INVERSE TIME-DEPENDENT SOURCE PROBLEMS IN EVOLUTION EQUATIONS

DILYARA KUANGALIYEVA AND DURVUDKHAN SURAGAN

Abstract. We solve inverse problems of determining continuous time-dependent source terms in evolution equations. As a particular case, one inverse initial-boundary value problem with observation data at a spatial point is sufficient to recover the coefficient explicitly. The concept is illustrated with analytical and numerical examples.

Contents

1. Introduction

Inverse problems are considered one of the youngest as well as complicated parts of Applied Mathematics [7]. Understanding the essence of such problems allows us to see the wide applicability of the topic in almost all areas of science. In a simple manner, the aim of inverse problems is to study the system knowing its current state and change it to the desired one in the future [5]. Consequently, the topic has been widely used in physics, geophysics, astronomy, medicine, and other natural sciences [7].

Obviously, inverse problems are classified into subtypes and inverse source problem is one of its subtypes. According to the Kabanikhin book [7], there are unknown

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functions included in the direct problem, in addition to unknown $u(t,x)$. Based on this unknown function in direct problems, inverse problems are classified. Thus, inverse source problems have such classification since it is required to determine the *source* or function $F(t,x)$ in the mathematical model further (2.1).

Inverse problems of time-dependent source terms identification have been an area of interest for many studies in different fields [1], [11]. As an example, inverse problems in parabolic equations with nonlocal boundary conditions and an integral observation [6] have been used for the population model investigation. Hazanee et al. [6] observed a one-dimensional heat equation with arbitrary fixed time with which they aimed to find the source term. Furthermore, there is a reference [4] that the method of determining continuous time-dependent source coefficients in inverse problems was used in the biology field as well, namely helping to construct a mathematical model for electroencephalography. To reach this, authors [4] used a common elliptic equation as an inverse problem to localize the source in space and time. Localization of the source implies an inverse problem depending on the chosen direct model of the electromagnetic field [2].

In the previous paper [8], multidimensional inverse problems were studied while *the main feature of the method of recovering source terms is to solve inverse problems by considering one initial-boundary value problem with observation data at a spatial point*. In the general case, we consider two initial-boundary value problems which contain unknown coefficients and each problem requires observation data at a fixed spatial point. This paper is the continuation of the previous results arising from [8] and extending to fractional differential equations [3], [10] and abstract Cauchy problems [9]. However, it studies inverse source problems and in contrast to the aforementioned papers, initial-boundary value problems are considered.

Let Ω be an open bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with piecewise smooth boundary $\partial\Omega$ and let T be a positive real number. For convenience, throughout the paper we use the following notations: $\Omega_T := (0,T) \times \Omega$ and $S_T := (0,T) \times \partial\Omega$.

Let $p \in C^1(\bar{\Omega})$ be such that $p > 0$ in Ω. We define an elliptic operator $L : C^2(\Omega) \to C(\Omega)$ given by

$$Lu(x) := \nabla_x \cdot (p(x)\nabla_x u(x)) \quad , \quad x \in \Omega.$$

Then if $u,\, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$, Green's second identity holds (see, [12])

$$\begin{aligned} &\int_\Omega \Big(v(\xi)Lu(\xi) - u(\xi)Lv(\xi) \Big) d\xi \\ &= \int_{\partial\Omega} p(\xi) \left(u(\xi)\frac{\partial v}{\partial n} - v(\xi)\frac{\partial u}{\partial n} \right) dS. \end{aligned} \tag{1.1}$$

This work has a five-part structure. In Section 2, we study divergence form parabolic equations considering particular and general cases. The particular case observes the initial-boundary value problem and, by solving an inverse problem of finding a unique pair *(u, r)*, states a theorem being able to express an unknown term *r(t)*. In contrast, the general case already contains the unknown term *r(t)* in the auxilary problem. Solving this inverse problem, we also provide another theorem. In Section 3, divergence form hyperbolic equations are calculated considering two cases as well.

The same as in the previous section approach is applied to study general and particular cases. As an examples, in sections 4 and 5, we provide numerical experiments for diffusion and wave equations.

2. Divergence form parabolic equations

2.1. Particular case. We consider the following initial-boundary value problem

$$\begin{aligned} \rho(x)\partial_t u(t,x) - \nabla_x \cdot \big(p(x)\nabla_x u(t,x)\big) &= F(t,x), \qquad (t,x) \in \Omega_T, \\ u(0,x) &= u_0(x), \qquad x \in \Omega, \\ u(t,x) &= 0, \qquad (t,x) \in S_T, \end{aligned} \tag{2.1}$$

where $\rho \in C(\overline{\Omega})$, $p \in C^1(\overline{\Omega})$ be such that $\rho > 0$, $p > 0$ in Ω, $F \in C(\Omega_T)$ and $u_0 \in C(\overline{\Omega})$. Then there exists a unique solution $u \in C^{1,2}(\Omega_T) \cap C(\overline{\Omega_T})$ of (2.1), see [12]. The solution is given by

$$\begin{aligned} u(t,x) = &\int_0^t \int_\Omega G(t-\tau,x,\xi)F(\tau,\xi)d\xi d\tau \\ &+ \int_\Omega G(t,x,\xi)u_0(\xi)\rho(\xi)d\xi, \quad (t,x) \in \Omega_T, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} G(t,x,\xi) &= \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\|\varphi_n\|^2} e^{-\lambda_n t}, \quad x,\xi \in \Omega, t > 0, \\ \|\varphi_n\|^2 &= \int_\Omega \rho(\xi)\varphi_n^2(\xi)d\xi, \end{aligned} \tag{2.3}$$

λ_n and φ_n are the nth eigenvalue and corresponding eigenfunction of the problem

$$\begin{aligned} \nabla \cdot \big(p\nabla\varphi\big) + \lambda s\varphi &= 0, \quad \text{in} \quad \Omega \\ \varphi &= 0, \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Following the paper [8] where our method was introduced, we solve the following inverse problem of finding a unique pair (u, r) explicitly

$$\begin{aligned} \rho(x)\partial_t u(t,x) - \nabla_x \cdot \big(p(x)\nabla_x u(t,x)\big) &= r(t), \qquad (t,x) \in \Omega_T, \\ u(0,x) &= u_0(x), \qquad x \in \Omega, \\ u(t,x) &= 0, \qquad (t,x) \in S_T, \end{aligned} \tag{2.4}$$

from the given observation data at a point $x_0 \in \Omega$

$$h_1(t) := u(t,x_0), \tag{2.5}$$

which is continuously differentiable in $[0, T]$.

To find (u, r) we solve an auxiliary (direct) problem

$$\begin{aligned} \partial_t v(t,x) - \nabla_x \cdot \big(p(x)\nabla_x v(t,x)\big) &= 0, \qquad (t,x) \in \Omega_T, \\ v(0,x) &= \nabla_x \cdot \Big(p(x)\nabla_x\big(\rho(x)u_0(x)\big)\Big), \qquad x \in \Omega, \\ v(t,x) &= 0, \qquad (t,x) \in S_T, \end{aligned} \tag{2.6}$$

and we denote the solution of (2.6) $x = x_0 \in \Omega$ by h_2, that is,

$$h_2(t) := v(t, x_0), \tag{2.7}$$

for all $t \in [0, T]$. Moreover, the solution of (2.6) is given by

$$v(t,x) = \int_\Omega G(t,x,\xi)\nabla_\xi \cdot \Big(p(\xi)\nabla_\xi\big(\rho(\xi)u_0(\xi)\big)\Big)\rho(\xi)d\xi, \quad (t,x) \in \Omega_T. \tag{2.8}$$

Now we state a theorem, which is the main feature of our method.

Theorem 2.1. *Assume that*

(i) *$\rho \in C^2(\bar{\Omega})$ be such that $\rho > 0$ in Ω;*
(ii) *$p \in C^1(\bar{\Omega})$ be such that $p > 0$ in Ω and $p = 0$ on $\partial\Omega$;*
(iii) *$u_0 \in C^2(\bar{\Omega})$;*
(iv) *the unknown coefficient $r \in C[0, T]$;*
(v) *h_1 defined by (2.5) belongs to $C^1[0, T]$.*

Then there exists a unique pair (u, r) of the problem (2.4)-(2.5) where r is given by

$$r(t) = h_1'(t) - h_2(t) \quad \text{for all } t \in [0, T].$$

Note that h_2 is derived from the direct problem (2.6), that is, from the solution of (2.6) at $x = x_0 \in \Omega$ and h_2 is a continuous function in $[0, T]$.

Proof. Assumptions (i-iii) allow to find representation of a solution of (2.4) and (2.6), which are

$$\begin{aligned} u(t,x) = &\int_0^t \int_\Omega G(t-\tau,x,\xi)r(\tau)d\xi d\tau \\ &+ \int_\Omega G(t,x,\xi)\rho(\xi)u_0(\xi)d\xi, \quad (t,x) \in \Omega_T, \end{aligned} \tag{2.9}$$

and

$$v(t,x) = \int_\Omega G(t,x,\xi)\nabla_\xi \cdot \Big(p(\xi)\nabla_\xi\big(\rho(\xi)u_0(\xi)\big)\Big)\rho(\xi)d\xi, \quad (t,x) \in \Omega_T. \tag{2.10}$$

correspondingly. By differentiating (2.9) in time, then using

$$\partial_t G(t-\tau,x,\xi) = \nabla_\xi \cdot \big(p(\xi)\nabla_\xi G(t-\tau,x,\xi)\big), \quad t > \tau,$$

$$\partial_t G(t,x,\xi) = \nabla_\xi \cdot \big(p(\xi)\nabla_\xi G(t,x,\xi)\big), \quad t > 0,$$

and from (2.3), x and ξ are interchangeable that is why

$$\nabla_y(p(\xi)\nabla_\xi G(t-\tau,x.\xi)) = \nabla_x(p(x)\nabla_x G(t-\tau,x.\xi))$$

we get

$$\begin{aligned} \partial_t u(t,x) = &\int_0^t \int_\Omega \partial_t G(t-\tau,x,\xi)r(\tau)d\xi d\tau + r(t) \\ &+ \int_\Omega \partial_t G(t,x,\xi)\rho(\xi)u_0(\xi)d\xi \\ = &\int_0^t \int_\Omega \nabla_\xi \cdot \big(p(\xi)\nabla_\xi G(t-\tau,x,\xi)\big)r(\tau)d\xi d\tau + r(t) \\ &+ \int_\Omega \nabla_\xi \cdot \big(p(\xi)\nabla_\xi G(t,x,\xi)\big)\rho(\xi)u_0(\xi)d\xi \end{aligned} \tag{2.11}$$

Now we can use Green's second identity (1.1) (continuation of (2.11))

$$
\begin{aligned}
\partial_t u(t,x) &= \int_0^t \int_\Omega G(t-\tau,x,\xi) r(\tau) \nabla_\xi \cdot \big(p(\xi)\nabla_\xi f(\tau,\xi)\big) d\xi d\tau + r(t) \\
&\quad + \int_\Omega G(t,x,\xi)\nabla_\xi \cdot \Big(p(\xi)\nabla_\xi\big(\rho(\xi)u_0(\xi)\big)\Big) d\xi \\
&= v(t,x) + r(t), \quad (t,x) \in \Omega_T.
\end{aligned} \tag{2.12}
$$

□

2.2. **General case.** When $F(t,x) = r(t)f(t,x)$ for all $(t,x) \in \Omega_T$, where f is given, the auxiliary problem (2.6) contains the unknown term r and it requires observation data at a given point. Thus, we consider the following inverse problem of finding the unique pair (u,r) explicitly

$$
\begin{aligned}
\rho(x)\partial_t u(t,x) - \nabla_x \cdot \big(p(x)\nabla_x u(t,x)\big) &= r(t)f(t,x), && (t,x) \in \Omega_T, \\
u(0,x) &= u_0(x), && x \in \Omega, \\
u(t,x) &= 0, && (t,x) \in S_T,
\end{aligned} \tag{2.13}
$$

and

$$
\begin{aligned}
\partial_t v(t,x) - \nabla_x \cdot \big(p(x)\nabla_x v(t,x)\big) &= r(t)\nabla_x \cdot \big(p(x)\nabla_x f(t,x)\big), && (t,x) \in \Omega_T, \\
v(0,x) &= \nabla_x \cdot \Big(p(x)\nabla_x\big(\rho(x)u_0(x)\big)\Big), && x \in \Omega, \\
v(t,x) &= 0, && (t,x) \in S_T,
\end{aligned} \tag{2.14}
$$

from the observation data at a point $x_0 \in \Omega$

$$
h_1(t) := u(t,x_0), \tag{2.15}
$$

which is continuously differentiable in $[0,T]$, and

$$
h_2(t) := v(t,x_0), \tag{2.16}
$$

which is continuous in $[0,T]$.

Theorem 2.2. *Assume that*

(i) $\rho \in C^2(\bar\Omega)$ *be such that* $\rho > 0$ *in* Ω*;*
(ii) $p \in C^1(\bar\Omega)$ *be such that* $p > 0$ *in* Ω *and* $p = 0$ *on* $\partial\Omega$*;*
(iii) $u_0 \in C^2(\bar\Omega)$*;*
(iv) $f \in C^{0,2}(\Omega_T)$*;*
(v) *the unknown coefficient* $r \in C[0,T]$*;*
(vi) h_1 *defined by (2.15) belongs to* $C^1[0,T]$*;*
(vii) h_2 *defined by (2.16) belongs to* $C[0,T]$*.*

Then there exists a unique pair (u,r) *of the problem (2.13)-(2.16) where* r *is given by*

$$
h_1'(t) = h_2(t) + r(t)f(t,x_0) \quad \textit{for all } t \in [0,T].
$$

Proof. Assumptions (i-iv) allow us to find representation of a solution of (2.13) and (2.14), which are

$$\begin{aligned} u(t,x) = &\int_0^t \int_\Omega G(t-\tau,x,\xi)r(\tau)f(\tau,\xi)d\xi d\tau \\ &+ \int_\Omega G(t,x,\xi)\rho(\xi)u_0(\xi)d\xi, \quad (t,x)\in\Omega_T, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} v(t,x) = &\int_0^t \int_\Omega G(t-\tau,x,\xi)r(\tau)\nabla_\xi\cdot\big(p(\xi)\nabla_\xi f(\tau,\xi)\big)d\xi d\tau \\ &+ \int_\Omega G(t,x,\xi)\nabla_\xi\cdot\Big(p(\xi)\nabla_\xi\big(\rho(\xi)u_0(\xi)\big)\Big)d\xi, \quad (t,x)\in\Omega_T, \end{aligned} \tag{2.18}$$

correspondingly. By differentiating (2.17) in time, then using

$$\partial_t G(t-\tau,x,\xi) = \nabla_\xi\cdot\big(p(\xi)\nabla_\xi G(t-\tau,x,\xi)\big), \quad t>\tau,$$

$$\partial_t G(t,x,\xi) = \nabla_\xi\cdot\big(p(\xi)\nabla_\xi G(t,x,\xi)\big), \quad t>0,$$

and the Green's second identity (1.1), we get

$$\begin{aligned} \partial_t u(t,x) = &\int_0^t \int_\Omega \partial_t G(t-\tau,x,\xi)r(\tau)f(\tau,\xi)d\xi d\tau + r(t)f(t,x) \\ &+ \int_\Omega \partial_t G(t,x,\xi)\rho(\xi)u_0(\xi)d\xi \\ &= \int_0^t \int_\Omega \nabla_\xi\cdot\big(p(\xi)\nabla_\xi G(t-\tau,x,\xi)\big)r(\tau)f(\tau,\xi)d\xi d\tau + r(t)f(t,x) \\ &+ \int_\Omega \nabla_\xi\cdot\big(p(\xi)\nabla_\xi G(t,x,\xi)\big)\rho(\xi)u_0(\xi)d\xi \\ &= \int_0^t \int_\Omega G(t-\tau,x,\xi)r(\tau)\nabla_\xi\cdot\big(p(\xi)\nabla_\xi f(\tau,\xi)\big)d\xi d\tau + r(t)f(t,x) \\ &+ \int_\Omega G(t,x,\xi)\nabla_\xi\cdot\Big(p(\xi)\nabla_\xi\big(\rho(\xi)u_0(\xi)\big)\Big)d\xi \\ &= v(t,x) + r(t)f(t,x), \quad (t,x)\in\Omega_T. \end{aligned} \tag{2.19}$$

Since we have data (2.15),(2.16) at $x_0\in\Omega$, at $x_0\in\Omega$ the formula (2.19) implies

$$h_1'(t) = h_2(t) + r(t)f(t,x_0),$$

for all $t\in[0,T]$. ☐

The homogeneous Dirichlet boundary condition can be replaced with the homogeneous Neumann boundary condition.

3. Divergence form hyperbolic equations

Moreover, this approach also works for the divergence form hyperbolic equations

$$\begin{aligned} \rho(x)\partial_t^2 u(t,x) - \nabla_x \cdot \big(p(x)\nabla_x u(t,x)\big) &= F(t,x) \quad \text{in} \quad \Omega_T, \\ u(0,x) &= u_0(x) \quad \text{in} \quad \Omega, \\ \partial_t u(0,x) &= u_1(x) \quad \text{in} \quad \Omega, \\ u(t,x) &= 0 \quad \text{on} \quad S_T, \end{aligned} \tag{3.1}$$

where $\rho \in C(\overline{\Omega})$, $p \in C^1(\overline{\Omega})$ be such that $\rho > 0$, $p > 0$ in Ω, $F \in C(\Omega_T)$, $u_0 \in C^1(\overline{\Omega})$ and $u_1 \in C(\overline{\Omega})$. Then there exists a unique solution $u \in C^{2,2}(\Omega_T) \cap C^{1,0}(\overline{\Omega_T})$ of (3.1), see [12]. The solution is given by

$$\begin{aligned} u(t,x) = &\int_0^t \int_\Omega G(t-\tau,x,\xi)F(\tau,\xi)d\xi d\tau \\ &+ \int_\Omega \partial_t G(t,x,\xi)u_0(\xi)\rho(\xi)d\xi + \int_\Omega G(t,x,\xi)u_1(\xi)\rho(\xi)d\xi, \end{aligned}$$

where G is defined by (2.3).

3.1. **Particular case.** We solve the following inverse problem of recovering r

$$\begin{aligned} \rho(x)\partial_t^2 u(t,x) - \nabla_x \cdot \big(p(x)\nabla_x u(t,x)\big) &= r(t) \quad \text{in} \quad \Omega_T, \\ u(0,x) &= u_0(x) \quad \text{in} \quad \Omega, \\ \partial_t u(0,x) &= u_1(x) \quad \text{in} \quad \Omega, \\ u(t,x) &= 0 \quad \text{on} \quad S_T. \end{aligned} \tag{3.2}$$

from the observation data at a point $x_0 \in \Omega$

$$h_1(t) := u(t,x_0), \tag{3.3}$$

which is twice continuously differentiable in $[0,T]$. To find (u,r) we solve an auxiliary (direct) problem

$$\begin{aligned} \partial_t^2 v(t,x) - \nabla_x \cdot \big(p(x)\nabla_x v(t,x)\big) &= 0 \quad \text{in} \quad \Omega_T, \\ v(0,x) &= \nabla_x \cdot \Big(p(x)\nabla_x\big(\rho(x)u_0(x)\big)\Big) \quad \text{in} \quad \Omega, \\ \partial_t v(0,x) &= \nabla_x \cdot \Big(p(x)\nabla_x\big(\rho(x)u_1(x)\big)\Big) \quad \text{in} \quad \Omega, \\ v(t,x) &= 0 \quad \text{on} \quad S_T. \end{aligned} \tag{3.4}$$

and we denote the solution of (3.4) $x = x_0 \in \Omega$ by h_2, that is,

$$h_2(t) := v(t,x_0), \tag{3.5}$$

which is continuous in $[0,T]$. Now we state a theorem, which is the main feature of our method.

Theorem 3.1. *Assume that*

(i) $\rho \in C^2(\bar{\Omega})$ *be such that* $\rho > 0$ *in* Ω*;*
(ii) $p \in C^1(\bar{\Omega})$ *be such that* $p > 0$ *in* Ω *and* $p = 0$ *on* $\partial\Omega$*;*

(iii) $u_0 \in C^2(\bar{\Omega})$;
(iv) $u_1 \in C^2(\bar{\Omega})$;
(v) *the unknown coefficient* $r \in C[0,T]$;
(vi) h_1 *defined by (3.3) belongs to* $C^2[0,T]$;

Then there exists a unique pair (u, r) *of the problem (3.2)-(3.4) where* r *is given by*

$$r(t) = h_1''(t) - h_2(t) \quad \textit{for all } t \in [0,T].$$

Proof. Proof of the theorem is a particular case of the following theorem. □

3.2. **General case.** We solve the following inverse problem of recovering r

$$\begin{aligned} \rho(x)\partial_t^2 u(t,x) - \nabla_x \cdot \big(p(x)\nabla_x u(t,x)\big) &= r(t)f(t,x) \quad \text{in} \quad \Omega_T, \\ u(0,x) &= u_0(x) \quad \text{in} \quad \Omega, \\ \partial_t u(0,x) &= u_1(x) \quad \text{in} \quad \Omega, \\ u(t,x) &= 0 \quad \text{on} \quad S_T. \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \partial_t^2 v(t,x) - \nabla_x \cdot \big(p(x)\nabla_x v(t,x)\big) &= r(t)\nabla_x \cdot \big(p(x)\nabla_x f(t,x)\big) \quad \text{in} \quad \Omega_T, \\ v(0,x) &= \nabla_x \cdot \Big(p(x)\nabla_x\big(\rho(x)u_0(x)\big)\Big) \quad \text{in} \quad \Omega, \\ \partial_t v(0,x) &= \nabla_x \cdot \Big(p(x)\nabla_x\big(\rho(x)u_1(x)\big)\Big) \quad \text{in} \quad \Omega, \\ v(t,x) &= 0 \quad \text{on} \quad S_T. \end{aligned} \tag{3.7}$$

from the observation data at a point $x_0 \in \Omega$

$$h_1(t) := u(t,x_0), \tag{3.8}$$

which is twice continuously differentiable in $[0,T]$, and

$$h_2(t) := v(t,x_0), \tag{3.9}$$

which is continuous in $[0,T]$.

Theorem 3.2. *Assume that*

(i) $\rho \in C^2(\bar{\Omega})$ *be such that* $\rho > 0$ *in* Ω;
(ii) $p \in C^1(\bar{\Omega})$ *be such that* $p > 0$ *in* Ω *and* $p = 0$ *on* $\partial\Omega$;
(iii) $u_0 \in C^2(\bar{\Omega})$;
(iv) $u_1 \in C^2(\bar{\Omega})$;
(v) $f \in C^{0,2}(\Omega_T)$;
(vi) *the unknown coefficient* $r \in C[0,T]$;
(vii) h_1 *defined by (3.8) belongs to* $C^2[0,T]$;
(viii) h_2 *defined by (3.9) belongs to* $C[0,T]$.

Then there exists a unique pair (u, r) *of the problem (3.6)-(3.7) where* r *is given by*

$$h_1''(t) = h_2(t) + r(t)f(t,x_0) \quad \textit{for all } t \in [0,T].$$

The proof is almost the same as in Theorem (2.2). Note that the homogeneous Dirichlet boundary condition can be replaced with the homogeneous Neumann boundary condition.

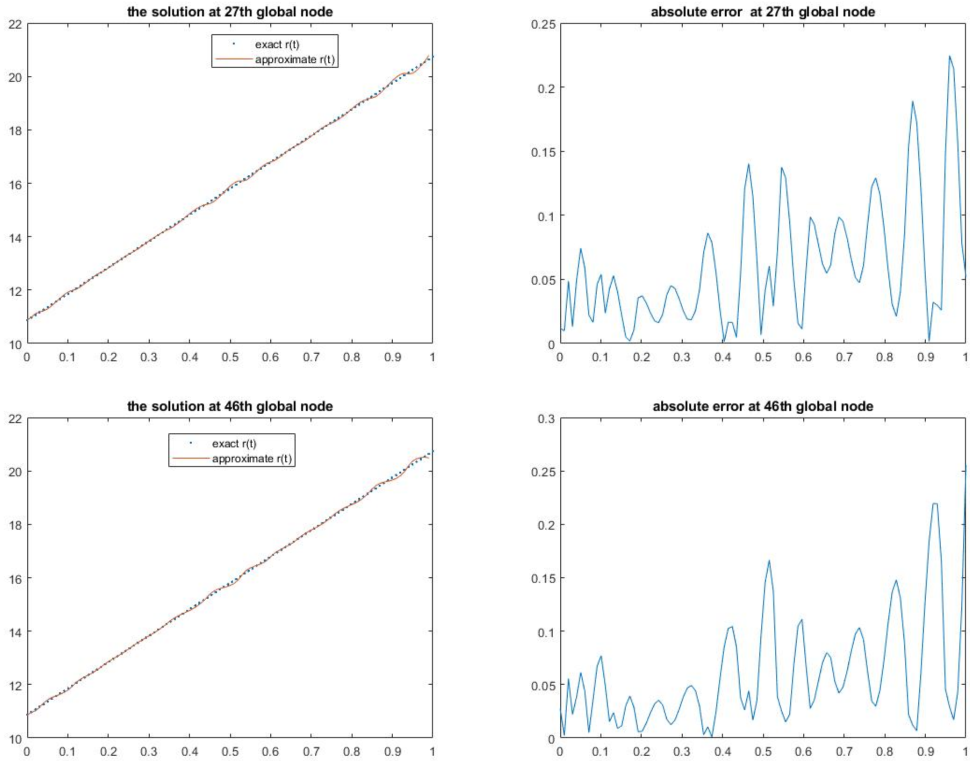

FIGURE 1. comparison and absolute error of $r(t) = \pi^2 t + \pi^2 + 1$ in Example (4.1), FEM, $\Delta t = 0.01$, $\Delta x = 0.02$

4. Numerical experiments

To illustrate the concept, we provide structural analysis using Finite Element Method (FEM) in MATLAB.

4.1. **Diffusion equations.**

Example 4.1.

$$\begin{aligned}
&\partial_t u(t,x) - \partial_x^2 u(t,x) = (\pi^2 t + \pi^2 + 1)\sin(\pi x), \quad 0 < t < 1, \quad 0 < x < 1\\
&u(t,x)|_{t=0} = \sin(\pi x), \quad 0 < x < 1\\
&u(t,0) = 0, \quad 0 < t < 1,\\
&u(t,1) = 0, \quad 0 < t < 1,
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\partial_t v(t,x) - \partial_x^2 v(t,x) = -\pi^2(\pi^2 t + \pi^2 + 1)\sin(\pi x), \quad 0 < t < 1, \quad 0 < x < 1,\\
&v(t,x)|_{t=0} = -\pi^2 \sin(\pi x), \quad 0 < x < 1,\\
&v(t,0) = 0, \quad 0 < t < 1,\\
&v(t,1) = 0, \quad 0 < t < 1,
\end{aligned} \tag{4.2}$$

Their solutions are $u = (t+1)\sin(\pi x)$, $v = -\pi^2(t+1)\sin(\pi x)$.

$$r(t) = \pi^2 t + \pi^2 + 1.$$

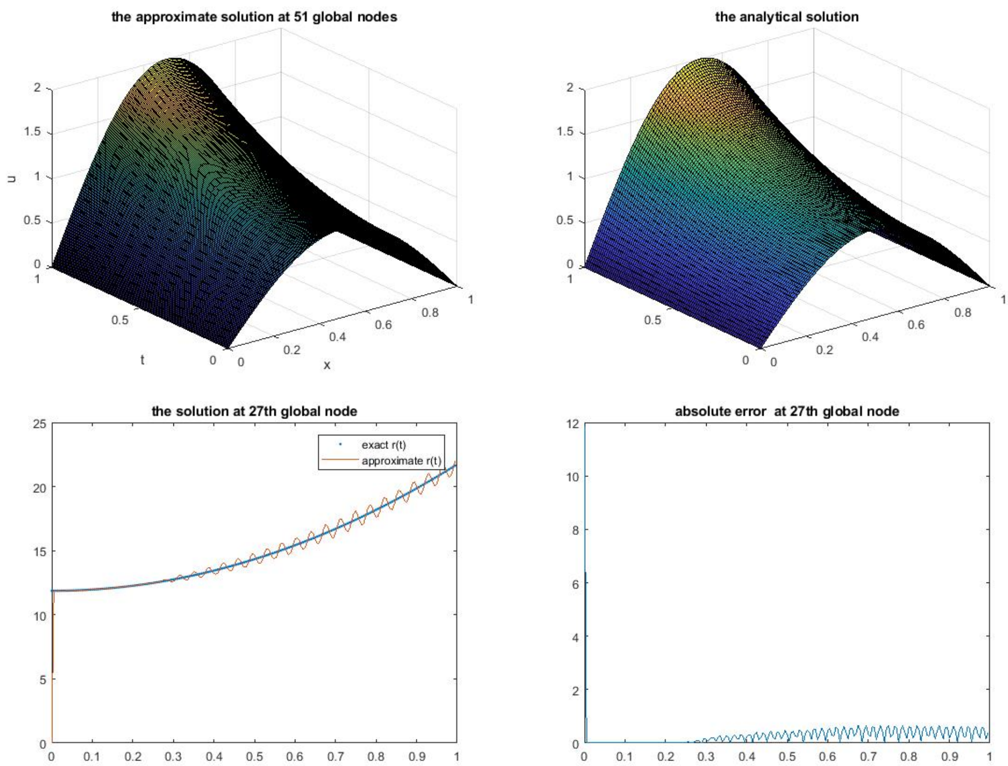

FIGURE 2. $r(t) = \pi^2 t^2 + \pi^2 + 2$ in Example (4.2), FEM, $\Delta t = 0.01$, $\Delta x = 0.02$

4.2. **Wave equations.**

Example 4.2.

$$
\begin{aligned}
&\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = (\pi^2 t^2 + \pi^2 + 2)\sin(\pi x), \quad 0 < t < 1, \quad 0 < x < 1 \\
&u(0,x) = \sin(\pi x), \quad 0 < x < 1 \\
&\partial_t u(0,x) = 0, \quad 0 < x < 1 \\
&u(t,0) = 0, \quad 0 < t < 1, \\
&u(t,1) = 0, \quad 0 < t < 1,
\end{aligned} \tag{4.3}
$$

and

$$
\begin{aligned}
&\partial_t^2 v(t,x) - \partial_x^2 v(t,x) = -\pi^2(\pi^2 t^2 + \pi^2 + 2)\sin(\pi x), \quad 0 < t < 1, \quad 0 < x < 1, \\
&v(0,x) = -\pi^2 \sin(\pi x), \quad 0 < x < 1, \\
&\partial_t v(0,x) = 0, \quad 0 < x < 1 \\
&v(t,0) = 0, \quad 0 < t < 1, \\
&v(t,1) = 0, \quad 0 < t < 1,
\end{aligned} \tag{4.4}
$$

Their solutions are $u = (t^2 + 1)\sin(\pi x)$, $v = -\pi^2(t^2 + 1)\sin(\pi x)$.

$$r(t) = \pi^2 t^2 + \pi^2 + 2.$$

5. Conclusion

In this paper, we continued the investigation of inverse problems by identifying time-dependent source terms. Considering one initial-boundary value problem with observation data at a spatial point and thus solving inverse problems, one can solve inverse problems by recovering source terms. There are two main features of the paper. Firstly, the theorems for each case were derived from the found data in observed inverse problems. Secondly, we could consider inverse problems with observation data at (1,0) which allows observing problems an uncountable number of cases. In contrast, in the previous paper [8] while considering partial differential equations, non-zero coefficients were used.

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Dilyara Kuangaliyeva:
Department of Mathematics
Nazarbayev University, Kazakhstan
E-mail address dilyara.kuangaliyeva@nu.edu.kz

Durvudkhan Suragan:
Department of Mathematics
Nazarbayev University, Kazakhstan
E-mail address durvudkhan.suragan@nu.edu.kz