggested analytical solution is suitable for modelling magnetic interaction and can be used in further research on engineering problems.

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Appendix A.

Bessel and Hankel Integrals

Definition 1. The *Bessel's differential equation* is given by

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2)w = 0. \quad (\text{A.1})$$

This differential equation has a regular singularity at $z = 0$ with indices $\pm\nu$, and an irregular singularity at $z = \infty$ of rank 1.

Definition 2. The *Bessel function of the first kind*, denoted by $J_\nu(z)$, is defined by the power series

$$J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu + k + 1)} \left(\frac{1}{4}z^2\right)^k. \quad (\text{A.2})$$

This solution is an analytic function of $z \in \mathbb{C}$, except for a branch point at $z = 0$ when ν is not an integer. The principal branch of $J_\nu(z)$ corresponds to the principal value of $(\frac{1}{2}z)^\nu$ and is analytic in the z -plane cut along the interval $(-\infty, 0]$.

When $\nu = n \in \mathbb{Z}$, $J_\nu(z)$ is entire in z . For fixed $z \neq 0$, each branch of $J_\nu(z)$ is entire in ν .

In this work, J_0 and J_1 denote the Bessel functions of the first kind of integer orders 0 and 1, respectively.

Gamma and psi Functions

Definition 3. The *Gamma function*, denoted by $\Gamma(z)$, is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \Re z > 0.$$

When $\Re z \leq 0$, $\Gamma(z)$ is defined by analytic continuation. It is a meromorphic function with no zeros, and with simple poles of residue $(-1)^n/n!$ at $z = -n$. The function $1/\Gamma(z)$ is entire, with simple zeros at $z = -n$.

Definition 4. The *psi function* (or digamma function), denoted by $\psi(z)$, is defined as the logarithmic derivative of the Gamma function:

$$\psi(z) = \Gamma'(z)/\Gamma(z), \quad z \neq 0, -1, -2, \dots$$

The function $\psi(z)$ is meromorphic with simple poles of residue -1 at $z = -n$.

Maple and Mathematica results

```
# Define the integral
L_y := Int( (1/q) * exp(-y*q) * BesselJ(1,q)^2, q = 0 .. infinity) :

# Evaluate to get the hypergeometric form
result := value(L_y) assuming y > 0;

result := 
$$\frac{\text{hypergeom}\left(\left[1, \frac{3}{2}, \frac{3}{2}\right], [2, 3], -\frac{4}{y^2}\right)}{4y^2}$$
 (1)
```

Figure A.8: Maple result: Expression of the integral as hypergeometric function





$$\frac{1}{4y^2} \text{HypergeometricPFQ}\left[\left\{1, \frac{3}{2}, \frac{3}{2}\right\}, \left\{\dots\right\}, \dots\right]$$




Assuming "y" is a variable ...

Input

$$\frac{1}{4y^2} {}_3F_2\left(1, \frac{3}{2}, \frac{3}{2}; 2, 3; -\frac{4}{y^2}\right)$$



Result

$$\frac{2y^2 - 2y^2 {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; -\frac{4}{y^2}\right)}{4y^2}$$

Series representation

$$\frac{{}_3F_2\left(1, \frac{3}{2}, \frac{3}{2}; 2, 3; -\frac{4}{y^2}\right)}{4y^2} = \frac{\sum_{k=0}^{\infty} \frac{{}_4F_3\left(1, 1, \frac{3}{2}, \frac{3}{2}; 1-k, 2, 3; z_0\right) \left(-\frac{4}{y^2} - z_0\right)^k z_0^{-k}}{k!}}{2y^2}$$

for (not ($z_0 \in \mathbb{R}$ and $1 \leq z_0 < \infty$))

$$\frac{{}_3F_2\left(1, \frac{3}{2}, \frac{3}{2}; 2, 3; -\frac{4}{y^2}\right)}{4y^2} = \frac{2 \sum_{k=0}^{\infty} \frac{\left(-x - \frac{4}{y^2}\right)^k \Gamma(1+k) \Gamma\left(\frac{3}{2}+k\right)^2 {}_3F_2\left(1+k, \frac{3}{2}+k, \frac{3}{2}+k; 2+k, 3+k; x\right)}{k!}}{\pi v^2}$$

Figure A.9: Mathematica result: Simplified Hypergeometric function

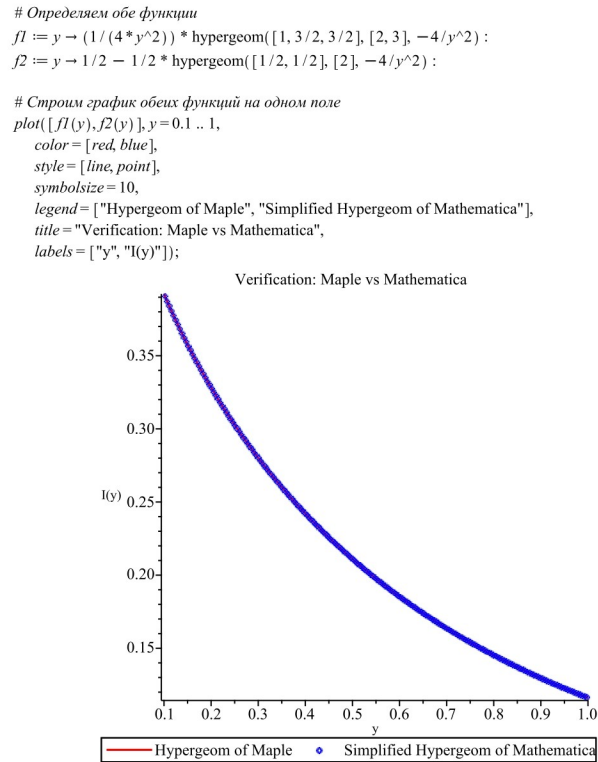


Figure A.10: Hypergeometric function taken from Maple vs Simplified Hypergeometric function of Mathematica graph construction

Appendix B.

Magpy simulation codes

The code for Magpy simulation for first magnet. For the second magnet we need to change only values of three parameters:

```

import numpy as np
import pandas as pd
import magpylib as magpy

```

```

diameter_mm = 9.3

```

```

height_mm = 1.7
Br = 0.5579327296589215

diameter = diameter_mm * 1e-3
height = height_mm * 1e-3
distances_mm = np.array([5, 10, 15, 20, 25, 30, 35, 40, 45, 50], dtype=float)

magnet1 = magpy.magnet.Cylinder(
    dimension=(diameter, height),
    polarization=(0, 0, Br),
)

results = []
for gap_mm in distances_mm:
    gap = gap_mm * 1e-3
    magnet2 = magpy.magnet.Cylinder(
        dimension=(diameter, height),
        polarization=(0, 0, Br),
        meshing=40,
    )

    magnet1.position = (0, 0, 0)
    magnet2.position = (0, 0, height + gap)
    F, T = magpy.getFT(magnet1, magnet2)

    results.append({
        "Distance, mm": gap_mm,
        "Fx, N": F[0],
        "Fy, N": F[1],
        "Fz, N": F[2],
        "Force magnitude, N": abs(F[2]),
    })

```

```
})
```

```
df = pd.DataFrame(results)
```

```
print(df)
```