Trinomial Tree Method for Option Pricing with Transaction Cost

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MATH 499: Capstone Project

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Abstract

This paper describes the progression of research initiated in MATH 424: Mathematical Finance course, with an emphasis on the creation and development of the Trinomial Tree Method for option pricing, particularly in the presence of transaction costs. We begin by reviewing the Cox-Ross-Rubenstein binomial scheme and then go on to trinomial approaches in financial literature, demonstrating their enhanced effectiveness over the binomial method. Our study includes complex models such as the Boyle and Vorst model, widening the scope beyond the standard Black-Scholes model covered in the course. However, the Boyle and Vorst's method only covers transaction cost for binomial models. Our objective is to review the literature on the Trinomial Tree Method considering transaction cost which requires solving the absolute value matrix equation $Ax - |X| = b$ along the tree. We have extended the results of the Boyle and Vorst from a binomial to trinomial method for a European call option. We implemented numerically our method and these results were in consistent with the results of the Boyle and Vorst's method comparable to the binomial results.

The research extends the practical application of option pricing models by providing a complete framework for solving absolute value equations in the setting of trinomial trees, yielding useful insights for the Trinomial Tree Method with transaction cost.

Keywords: Trinomial Tree Method, option pricing, transaction cost, financial mathematics

Contents

1 Introduction

1.1 Literature Review

The field of mathematical finance has undergone significant progress in modeling and pricing financial derivatives over time in complete markets. The pursuit of more accurate and sophisticated methods for incomplete markets is motivated by the dynamic nature of financial markets and the necessity to consider various factors influencing option prices. This section traces the development of option pricing models, highlighting crucial milestones that have shaped the landscape of financial mathematics.

In the early 1970s, the groundbreaking work of Fischer Black, Myron Scholes and Robert C. Merton laid the groundwork for contemporary option pricing theory. The Black-Scholes model, presented in their paper "The Pricing of Options and Corporate Liabilities" (Black and Scholes, 1973), provided an innovative analytical solution for European call and put options. Despite its elegance, this model assumes constant volatility and complete markets, constraining its practical applicability.

Cox, Ross, and Rubinstein first proposed the binomial option pricing model in 1979 and hereafter referred to as the CRR model (Cox et al., 1979). This discrete-time model allowed for the consideration of varying volatility and discrete dividend payments, enhancing its versatility for option pricing. The binomial approach served as a stepping stone toward more intricate and realistic models. Expanding upon the binomial framework, researchers explored Trinomial Tree Methods to achieve even greater accuracy. It is well-known that the binomial method's solution, as time period approaches to infinity, converges to the exact solution of the Black-Scholes equation.

In general, it is assumed that trading involves no transaction costs, meaning that the buy and sale prices of stocks are identical. Such models without transaction costs are described as friction-free. However, in actual market settings, transaction costs come in various forms. Fixed costs, where a consistent price is charged for each transaction regardless of its amount, and proportional costs, where fees are scaled according to transaction size, are the two most basic forms. For simplicity, in 1992, Boyle and Vorst focused on proportional transaction costs in their analysis and enhanced option pricing by integrating transaction costs into the binomial model (Boyle and Vorst, 1992). Although not originally focused on trinomial trees, their methodology can be adapted to these models, providing a more accurate reflection of market conditions. This work forms a bridge between simpler binomial methods and more complex trinomial models, particularly by adding real-world trading costs. Our goal is to extend these models to incorporate transaction costs, using the foundational work of Boyle and Vorst as a guide.

We will further detail the enhancements and practical implications of integrating transaction costs into Trinomial Tree Methods in the subsequent sections, advancing the field of financial mathematics.

1.2 Introduction to Options

An **option** is a contract that offers the owner a right, rather than an obligation, to trade the underlying asset at a fixed price, denoted by K-called the strike price, and at a specified future time, denoted by T-called the expiration date (Cox et al., 1979). There are two types of common options: a call and a put. The **call option** allows the holder to purchase the asset; if it is exercised at the expiration date, then it is considered as a **European call option** (Black and Scholes, 1973). The further explanations are considered in the following example based on the idea derived from the chapter "A Review of Options" in the book *A Discussion of Financial Economics in Actuarial Models* (Finan, 2016). For simplicity we only consider the European call option in this project. The following is a simple example to explain how profits can be made for a European long-call option.

Example 1:

Suppose the current stock price, denoted by S, on the market is \$100. An investor chooses to purchase a call option with strike price $K = \$105$. The investor, as a buyer of the option, takes a long position and pays a premium for the option at a price of \$5, denoted by P_c. On the other side, the seller of the option takes a short-sale position for the asset and is obligated to sell the stock and deliver to the option holder when it is exercised. At the expiration date, suppose that the stock price increases to $S_T = \$120$. What is the profit of the investor?

Solution:

The payoff and profit of the call option at expiration date can be expressed in the following equations:

$$\text{Payoff to Call Option} = \max(0, S_T - K) \tag{1}$$

$$\text{Call Profit} = \max(0, S_T - K) - P_c \tag{2}$$

The payoff and profit diagrams of the call option are shown in Figure 1 below. The call option can be exercised **if, and only if,** $S_T > K$ at the expiration date (Finan, 2016). Otherwise, if the stock price is less than K, the investor loses the premium.

Therefore, the investor's call profit $= \max(0, S_T - K) - P_c =$ max(0, \$120 - \$105) - \$5 = \$10.

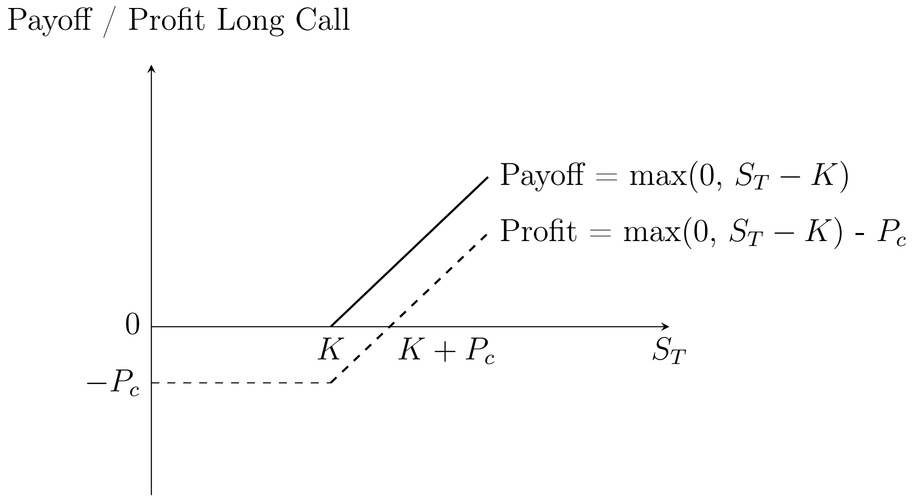

Figure 1: Payoff and Profit of a Long Call Option

The central question: What is the price of a European call option at time 0? In the following we will explain how to determine this price by introducing Binomial models.

2 Binomial Models

We introduce the CRR binomial method to solve for the option price and associate it with risk-neutral probability. In the following, for this we introduce two main approaches: Replicating portfolio in Section 2.1 and a Risk-neutral approach in Section 2.2. Moreover, we introduce the concept of transaction costs, which brings a layer of real-world complexity to the binomial model in Section 2.3. This section will discuss how incorporating transaction costs affects the pricing and replication strategies, necessitating adjustments to the standard binomial approach to accurately reflect the impact of these costs on trading strategies and option valuations. Through these discussions, our aim is to provide a nuanced understanding of both the theoretical underpinnings and practical applications of the binomial method in financial mathematics.

2.1 Replicating Portfolio Approach of the Binomial Model

The binomial model for option pricing represents one of the foundational methodologies in financial mathematics, offering a discrete-time framework for valuing options. This model, introduced by Cox et al. (1979), conceptualizes the evolution of stock prices as a binomial tree, where each node represents a possible price at a given time in the future.

At its core, **the binomial model** assumes that the price of the underlying asset, S, can move to one of two possible prices in the next time step: an up state (u) with a probability p or a down state (d) with a probability $1-p$. Let $C_u = \max(0, uS-K)$ and $C_d = \max(0, dS-K)$ be two possible values of the payoff values of the option at time period n = 1 (Cox et al., 1979). At two-time period, we assume $udS = duS$ and $C_{ud} = C_{du}$ as illustrated in the following Figure 2:

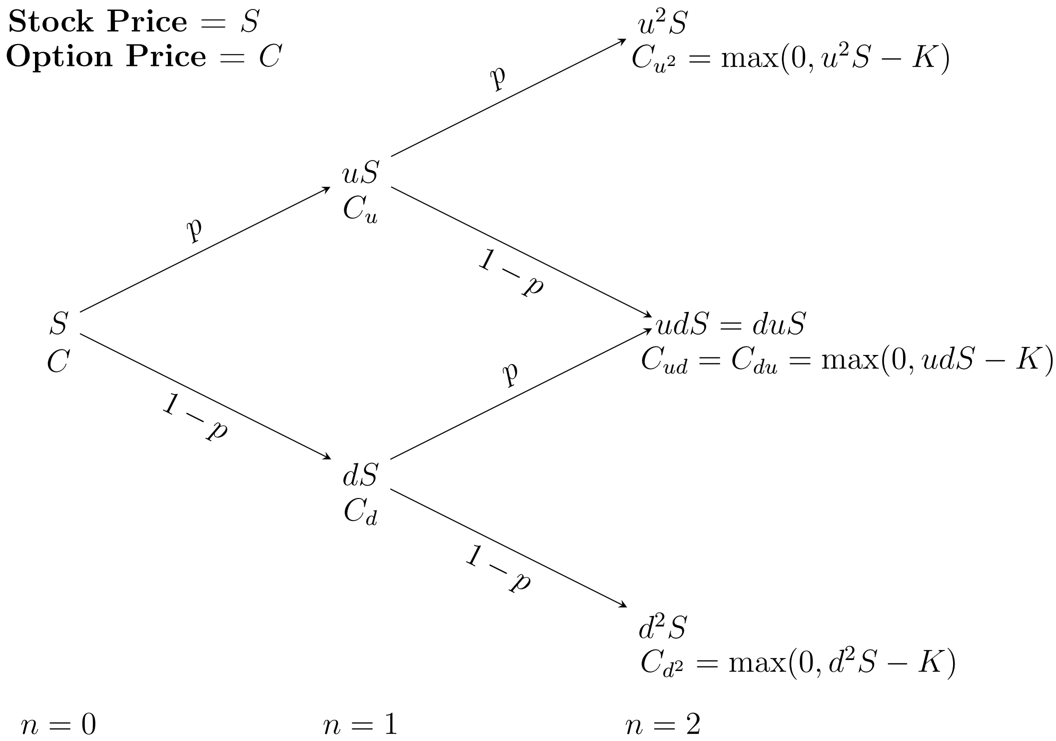

Figure 2: Two-step Binomial tree method

We assume there is no transaction cost and no arbitrage opportunity. The binomial model's flexible feature allows us to incorporate with delta hedging type framework. The seller of the option needs to form a portfolio to match the value of the option so that at the expiration date he can fulfill the obligation. The portfolio is replicated with the number of shares of the stock denoted by Δ and the amount of invested money in risk-free bonds identified as B (Cox et al., 1979). The cost of the portfolio, when the stock price goes up, is $C_u = uS\Delta + B$, and when the stock price goes down, is $C_d = dS\Delta + B$ (Hull, 2022).

Regardless of the outcome, based on the assumption that definition of complete market is complete, the value of the portfolio is the same (Cutland and Roux, 2013), so we ensure that it is risk-free and it follows:

$$\Delta = \frac{C_u - C_d}{uS - dS} \tag{3}$$

$$B = e^{-rT}\frac{uC_d - dC_u}{u - d} \tag{4}$$

$$C = \Delta S + B \tag{5}$$

The Equation (3) shows that Δ is a hedge ratio of the price that measures the sensitivity of the option price in general trading to the change of the option price. The risk-free portfolio with no-arbitrage opportunity must earn a risk-free interest rate, which is discussed in the following section.

2.2 Risk-Neutral Approach of the Binomial Method

Let r be the risk-free interest rate and the condition $d < r < u$ is required to mitigate arbitrage opportunities (Cox et al., 1979). The cost of setting up the portfolio at time T can be obtained by substituting Δ and B from Equations (3) and (4):

$$C = e^{-rT}(p_u C_u + (1 - p_u)C_d) \tag{6}$$

$$p_u = \frac{e^{rT} - d}{u - d} \tag{7}$$

The binary outcome is derived from the probability of up and down movements. However, instead of a real-world probability, we use a **risk-neutral probability**, denoted as p_u, where $p_d = 1 - p_u$ (Bjorefeldt et al., 2016). It avoids complex risk-evaluation by ensuring that the expected payoff of the derivative is equal to the current price with the risk-free rate. In case of two-step, illustrated in Figure 3, we separately consider the value option at each node, so the length of the time step is Δt years, instead of T (Hull, 2022). Equations (6) and (7) become:

$$p_u = \frac{e^{r\Delta t} - d}{u - d} \tag{8}$$

$$C = e^{-r\Delta t}(p_u C_u + (1 - p_u)C_d) \tag{9}$$

We can also calculate and simplify the call option pricing for a two-period:

$$C = e^{-2r\Delta t}({p_u}^2 C_{u^2} + 2p_u p_d C_{ud} + {p_d}^2 C_{d^2}) \tag{10}$$

$$\begin{tikzcd}[column sep=small]
 & & C_{u^2} = max(0, u^2S - K) \\
 & C_u = e^{-r\Delta t}(p_u C_{u^2} + p_d C_{ud}) \arrow[ur] \arrow[dr] & \\
C = e^{-r\Delta t}(p_u C_u + p_d C_d) \arrow[ur] \arrow[dr] & & C_{ud} = max(0, udS - K) \\
 & C_d = e^{-r\Delta t}(p_u C_{ud} + p_d C_{d^2}) \arrow[ur] \arrow[dr] & \\
 & & C_{d^2} = max(0, d^2S - K)
\end{tikzcd}$$

Figure 3: Call Option Pricing in a two-step Binomial method

The parameters u and d are determined by the implied volatility, denoted as σ (Black and Scholes, 1973). Hence, the standard deviation of the return is $\sigma\sqrt{\Delta t}$, and Cox et al. (1979) suggested the following equations:

$$u = e^{\sigma\sqrt{\Delta t}} \tag{11}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{12}$$

The DerivaGem software was used in the book *Options, Futures, and other derivatives* by Hull, 2022 and it can calculate the option price for trees up to 500 steps. This software is used for valuing the following example for a European call option.

Example 2:

A stock is currently \$100. The rate of risk-free interest is 10% per annum, and the rate of olatility is 20% per annum. Consider a 1-year European call option, when the strike price is \$120 and find the option price at time 0.

Solution:

In order to construct a binomial tree, we divide the life of the option into six periods and the length of one time step is approximately 61 days or $\Delta t = 0.1667$. Using Equations from (8) to (12):

$$u = e^{\sigma\sqrt{\Delta t}} = 1.0851,\ d = e^{-\sigma\sqrt{\Delta t}} = 0.9216,\ p_u = 0.5824$$

By working backwards through the tree, as shown in Figure 4, the value of the option at time 0 is \$4.355255. The stock prices and values of the options at each node of the binomial tree were calculated using DerivaGem software.

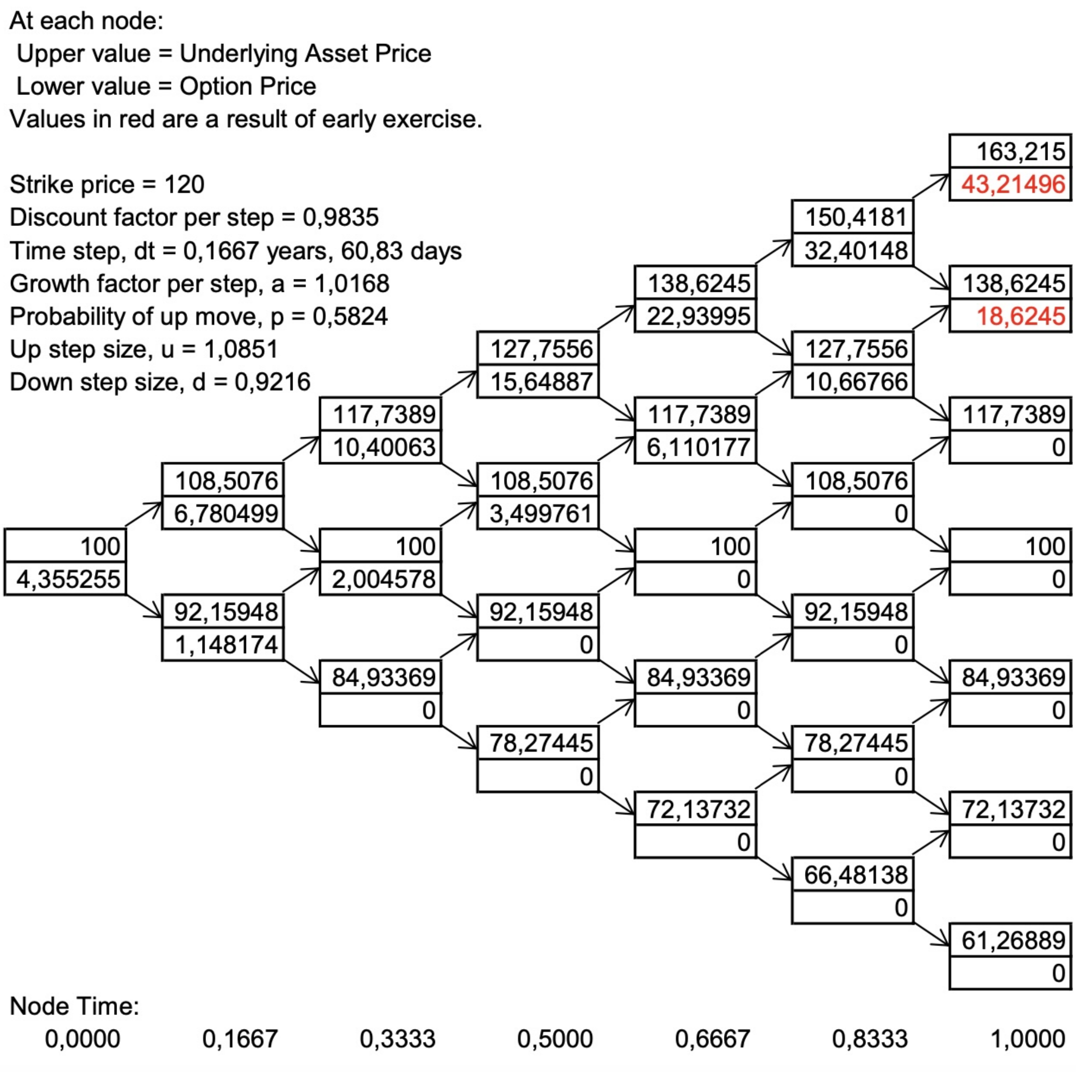

Figure 4: Binomial tree for European Call option using DerivaGem

In the real world, to correctly estimate the current option value, a smaller Δt value is used as the error approaches to zero. Therefore, after implementing 500 tree steps using the DerivaGem software, the realistic option price in Example 2 is \$4.7059.

To demonstrate this graphically, Figure 5 shows the convergence of the option price from the binomial tree method to the exact solution of the Black-Scholes partial differential equation (PDE) model for a large number of time steps.

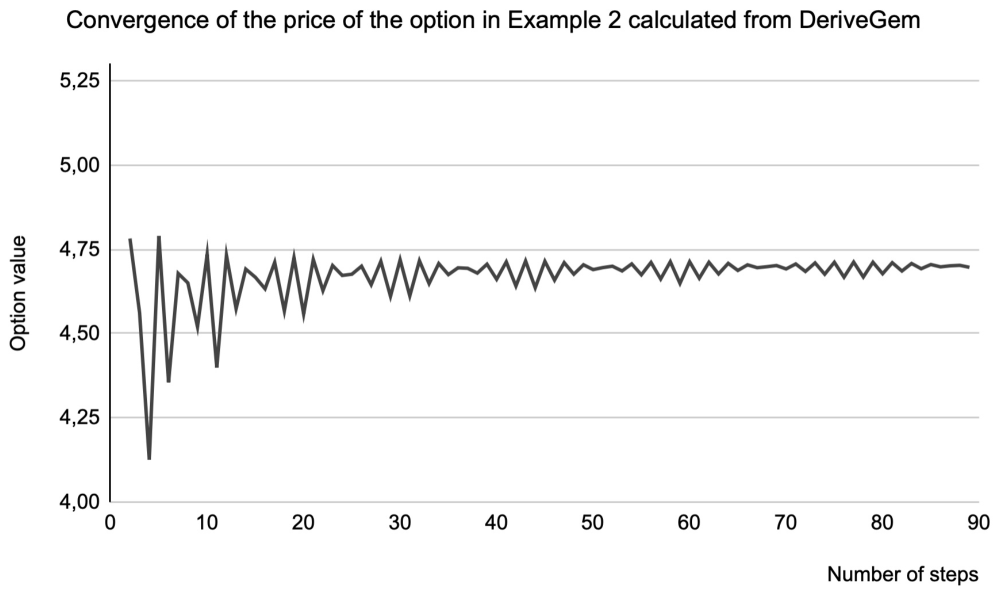

Figure 5: Convergence of the Price of the Option using DerivaGem

This binary outcome reflects the fundamental uncertainty in price movements, capturing the essence of market dynamics in a simplified manner. Model provides the ability to evaluate call and put options by monitoring stock volatility, price movements, strike price, interest rate and time value of money, respectively. The strength of the binomial model lies in its iterative nature, allowing for the backward induction process to determine the option's fair value at the initial time. By considering the payoff of an option at different nodes and discounting it back to the present using a risk-neutral probability, the model can account for different option types and payoff structures.

2.3 Completeness in Multi-Stock Scenarios

The primary objective of this section is to elucidate the construction of a matrix system essential for pricing options within a binomial model framework, and to reveal the possibility of encountering multiple solutions. Our focus is to dissect the model's completeness when dealing with portfolios comprising various stocks. Through the application of linear algebraic principles, we aim to assess whether every derivative within the model can be replicated—a marker of the model's completeness. Conversely, the emergence of multiple risk-neutral probabilities would signal a lack of completeness. The strategic arrangement into a matrix system is pivotal for this analysis, as it allows for the inspection of the equation system, which consists of stock and bond price data over different market scenarios. It depends on discerning whether the equations yield a single solution, thereby affirming the model's ability to replicate derivatives uniquely, or if they admit multiple solutions, reflecting the model's incompleteness.

We delve into the notion of completeness within the framework of a multi-stock scenario in a single-period model. Cutland and Roux (2013) characterized such models by the presence of risk-neutral probabilities, which may not necessarily be unique. A model is considered complete when every derivative within it can be perfectly replicated. This completeness is assessed in terms of scenarios, denoted by $\Omega = \{\omega_1, ..., \omega_n\}$, through a system of equations $xB_1 + yS_1(\omega_i) = D(\omega_i)$ for each derivative D (Cutland and Roux, 2013). From a linear algebra standpoint, completeness is contingent upon the rank of the matrix A, constructed with rows representing bond and stock prices across scenarios.

$$A = \begin{bmatrix} B_1 & S_1^1(\omega_1) & \cdots & S_1^m(\omega_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & S_1^1(\omega_n) & \cdots & S_1^m(\omega_n) \end{bmatrix} \tag{13}$$

If A has a rank equal to the number of scenarios n, indicating n linearly independent columns or rows, the model is deemed complete. Thus, a single-period model with n scenarios achieves completeness when it comprises at least n independent assets, encompassing both risky and risk-free types.

Here, B_1 is the price of a bond and $S^j(u_j)$ is the price of the j-th stock in state u_j. Then, we need to introduce the vector $q = (q_1, ..., q_n)$ with $\sum_{i=1}^{n} q_i = 1$ and $q_i \geq 0$ for all i, representing a portfolio of securities (Cutland and Roux, 2013). Laying the groundwork for assessing the model's capacity to ensure the derivative's replication, the expanded matrix form that mirrors the valuation equations for two distinct scenarios within the market can be written as follows:

$$A = \begin{bmatrix} B_1 S(q_1) \\ B_1 S(q_2) \end{bmatrix} \tag{14}$$

The interaction between the transpose of matrix A and the portfolio vector q, equating it to the anticipated returns from the bond and the stock, adjusted for the risk-free interest rate r is illustrated in the following equation:

$$A^T q = \begin{bmatrix} B_1 B_1 \\ S(q_1) S(q_1) \end{bmatrix} \times \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = (1 + r) \begin{bmatrix} B_0 \\ S_0 \end{bmatrix} \tag{15}$$

Provided this, we observe a simplification where the investment returns from the bond in both scenarios are collectively equated to the present value of the bond, compounded at the risk-free rate:

$$B_1 q_1 + B_1 q_2 = (1+r)B_0 \tag{16}$$

$$q_2 = 1 - q_1 \tag{17}$$

$$B_1 = (1+r)B_0 \tag{18}$$

$$q_1 S(q_1) + q_2 S(q_2) = (1+r)S_0 \tag{19}$$

$$pSu + (1-p)Sd = (1+r)S_0 \tag{20}$$

Further expanding the model, we include the derivative's value D, incorporating it into the linear system that equates the combined returns of the bond, the stock, and the derivative to their present values factoring in the risk-free rate:

$$\begin{bmatrix} A^T \\ D \end{bmatrix} q = (1+r) \begin{bmatrix} B_0 \\ S_0 \\ D_0 \end{bmatrix} \tag{21}$$

$$D(q_1)q_1 + D(q_2)q_2 = (1+r)D_0 \tag{22}$$

Using the obtained formulas, we confront a parameter λ, which introduces a family of risk-neutral probabilities, denoted as Q_λ. This parameter's existence exemplifies the notion that, unlike binomial models, trinomial models can offer a spectrum of risk-neutral valuations, reflective of the multifaceted states that a stock price can assume in reality.

$$Q_\lambda = (\lambda, \frac{5}{3}, -5\lambda, -\frac{4}{5} + 6\lambda) = (0, \frac{5}{3}, -\frac{4}{5}) + \lambda(1, -5, 6) \tag{23}$$

As we can see in Equation (23), there is one parameter λ of risk-neutral probabilities Q_λ which lead to a range of risk-neutral option prices. This is the major difference between binomial and trinomial methods. In real life, stock price can have multiple states.

2.4 Transaction Cost

In the previous section the binomial method based on risk-neutral assumption the methods works only on complete market. It only covers complete markets with risk-neutral probability and no-arbitrage method. Here we consider an incomplete market by taking into account transaction cost in replicating the portfolio. We use the equation derived from Boyle and Vorst, 1992, where k is the rate of transaction cost:

$$C_u = \Delta_u(uS) + B_u + k|\Delta - \Delta_u|(uS) \tag{24}$$

$$C_d = \Delta_d(dS) + B_d + k|\Delta - \Delta_d|(dS) \tag{25}$$

We derived formulas for option values extended to two-time period for up and down states:

$$C_{u^2} = \Delta_{u^2}(u^2S) + B_{u^2} + k|\Delta_u - \Delta_{u^2}|(u^2S) \tag{26}$$

$$C_{ud} = \Delta_{ud}(udS) + B_{ud} + k|\Delta_d - \Delta_{ud}|(udS) \tag{27}$$

$$C_{d^2} = \Delta_{d^2}(d^2S) + B_{d^2} + k|\Delta_d - \Delta_{d^2}|(d^2S) \tag{28}$$

Boyle and Vorst used the CRR model to replicate the payoff of a European call option. However, they suggested that the number of shares and bonds for each period is unique, such as the portfolio (Δ_u, B_u) and (Δ_d, B_d) can be maintained when the stock goes up and down. Therefore, the inequality assumes $\Delta_d \leq \Delta \leq \Delta_u$ (Boyle and Vorst, 1992) and Equations (24) and (25) become linear:

$$C_u = \Delta\bar{u}S + Be^{r\Delta t} = \Delta_1\bar{u}S + B_u \tag{29}$$

$$C_d = \Delta\bar{d}S + Be^{r\Delta t} = \Delta_2\bar{d}S + B_d \tag{30}$$

where:

$$\bar{u} = u(1+k) \text{ and } \bar{d} = d(1-k) \tag{31}$$

The option value formula with transaction cost at time 0 for a two-period can be derived from Equations (29) to (31):

$$C = \Delta S + B = e^{-r\Delta t}(\bar{p_u}[(1+k)\Delta_u S_u + B_u] + \bar{p_d}[(1-k)\Delta_d S_d + B_d]) \tag{32}$$

As they claimed that B and Δ are different at each node, it brings a challenge due to many unknown variables. Further research was conducted using the works of Tichy (2005), Melnikov (2005), and Cutland and Roux (2013).

Transaction costs encompass the expenses incurred during trading, including commissions and the gap between the actual trading price and the midpoint of the bid-offer spread. As a result, the price for buying an asset, denoted as S^{ask}, is invariably higher than the price for selling the same asset, S^{bid}, at the same moment (Melnikov, 2005; Tichy, 2005). The following formulas suggest how asset prices and transaction costs are mathematically linked in trading models:

$$S_T(1-\kappa) = S_T^{bid} \leq S_T \leq S_T^{ask} = S_T(1+\kappa) \tag{33}$$

$$p_u = \frac{1+r-d(1-\kappa)}{(1+\kappa)u - d(1-\kappa)} \tag{34}$$

$$C = \frac{1}{1+r}\left[C_u p_u + \kappa u S_0 q + C_d p_d\right] \tag{35}$$

Let's define x as bonds and y as shares, as suggested in the book *Derivative Pricing in Discrete Time* (Cutland and Roux, 2013). For the sake of simplicity, they consider only proportional transaction cost k which belongs to (0,1) interval. The cost of creating the portfolio at time T follows:

$$C = xB_T + y^+ S_T^{ask} - y^- S_T^{bid} = xB_t + yS_T + k|y|S_T \tag{36}$$

We use these standard notations:

$$y^+ = \begin{cases} y, & \text{if } y > 0 \\ 0, & \text{if } y < 0 \end{cases} \qquad y^- = \begin{cases} 0, & \text{if } y > 0 \\ -y, & \text{if } y < 0 \end{cases} \qquad y^+ - y^- = y$$

$$[x]^+ = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases} \qquad [x]^- = \begin{cases} 0, & \text{if } x > 0 \\ -x, & \text{if } x < 0 \end{cases} \qquad [x]^+ - [x]^- = [x]$$

Example 3:

The conditions are the same as in Example 2, but we need to consider the transaction cost rates, k, which are 0.5% and 2%.

Solution:

Using the Equations (29), (30), (31), and (32), we obtained the following results as illustrated in Table 1.

Strike Price	Number of Time Periods (n)			
	6	13	52	250
	$k = 0.5\%$			
120	4.663	5.084	5.820	7.161
	$k = 2\%$			
120	5.926	6.859	8.950	12.750

Table 1: Results for Example 3

3 Trinomial models

3.1 Trinomial Method

The trinomial method extends the concepts of the binomial models but reflects three potential stock price movements for each node, enhancing the model's complexity and ability to simulate more realistic market movements as shown in Figure 6. According to Josheski and Apostolov (2020) and Bjorefeldt et al. (2016), the pricing of European options through trinomial models reaches the valuation provided by the European Black-Scholes method faster than binomial models do. Moreover, unlike the deterministic framework of binomial models, the trinomial method introduces nondeterminism with two equations and three unknowns, incorporating a parameter that establishes a family of risk-neutral probabilities and thereby a range of fair prices for derivatives.

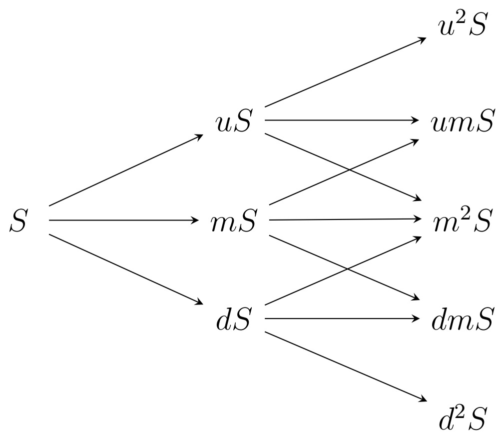

Figure 6: Stock Price for a two-step Trinomial Tree Method

Hull (2022) provided the following formulas for the value of a call option in the trinomial model:

$$C = e^{-r\Delta t}(p_u C_u + p_m C_m + p_d C_d) \tag{37}$$

where u and d represent the up and down factors for the stock price, respectively, and are given by:

$$u = e^{\sigma\sqrt{3\Delta t}}, \quad d = 1/u$$

The probabilities associated with each move—up, middle, and down—are calculated using:

$$p_u = \sqrt{\frac{\Delta t}{12\sigma^2}}(r - q - \frac{\sigma^2}{2}) + \frac{1}{6} \tag{38}$$

$$p_m = \frac{2}{3} \tag{39}$$

$$p_d = -\sqrt{\frac{\Delta t}{12\sigma^2}}(r - q - \frac{\sigma^2}{2}) + \frac{1}{6} \tag{40}$$

For the two period trinomial tree, the option values at the next time step are calculated by aggregating the discounted values of potential future states:

$$C_u = e^{-r\Delta t}(p_u C_{u^2} + p_m C_{um} + p_d C_{ud}) \tag{41}$$

$$C_m = e^{-r\Delta t}(p_u C_{mu} + p_m C_{m^2} + p_d C_{md}) \tag{42}$$

$$C_d = e^{-r\Delta t}(p_u C_{du} + p_m C_{dm} + p_d C_{d^2}) \tag{43}$$

These computations are visually supported by the trinomial tree diagrams for option prices presented below in Figure 7:

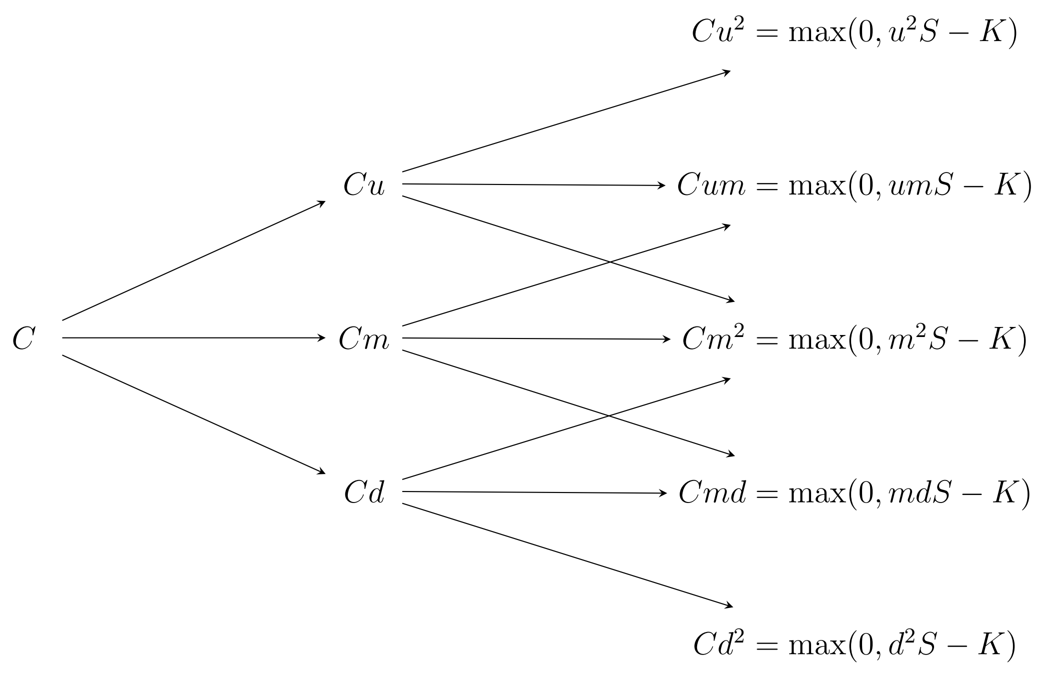

Figure 7: Call Option for a two-step Trinomial Tree Method

This model's mathematical formulation is encapsulated in the following system of equations, which provides a method for computing the fair value of derivatives based on risk-neutral valuation:

$$\begin{cases} q_1 + q_2 + q_3 = 1 \\ C_0 = (1+r)^{-1}\sum_{i=1}^{3} q_2 * C(q_i) \end{cases} \tag{44}$$

The system, a 2x3 matrix of two equations with three unknowns, demonstrates the potential for multiple risk-neutral measures, emphasizing the model's capacity to accommodate a spectrum of market conditions and theoretical prices.

$$\begin{bmatrix} 1 & 1 & 1 \\ C(q_1) & C(q_2) & C(q_3) \end{bmatrix} \times \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ C_0(1+r) \end{bmatrix} \tag{45}$$

Here, we get $q_1 = \lambda$ and $\frac{2}{9} < \lambda < \frac{1}{3}$

$$C(u)p + C(d)(1-p) = (1+r)C_0 \tag{46}$$

$$C_0 = \frac{1}{1+r}[pC_u + qC_d] = (1+r)^{-1}(pC_u + qC_d) \tag{47}$$

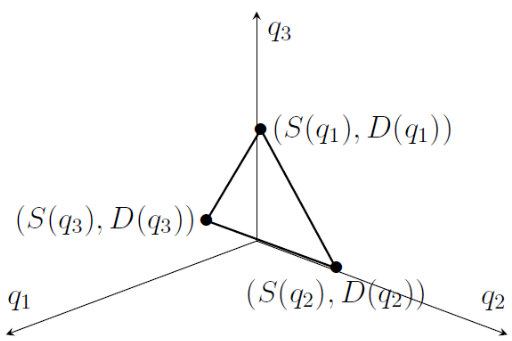

Figure 8: 3D plot with triangle vertices at given coordinates.

The trinomial model not only facilitates a deeper understanding of option pricing dynamics but also allows the exploration of scenarios where traditional models may not provide complete solutions. This complexity and flexibility make it particularly useful in financial markets where the assumptions of simpler models are too restrictive.

3.2 Trinomial Method with Transaction Cost: the Boyle and Vorst model

In this section, we explore the Trinomial Tree Method considering the transaction cost which is informed by the seminal work of Boyle and Vorst (1992). We have found several ways to balance the replicating portfolio and corresponding formulas for B. First, Melnikov (2005) proposed the formula for the bond price $B_T = (1+r)^T$. Second, Cutland and Roux (2013) indicated as $B_T = B_0(1+r)^T$. On the other hand, as shown in Figure 9, Boyle and Vorst (1992) suggested that the number of bonds for up, middle, down states and for each period is different:

$$
\begin{tikzcd}
 & & \Delta_4, B_4 \\
 & \Delta_1, B_1 \arrow[ur] \arrow[r] \arrow[dr] & \Delta_5, B_5 \\
\Delta, B \arrow[ur] \arrow[r] \arrow[dr] & \Delta_2, B_2 \arrow[ur] \arrow[r] \arrow[dr] & \Delta_6, B_6 \\
 & \Delta_3, B_3 \arrow[ur] \arrow[r] \arrow[dr] & \Delta_7, B_7 \\
 & & \Delta_8, B_8
\end{tikzcd}
$$

Figure 9: The number of bonds and shares for the Trinomial Tree Method

As interpretations are different in each paper, our goal is to specify the formula for each period. The model suggested by Boyle and Vorst (1992) is challenging, as there are too many unknown variables. To minimize the number of unknown variables, we assume and suggest that the number of bonds is the same for potential nodes in the same time period, such as $B_1 = B_2 = B_3$ and $B_4 = B_5 = B_6 = B_7 = B_8$. Given the multiplicity of formulas across the literature, our objective is to consolidate the notations and distill a definitive formula. We propose a unified approach to ensure that bond prices, regardless of the chosen path, converge to a single value at time T:

$$B_T = B_0 e^{rT} \tag{48}$$

Accompanied by Equations (24) - (28), we demonstrate the valuation of a call option within a trinomial framework, including transaction costs:

$$C_u = \Delta uS + Be^{rT} = B_1 + \Delta_1 uS + k|\Delta - \Delta_1|uS \tag{49}$$

$$C_m = \Delta mS + Be^{rT} = B_2 + \Delta_2 mS + k|\Delta - \Delta_2|mS \tag{50}$$

$$C_d = \Delta dS + Be^{rT} = B_3 + \Delta_3 dS + k|\Delta - \Delta_3|dS \tag{51}$$

The Trinomial Tree Method, a versatile tool for pricing derivatives, must be adapted to account for the impact of transaction costs on trading strategies and derivative valuation. However, the inequality $\Delta_d \leq \Delta \leq \Delta_u$ was proven only for the binomial model in Boyle and Vorst (1992) and not for the 3x3 system of absolute value equations. This leads to further research to make the above equations linear.

3.3 Further Research

As we continue our research on option pricing models, our future goal is to capture this complexity in a set of absolute value equations of the type $Ax - |X| = b$. Unraveling the answers to this system will be the foundation of our future research, potentially leading to more precise and realistic option valuation approaches that reflect the multidimensional structure of financial markets. We also plan to improve the Trinomial Tree Method by including transaction costs into the Heston model's robust system. Based on Heston model, Rouah and Vainberg (2007) formulated an expression for the valuation, at time t, of a European call option expiring at $t + T$, featuring a strike price K:

$$Call = e^{rT} E_t^*[(S_{t+T} - k, 0)] = S_t P_1 - k e^{-rT} P_2 \tag{52}$$

$$P_1 = \frac{1}{2} + \frac{e^{-rT}}{\pi S_t} \left(\int_0^\infty \mathrm{Re} \left[\frac{k^{-i\varphi} f^*(i\varphi + 1)}{i\varphi} \right] d\varphi \right) \tag{53}$$

$$P_2 = \frac{1}{2} + \frac{1}{\pi} \left(\int_0^\infty \mathrm{Re} \left[\frac{k^{-i\varphi} f^*(i\varphi)}{i\varphi} \right] d\varphi \right) \tag{54}$$

T represents the duration until maturity, $E_t^*[(S_{t+T} - k, 0)]$ denotes the expectation at time t following the risk-neutral distribution, S_t stands for the asset's price at time t, and P_1 and P_2 represent probabilities adjusted for risk-neutrality.

This model enhances our pricing methodology by introducing stochastic volatility into our trinomial framework, while also factoring in the intricacies of transaction costs. Unlike the Black-Scholes model and its trinomial tree modifications, which assume constant volatility, the Heston model demonstrates that volatility is a dynamic entity capable of mean reversion and volatility clustering, both of which are observed in real financial markets. The paper by Yan (2021) should be reviewed for future studies as it considers Heston model for European option pricing. This investigation will result in the creation of a system of three equations, each associated with separate Δ values representing the hedging methods used in the trinomial configuration.

4 Conclusion

In conclusion, this paper extended the fundamental concepts of the CRR binomial model by introducing the Trinomial Tree Method to account for transaction costs, which are a key element in real-world trading. Our research found that including transaction costs into the trinomial tree system improves the model's realism while also aligning it more closely with observed market behaviors and price anomalies. This method's numerical implementation has shown comparable accuracy to existing models, while also giving deeper insights into the effects of transaction costs on option prices. Future research will focus on improving these models, including incorporating more complicated market situations, and investigating the consequences for financial strategy and risk management.

This adaptation involves adjusting the tree's parameters to incorporate the costs associated with buying and selling the underlying asset, thereby altering the probabilities and expected returns used in the model. The key challenge lies in accurately reflecting the bid-ask spread induced by transaction costs, which affects the liquidity and price movement assumptions inherent in the model. By incorporating transaction costs, the Trinomial Tree Method becomes a more realistic tool for financial analysis, offering insights into the cost implications of trading strategies and the true value of derivatives in a less-than-ideal market environment.

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