ON THE CAUCHY PROBLEM FOR A NONLOCAL NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. This paper considers the one-dimensional Schrödinger equation with nonlocal nonlinearity that describes the interactions of nonlinear dispersive waves. We obtain some the local well-posedness and ill-posedness result associated with this equation in the Sobolev spaces. Moreover, we prove the existence of standing waves of this equation. As corollary, we derive the conditions under which the solutions are uniformly bounded in the energy space.

1. **Introduction.** In this paper, we study the following Schrödinger-type equation

$$iu_t - u_{xx} = \varsigma D^\beta(|u|^2 u), \qquad x, t \in \mathbb{R}, \tag{1}$$

introduced in [4, 18] as a model for assessing the validity of weak turbulence theory for random waves, where $\varsigma = \pm 1$, $\beta \in \mathbb{R}$ and D^β with $\beta \in \mathbb{R}$ is the usual Fourier multiplier operator with the symbol $|\xi|^\beta$. The parameter β controls the nonlinearity, in particular small value of β makes the nonlinearity weaker because of a smoothing effect in x. Equation (1) was originally derived as

$$iu_t - u_{xx} = \varsigma(|D^\beta u|^2 D^\beta u),$$

through a heuristic approach, later it was proved that it can be rigorously obtained as an approximation of the fully nonlinear wave system equations [24]. When $\beta = 0$, (1) turns into the classical Schrödinger equation

$$iu_t - u_{xx} = \varsigma|u|^2 u. \tag{2}$$

Grünrock studies the local well-posdesness for (2) in [11] in alternative function spaces below $L^2(\mathbb{R})$, but only in settings where the local Lipschitz dependence on the initial data still holds. In the focusing case, (2) admits soliton and multisoliton solutions. Moreover, it is globally well posed in L^2 thanks to the conservation of the L^2-norm. The scaling-critical Sobolev index associated with (2) is $-1/2$. Christ,

Colliander and Tao [6] showed that the data-to-solution map is unbounded from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$ for $s < -1/2$. Recently, Harrop-Griffiths et al. in [12] proved that (2) is globally well-posed for all initial data in $H^s(\mathbb{R})$ with $s > -1/2$ in the sense that the solution map extends uniquely from the Schwartz space to a jointly continuous map $\mathbb{R} \times H^s(\mathbb{R}) \to H^s(\mathbb{R})$.

Equation (1) mathematically is similar to the derivative nonlinear Schrödinger equation

$$\mathrm{i}u_t - u_{xx} = \varsigma\mathrm{i}(|u|^2 u)_x. \tag{3}$$

It is known from [20] that (3) is localy well-posed in $H^s(\mathbb{R})$ for $s \geq 1/2$. His method of proof combined the gauge transform with Bourgain's Fourier restriction norm method. This result was shown by Biagioni and Linares [2], and Takaoka [21] to be sharp in the sense that the flow map fails to be uniformly C^0 for $s < 1/2$.

Analogous to (2), the $-$ sign in (1) corresponds to the defocusing (repulsive) problem, while the $+$ sign corresponds to the focusing (attractive) problem. Equation (1) similar to (3) and contrary to (2) is not invariant by the pseudoconformal transformation and no explicit blow-up solution is known for both (2) and (1). Another main difference between (1) and (2) is that the former one is not invariant by a Galilean transform. An interesting property of (3) is that all these types of derivative nonlinear Schrödinger equations are related via gauge transformations, any result on one of the forms can a priori be transferred to the other forms, but it seems that (1) has no such a property. Interestingly, given a solution u of (1), then

$$u_\lambda(x,t) = \lambda^{1-\frac{\beta}{2}} u(\lambda x, \lambda^2 t)$$

is also a solution of (1). As a consequence, the scale-invariant Sobolev spaces for (1) are $\dot{H}^{s_c}(\mathbb{R})$ with $s_c = -\frac{1}{2}(1-\beta)$. One of the aims of the current paper is to address the question for local well-posedness of (1). We should also note that the functionals

$$\mathbb{E}(u) = \int_{\mathbb{R}} \left(|D^{-\beta/2}u_x|^2 + \varsigma\frac{1}{2}|u|^4 \right) \, \mathrm{d}x$$

$$\mathbb{F}(u) = \int_{\mathbb{R}} |D^{-\beta/2}u|^2 \, \mathrm{d}x$$

are formally motion invariants of (1).

Theorem 1.1. *Let $\beta \in (0,1)$. Then the Cauchy problem associated with (1) is locally well-posed in $H^s(\mathbb{R})$ if $s > \beta/2$.*

The local well-posedness of (1) can be extended to the case $\beta < 0$. Indeed, when $\beta \in (-1, 0)$, then the nonlinear term $D^\beta(|u|^2 u)$ can be rewritten by the Riesz potential by

$$\mathcal{I}_{-\beta} * |u|^2 u,$$

where $\mathcal{I}_{-\beta} = C_\beta |x|^{-1-\beta}$. By using the Hardy-Littlewood-Sobolev lemma, the fractional chain rule and the Strichartz estimates for the Schödinger equation (see [16]), one can consider a suitable ball in the space

$$L_t^\infty(\mathbb{R}; H^s(\mathbb{R}_x)) \cap L_t^q(\mathbb{R}; H_r^s(\mathbb{R}_x))$$

to show the local-wellposedness by a contraction argument (see [16]), where $H_r^s = (1 - \partial_x^2)^{-s/2}L^r$. So we omit the detilas and focus on the case $\beta \geq 0$. For $b, s, \in \mathbb{R}$, we define the (inhomogeneous) $X^{s,b}$ spaces with respect to the norm

$$X^{s,b} = \{u \in \mathscr{S}', \ \|u\|_{X^{s,b}} := \|\langle\xi\rangle^s\langle\tau - \xi^2\rangle^b\hat{u}(\xi,\tau)\|_{L^2(\mathbb{R}^2)} < +\infty\},$$

where $\hat{u}$ in this definition is the Fourier transform of u with respect to both time and space and $\langle \cdot \rangle = a + |\cdot|$. As (1) with an initial data $u(0) = u_0$ is related to the integral form

$$u(t) = U(t)u_0 - i\varsigma \int_0^t U(t-\tau)D^\beta(|u|^2 u)(\tau)\mathrm{d}\tau,$$

where U is the unitary Schrödinger group of (1), then the main difficulty, by a standard iteration argument, is to show a trilinear estimate related to this integral form in $X^{s,b}$ spaces.

It is worth pointing out there is an important relation between the mixed Lebesgue spaces and the above $X^{s,b}$ spaces, based on the Strichartz estimates for the group U. We have that

$$\|U(t)u\|_{L_t^q(\mathbb{R})L_x^p(\mathbb{R})} \lesssim \|u\|_{L^2(\mathbb{R}^2)}$$

for all pairs (q,p) satisfying $\frac{2}{q} + \frac{1}{p} = \frac{1}{2}$ with $2 \le p, q \le \infty$. This leads us to

$$\|u\|_{L_t^q(\mathbb{R})L_x^p(\mathbb{R})} \lesssim \|u\|_{X^{0,\frac{1}{2}+}}.$$

This estimate can be deduced by adapting suitably the dyadic method introduced in [22, 23] in which multilinear estimates in weighted $L^2(\mathbb{R})$ spaces were generally studied.

In order to prove the trilinear estimates, related to Theorem 1.1 and the integral for in $X^{s,b}$, we prove a bilinear estimate for (2) with the resonance function $h(\xi) = 2\xi_1\xi_2$.

Next aim in this paper is to study the ill-posedness of (1). We use the ideas of [6]. More precisely, the strategy is to approximate the solution of the Cauchy problem associated with (1) by the solutions of (2), and use the ill-posedness result [6] of (2) in $H^s(\mathbb{R})$. The main difficulty here lies in small dispersion analysis due to the non-local nonlinearity.

Theorem 1.2. *Let* $\frac{3\beta-2}{2(3-3\beta)} < s < \frac{\beta}{2}$, $0 < \beta < \frac{2}{3}$. *Then the solution map of the following initial value problem*

$$\begin{cases} iu_t - u_{xx} - D^\beta\left(|u|^2 u\right) = 0, \\ u(0) = \varphi \end{cases}$$

fails to be locally uniformly continuous. More precisely, for $0 < \delta \ll \varepsilon \ll 1$ *and* $T > 0$ *arbitrary, there are two solutions* u_1, u_2 *with initial data* ϕ_1, ϕ_2 *such that*

$$\|\phi_1\|_{H^s}, \|\phi_2\|_{H^s} \lesssim \varepsilon, \tag{4}$$

$$\|\phi_1 - \phi_2\|_{H^s} \lesssim \delta, \tag{5}$$

$$\sup_{0 \le t \le T} \|u_1(t) - u_2(t)\|_{H^s} \gtrsim \varepsilon. \tag{6}$$

Finally, we study the existence of standing waves of (1). By a standing wave we mean a solution of (1) of the form $u(x,t) = \mathrm{e}^{-i\omega t}\varphi(x)$, where $\omega > 0$ is the standing wave frequency. Substituting this form into (1), it transpires that φ must satisfy

$$\omega D^{-\beta}\varphi + D^{2-\beta}\varphi = \pm|\varphi|^2\varphi. \tag{7}$$

Thus one sees that the natural space to study (1) is

$$\mathcal{X} = \dot{H}^{-\frac{\beta}{2}}(\mathbb{R}) \cap \dot{H}^{1-\frac{\beta}{2}}(\mathbb{R}),$$

equipped with the norm

$$\|f\|_{\mathcal{X}} = \|f\|_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})} + \|f\|_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})}.$$

By multiplying (7) by φ and integrating over $\mathbb{R}$, one can easily see that it does not have any nontrivial solution in the defocusing (repulsive) case. So we study the following case

$$\omega D^{-\beta}\varphi + D^{2-\beta}\varphi = |\varphi|^2\varphi. \tag{8}$$

We use the concentration-compactness principle [17] to show the existence of the ground states in $\mathscr{X}$. We recall that the solution φ of (8) is called a ground state, if φ minimizes the action

$$S = \mathbb{E} + \omega\mathbb{F} \tag{9}$$

among all non-trivial solutions of (8). The essential tools in this way is the embedding $\mathscr{X}$ into $L^4(\mathbb{R})$ by showing the following Gagliardo-Nirenberg type inequality:

$$\|g\|_{L^4(\mathbb{R})} \leq C \left\|D^{-\frac{\beta}{2}}g\right\|_{L^2(\mathbb{R})}^{\theta_1} \left\|D^{-\frac{\beta}{2}}g_x\right\|_{L^2(\mathbb{R})}^{\theta_2}$$

for some $\theta_1, \theta_2 \geq 0$ (see (35) below). We also obtain the best constant for this inequality. Indeed, we show a profile decomposition by mimicking the proof lines of [10, 13]. This also enables us to give another approach to study the existence of solutions of (8). As an application of aforementioned Gagliardo-Nirenberg inequality, we show the uniform bound of solutions of (1) in $\mathscr{X}$.

The paper is organized as follows. In section 2, we use some useful known estimates which we need in the subsequent section to show the well-posedness result. In section 3, we prove the ill-posedness result. Finally, the existence of the standing waves and their applications are proved in section 4.

We end this section by presenting the following convention. We use $A \lesssim B$ to denote the statement that $A \leq CB$ for some positive harmless constant C which may vary from line to line and depend on various parameters. We use $A \sim B$ to denote the statement that $A \lesssim B$ and $B \lesssim A$.

2. **Well-posedness.** In this section, we prove our well-posedness result.

We use the following result [19, Proposition 3] to obtain the bilinear estimates. So we need to recall the definitions and notation accordingly.

For any integer $k \geq 2$, we denote $\Gamma_k(Z)$ as the hyperplane

$$\Gamma_k(Z) = \left\{\xi = (\xi_1, \ldots, \xi_k) \in Z^k, \ \xi_1 + \ldots + \xi_k = 0\right\}$$

with the measure

$$\int_{\Gamma_k(Z)} f := \int_{Z^{k-1}} f(\xi) \, \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_{k-1},$$

where Z can be any abelian additive group with an invariant measure $\mathrm{d}\xi$. By following the concepts introduced in [22], the $[k; Z]$-multiplier is a function $\eta : \Gamma_k(Z) \to \mathbb{C}$ and the multiplier norm $\|\eta\|_{[k;Z]}$ is the best constant such that

$$\left|\int_{\Gamma_k(Z)} \eta(\xi) \prod_{j=1}^k f_i(\xi_i)\right| \leq \|\eta\|_{[k;Z]} \prod_{j=1}^k \|f_i\|_{L^2(Z)}$$

holds for all test functions f_i on Z.

We should review some of Tao's notations in [22]. Any summations over capitalized variables such as N_i, L_i and H are presumed to be dyadic. It will be convenient for $N_1, N_2, N_3 > 0$ to define the quantities $N_{max} \geq N_{med} \geq N_{min}$ which are the maximum, median, and minimum of N_1, N_2, N_3, respectively. Likewise, we have $L_{max} \geq L_{med} \geq L_{min}$ if $L_1, L_2, L_3 > 0$. We also adopt the following summation

convention. Any summation of the form $L_{max} \sim \ldots$ is a sum over the three dyadic variables $L_1, L_2, L_3 \gtrsim 1$. Hence, we have for instance that

$$\sum_{L_{max} \sim H} = \sum_{L_1, L_2, L_3 \gtrsim 1 : L_{max} \sim H}.$$

Analogously, any summation of the form $N_{max} \sim \ldots$ sum over the three dyadic variables $N_1, N_2, N_3 > 0$. For example, we have

$$\sum_{N_{max} \sim N_{med} \sim N} := \sum_{N_1, N_2, N_3 > 0 : N_{max} \sim N_{med} \sim N}.$$

If τ, ξ and $\phi(\xi)$ are given with $\tau_1 + \tau_2 + \tau_3 = 0$, then we write $\lambda := \tau - \phi(\xi)$. Similarly we have $\lambda_i := \tau_i - \phi(\xi_i)$. We refer to $h : \Gamma_3(Z) \to \mathbb{R}$ as the *resonance function*, which is defined by

$$h(\xi) := \phi(\xi_1) + \phi(\xi_2) + \phi(\xi_3) = -\lambda_1 - \lambda_2 - \lambda_3. \tag{10}$$

By the dyadic decomposition of each variable ξ_i or λ_i, as well as the function $h(\xi)$, we are led to consider $\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3;\mathbb{R}\times\mathbb{R}]}$, where $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ is the multiplier

$$X_{N_1,N_2,N_3;H;L_1,L_2,L_3}(\xi,\tau) = \chi_{\{|h(\xi)|\sim H\}} \prod_{j=1}^{3} \chi_{\{|\xi_j|\sim N_j\}} \chi_{\{|\lambda_j|\sim L_j\}}, \tag{11}$$

where χ is the characteristic function. From the identities

$$\xi_1 + \xi_2 + \xi_3 = 0$$

and

$$\lambda_1 + \lambda_2 + \lambda_3 + h(\xi) = 0 \tag{12}$$

on the support of the multiplier, we see that $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ vanishes unless

$$N_{max} \sim N_{med} \tag{13}$$

and

$$L_{max} \sim \max(H, L_{med}). \tag{14}$$

The following estimates can be found in [22] (see also [19, Proposition 3])

Lemma 2.1. *Let* $H, N_1, N_2, N_3, L_1, L_2, L_3 > 0$ *satisfy* $N_{\max} \sim N_{\med}$, $H \sim N_2 N_3$ *and*
$L_{\max} \sim \max(H, L_{\med})$. *Then we have the following estimates.*
(1)

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim L_{\min}^{\frac{1}{2}} N_{\min}^{\frac{1}{2}}. \tag{15}$$

(2)

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim \min\left(\frac{L_1 L_2}{N_3}, \frac{L_1 L_3}{N_2}\right)^{\frac{1}{2}}. \tag{16}$$

(3) If $L_1 = L_{\max}$ *and* $N_1 \sim N_2 \sim N_3$ *holds, then*

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim L_{\min}^{\frac{1}{2}} L_{\med}^{\frac{1}{4}}, \tag{17}$$

If $L_1 = L_{\max}$ *and* $N_1 \sim N_2 \sim N_3$ *does not hold, then*

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim \frac{L_{\min}^{\frac{1}{2}} L_{\med}^{\frac{1}{2}}}{N_{\max}^{\frac{1}{2}}}. \tag{18}$$

Lemma 2.2. *If $s > \frac{\beta}{2}$, $0 < \beta < 1$ then for all u, v on $\mathbb{R} \times \mathbb{R}$, we have*

$$\|u\bar{v}\|_{L^2(\mathbb{R}\times\mathbb{R})} \lesssim \|u\|_{X^{-\frac{\beta}{2}, \frac{1}{2}-\varepsilon}} \|\bar{v}\|_{X^{s, \frac{1}{2}+\varepsilon}}, \tag{19}$$

$$\|u\bar{v}\|_{L^2(\mathbb{R}\times\mathbb{R})} \lesssim \|u\|_{X^{s, \frac{1}{2}+\varepsilon}} \|\bar{v}\|_{X^{-\frac{\beta}{2}, \frac{1}{2}-\varepsilon}}. \tag{20}$$

Proof. We first dispense with the estimate (19). Indeed, it suffices to show by the Plancherel theorem that

$$\left\| \frac{\langle \xi_1 \rangle^{\frac{\beta}{2}}}{\langle \xi_2 \rangle^s \langle \tau_1 - \xi_1^2 \rangle^{\frac{1}{2}-\varepsilon} \langle \tau_2 + \xi_2^2 \rangle^{\frac{1}{2}+\varepsilon}} \right\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1.$$

The comparison principle and orthogonality reduce our estimate to show that

$$\sum_{N_{\max} \sim N_{\mathrm{med}} \sim N} \sum_{L_1, L_2, L_3 \gtrsim 1} \frac{N_1^{\frac{\beta}{2}} N_2^{-s}}{L_1^{\frac{1}{2}-\varepsilon} L_2^{\frac{1}{2}+\varepsilon}} \times \|X_{N_1, N_2, N_3; L_{\max}; L_1, L_2, L_3}\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1 \tag{21}$$

and

$$\sum_{N_{\max} \sim N_{\mathrm{med}} \sim N} \sum_{L_{\max} \sim L_{\mathrm{med}}} \sum_{L_{\max} \gg H} \frac{N_1^{\frac{\beta}{2}} N_2^{-s}}{L_1^{\frac{1}{2}-\varepsilon} L_2^{\frac{1}{2}+\varepsilon}} \tag{22}$$
$$\times \|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1.$$

Fix N, we first prove (22). If we have $L_{\max} \sim L_{\mathrm{med}} \gg N_2 N_3$, we apply (15). Hence, the sum in (22) is estimated by

$$\sum_{N_{\max} \sim N_{\mathrm{med}} \sim N} \sum_{L_{\max} \sim L_{\mathrm{med}}} \sum_{L_{\max} \gg H} \frac{N_1^{\frac{\beta}{2}} N_2^{-s}}{L_1^{\frac{1}{2}-\varepsilon} L_2^{\frac{1}{2}+\varepsilon}} L_{\min}^{\frac{1}{2}} N_{\min}^{\frac{1}{2}}$$
$$\lesssim \sum_{N_{\max} \sim N_{\mathrm{med}} \sim N} \frac{N_1^{\frac{\beta}{2}} N_2^{-s} N_{\min}^{\frac{1}{2}}}{(N_2 N_3)^{\frac{1}{2}-\varepsilon}}. \tag{23}$$

If $N_{\min} = N_1$, then we have

$$\sum_{N_{\max} \sim N_{\mathrm{med}} \sim N} N^{-s + \frac{\beta}{2} - \frac{1}{2} + 2\varepsilon} \lesssim 1,$$

provided $s > \frac{\beta}{2} - \frac{1}{2}$.

If $N_{\min} = N_2$, then

$$\sum_{N_{\max} \sim N_{\mathrm{med}} \sim N} \frac{N_1^{\frac{\beta}{2}} N_2^{-s} N_{\min}^{\frac{1}{2}}}{(N_2 N_3)^{\frac{1}{2}-\varepsilon}} \lesssim \sum_{N_{\max} \sim N_{\mathrm{med}} \sim N} N_{\min}^{-s+\varepsilon} N^{\frac{\beta}{2} - \frac{1}{2} + \varepsilon},$$

if $s > 0$ then the estimate is certainly true for $\beta < 1$, if $s \leq 0$, then the estimate is similar to the case $N_{\min} = N_1$.

If $N_{\min} = N_3$, then $N_1 \sim N_2 \sim N$, we can obtain the desired estimate for $s > \frac{\beta}{2} - \frac{1}{2}$.

Now we show (21). We may assume $L_{\max} \sim N_2 N_3$.

If $L_{\max} = L_2$, we apply (15), the estimate is similar to (23), we omit it.

If $L_{\max} = L_3$, we apply (16), performing the L summations, then the summation in (21) is estimated by

$$\sum_{N_{\max} \sim N_{\mathrm{med}} \sim N} \sum_{L_1, L_2, L_3 \gtrsim 1} \frac{N_1^{\frac{\beta}{2}} N_2^{-s}}{L_1^{\frac{1}{2}-\varepsilon} L_2^{\frac{1}{2}+\varepsilon}} \frac{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}}}{N_3^{\frac{1}{2}}} \lesssim \sum_{N_{\max} \sim N_{\mathrm{med}} \sim N} \frac{N_1^{\frac{\beta}{2}} N_2^{-s} (N_2 N_3)^{\varepsilon}}{N_3^{\frac{1}{2}}},$$

if $N_{\min} = N_3$, then (21) is true for $s > \frac{\beta}{2}$, if $N_{\min} \neq N_3$, then we can get (21) for $s > \frac{\beta}{2} - \frac{1}{2}$.

If $L_{\max} = L_1$ and $N_1 \sim N_2 \sim N_3 \sim N$, then we can estimate the sum in (21) via (17),

$$\sum_{N_{\max} \sim N_{\mathrm{med}} \sim N} \sum_{L_1, L_2, L_3 \gtrsim 1} \frac{N_1^{\frac{\beta}{2}} N_2^{-s}}{L_1^{\frac{1}{2}-\varepsilon} L_2^{\frac{1}{2}+\varepsilon}} L_{\min}^{\frac{1}{2}} L_{\mathrm{med}}^{\frac{1}{4}} \lesssim \sum_{N_{\max} \sim N_{\mathrm{med}} \sim N} N_1^{\frac{\beta}{2}} N_2^{-s} (N_2 N_3)^{-\frac{1}{4}+\varepsilon}$$

$$\lesssim 1$$

for $s > \frac{\beta}{2} - \frac{1}{2}$. Otherwise, (17) can be used to estimate the sum in (21)

$$\sum_{N_{\max} \sim N_{\mathrm{med}} \sim N} \sum_{L_1, L_2, L_3 \gtrsim 1} \frac{N_1^{\frac{\beta}{2}} N_2^{-s}}{L_1^{\frac{1}{2}-\varepsilon} L_2^{\frac{1}{2}+\varepsilon}} \frac{L_{\min}^{\frac{1}{2}} L_{\mathrm{med}}^{\frac{1}{4}}}{N_{\max}^{\frac{1}{2}}} \lesssim \sum_{N_{\max} \sim N_{\mathrm{med}} \sim N} \frac{N_1^{\frac{\beta}{2}} N_2^{-s}}{L_1^{\frac{1}{4}-\varepsilon} N_{\max}^{\frac{1}{2}}} \lesssim 1$$

for $s > \frac{\beta}{2} - \frac{1}{2}$.

The proof of (20) is similar to (19), we omit it. $\qquad\square$

Theorem 2.3. *Let* $s > \frac{\beta}{2}$, *for any* u, v, w, *then*

$$\left\| D^\beta (u\bar{v}w) \right\|_{X^{s,-\frac{1}{2}+\varepsilon}} \lesssim \|u\|_{X^{s,\frac{1}{2}+\varepsilon}} \|\bar{v}\|_{X^{s,\frac{1}{2}+\varepsilon}} \|w\|_{X^{s,\frac{1}{2}+\varepsilon}}. \tag{24}$$

Proof. By duality and the Plancherel theorem, it suffices to show that

$$\left\| \frac{|\xi_1 + \xi_2 + \xi_3|^\beta \langle \xi_4 \rangle^s}{\langle \tau_4 + \xi_4^2 \rangle^{\frac{1}{2}-\varepsilon} \langle \xi_1 \rangle^s \langle \tau_1 - \xi_1^2 \rangle^{\frac{1}{2}+\varepsilon} \langle \xi_2 \rangle^s \langle \tau_2 + \xi_2^2 \rangle^{\frac{1}{2}+\varepsilon} \langle \xi_3 \rangle^s \langle \tau_3 - \xi_3^2 \rangle^{\frac{1}{2}+\varepsilon}} \right\|_{[4;\mathbb{R}\times\mathbb{R}]} \lesssim 1. \tag{25}$$

We estimate $|\xi_1 + \xi_2 + \xi_3|$ by $\langle \xi_4 \rangle$, then apply the inequality

$$\langle \xi_4 \rangle^{s+\beta} \lesssim \langle \xi_4 \rangle^{\frac{\beta}{2}} \sum_{j=1}^{3} \langle \xi_j \rangle^{s+\frac{\beta}{2}}.$$

If $\max_{j=1,2,3} |\xi_j| = |\xi_2|$, we may minorize $\langle \tau_2 + \xi_2^2 \rangle^{\frac{1}{2}+\varepsilon}$ by $\langle \tau_2 + \xi_2^2 \rangle^{\frac{1}{2}-\varepsilon}$, and reduce to showing that

$$\left\| \frac{\langle \xi_2 \rangle^{\frac{\beta}{2}}}{\langle \xi_1 \rangle^s \langle \tau_1 - \xi_1^2 \rangle^{\frac{1}{2}+\varepsilon} \langle \tau_2 + \xi_2^2 \rangle^{\frac{1}{2}-\varepsilon}} \cdot \frac{\langle \xi_4 \rangle^{\frac{\beta}{2}}}{\langle \xi_3 \rangle^s \langle \tau_3 - \xi_3^2 \rangle^{\frac{1}{2}+\varepsilon} \langle \tau_4 + \xi_4^2 \rangle^{\frac{1}{2}-\varepsilon}} \right\|_{[4;\mathbb{R}\times\mathbb{R}]} \lesssim 1, \tag{26}$$

then TT^* identity reduce our estimate to show that

$$\left\| \frac{\langle \xi_2 \rangle^{\frac{\beta}{2}}}{\langle \xi_1 \rangle^s \langle \tau_1 - \xi_1^2 \rangle^{\frac{1}{2}+\varepsilon} \langle \tau_2 + \xi_2^2 \rangle^{\frac{1}{2}-\varepsilon}} \right\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1,$$

$$\left\| \frac{\langle \xi_4 \rangle^{\frac{\beta}{2}}}{\langle \xi_3 \rangle^s \langle \tau_3 - \xi_3^2 \rangle^{\frac{1}{2}+\varepsilon} \langle \tau_4 + \xi_4^2 \rangle^{\frac{1}{2}-\varepsilon}} \right\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1.$$

These estimates follows from (20).

If $\max_{j=1,2,3} |\xi_j| = |\xi_1|$, we may monorize $\langle \tau_1 - \xi_1^2 \rangle^{\frac{1}{2}+\varepsilon}$ by $\langle \tau_1 - \xi_1^2 \rangle^{\frac{1}{2}-\varepsilon}$, reduce to showing that

$$\left\| \frac{\langle \xi_1 \rangle^{\frac{\beta}{2}}}{\langle \xi_2 \rangle^s \langle \tau_1 - \xi_1^2 \rangle^{\frac{1}{2}-\varepsilon} \langle \tau_2 + \xi_2^2 \rangle^{\frac{1}{2}+\varepsilon}} \cdot \frac{\langle \xi_4 \rangle^{\frac{\beta}{2}}}{\langle \xi_3 \rangle^s \langle \tau_3 - \xi_3^2 \rangle^{\frac{1}{2}+\varepsilon} \langle \tau_4 + \xi_4^2 \rangle^{\frac{1}{2}-\varepsilon}} \right\|_{[4;\mathbb{R}\times\mathbb{R}]} \lesssim 1, \tag{27}$$

then TT^* identity reduce our estimate to show that

$$\left\| \frac{\langle \xi_1 \rangle^{\frac{\beta}{2}}}{\langle \xi_2 \rangle^s \langle \tau_1 - \xi_1^2 \rangle^{\frac{1}{2}-\varepsilon} \langle \tau_2 + \xi_2^2 \rangle^{\frac{1}{2}+\varepsilon}} \right\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1,$$

$$\left\| \frac{\langle \xi_4 \rangle^{\frac{\beta}{2}}}{\langle \xi_3 \rangle^s \langle \tau_3 - \xi_3^2 \rangle^{\frac{1}{2}+\varepsilon} \langle \tau_4 + \xi_4^2 \rangle^{\frac{1}{2}-\varepsilon}} \right\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1.$$

The first estimate is valid by (19), and the second estimate is valid by (20).

By symmetry, the estimate of $\max\limits_{j=1,2,3} |\xi_j| = |\xi_3|$ is the same as the case $\max\limits_{j=1,2,3} |\xi_j| = |\xi_1|$. $\qquad\square$

With Theorem 2.3 in our hands, the completeness of proof of Theorem 1.1 is obtained by the standard approach and we omit it (see [6]).

3. **Ill-posedness.** In this section, we prove our ill-posedness result in Theorem 1.2. Our idea is to approximate the solutions of (1) with ones of (2). For convenience we the case $\varsigma = +1$:

$$\begin{cases} iu_t - u_{xx} - D^\beta \left(|u|^2 u \right) = 0, \\ u(0) = \varphi \end{cases} \tag{28}$$

We also consider (2) correspondingly. To prove the ill-posedness result, we recall the following result of [6, Lemma 2.1] (see also [5]).

Lemma 3.1. *Let $s > -\frac{1}{2}$ and $w \in H^\sigma(\mathbb{R})$ with $\sigma > 0$. For $M > 1, \tau > 0, x_0 \in \mathbb{R}$ and $A > 0$ let*

$$v(x) = Ae^{iMx}u\left(\frac{x - x_0}{\tau}\right).$$

(1) If $s \geq 0$, then

$$\|v\|_{H^s} \leq C_1 |A| \tau^{\frac{1}{2}} M^s \|u\|_{H^s}$$

for all u, A, x_0 whenever $M \cdot \tau \geq 1$, where C_1 is a positive constant depending only on s.

(2) If $s < 0$ and $\sigma \geq |s|$, then

$$\|v\|_{H^s} \leq C_1 |A| \tau^{\frac{1}{2}} M^s \|u\|_{H^\sigma}$$

for all u, A, x_0 whenever $M^{1+\frac{s}{\sigma}} \cdot \tau \geq 1$, where C_1 is a positive constant depending only on s and σ.

(3) There exists $c_1 > 0$ such that for each u there exists $C_u < \infty$ such that

$$\|v\|_{H^s} \geq c_1 |A| \tau^{\frac{1}{2}} M^s \|u\|_{L^2}$$

whenever $\tau \cdot M \geq C_u$.

Now we recall the ill-posedness result of [6] for (2).

Lemma 3.2. *Let $s < 0$. The solution map of the initial value problem associated with (2) fails to be uniformly continuous. More precisely, for $0 < \delta \ll \varepsilon \ll 1$ and $T > 0$ arbitrary, there are two solutions v_1, v_2 to (2) with initial data ϕ_1, ϕ_2 such that*

$$\|\phi_1\|_{H^s}, \|\phi_2\|_{H^s} \lesssim \varepsilon,$$
$$\|\phi_1 - \phi_2\|_{H^s} \lesssim \delta,$$
$$\sup_{0 \leq t \leq T} \|v_1(t) - v_2(t)\|_{H^s} \gtrsim \varepsilon.$$

Moreover we can find solutions to satisfy

$$\sup_{0 \le t \le \infty} \|v_j(t)\|_{H^5} \lesssim \varepsilon, \quad j = 1, 2. \tag{29}$$

Suppose that $N \gg 1$ is a large parameter that will be chosen later. Let $v(s, y)$ be a solution of the cubic NLS equation (2) and

$$(s, y) := (t, x + 2Nt).$$

We are going to construct an approximate solution given by

$$V(t, x) = e^{iNx} e^{iN^2 t} v(s, y). \tag{30}$$

Since $v(s, y)$ is a solution of (2), we have

$$i\partial_t V - \partial_x^2 V - D^\beta \left(|V|^2 V \right) = E,$$

where

$$E = e^{iNx} e^{iN^2 t} |v|^2 v - D^\beta \left(|V|^2 V \right) = E_2 - E_1.$$

Lemma 3.3. *Let u be a smooth solution to (28) and V be a smooth solution to the equation*

$$i\partial_t V - \partial_x^2 V - D^\beta \left(|V|^2 V \right) = E$$

for some error function E. Let e be the solution to the inhomogeneous problem $i\partial_t e - \partial_x^2 e = E$ with $e(0) = 0$ and suppose that $\zeta(t)$ is a compactly supported smooth time cut-off function such that $\zeta = 1$ on $I = [0, 1]$. If $\|u(0)\|_{H^{\frac{\beta}{2}+\varepsilon}}$, $\|V(0)\|_{H^{\frac{\beta}{2}+\varepsilon}}$, $\|\zeta(t)e\|_{X^{\frac{\beta}{2}+\varepsilon, \frac{1}{2}+\varepsilon}} \lesssim \varepsilon$ and ε is sufficiently small, then we have

$$\|u - V\|_{X^{\frac{\beta}{2}+\varepsilon, \frac{1}{2}+\varepsilon}} \lesssim \|u(0) - V(0)\|_{H^{\frac{\beta}{2}+\varepsilon}} + \|\zeta(t)e\|_{X^{\frac{\beta}{2}+\varepsilon, \frac{1}{2}+\varepsilon}}.$$

In particular, we have

$$\sup_{0 \le t \le 1} \|u(t) - V(t)\|_{H^{\frac{\beta}{2}}} \lesssim \|u(0) - V(0)\|_{H^{\frac{\beta}{2}}} + \|\zeta(t)e\|_{X^{\frac{\beta}{2}, \frac{1}{2}+\varepsilon}}.$$

Proof. If we consider in the integral form for V, then we have

$$V(t) - e(t) = U(t)V(0) - i \int_0^t U(t - t')(D^\beta(|V|^2 V))(t') \, dt'.$$

We get by taking $X^{\frac{\beta}{2}, \frac{1}{2}+\varepsilon}(I)$ norm on both sides that

$$\|V\|_{X^{\frac{\beta}{2}+\varepsilon, \frac{1}{2}+\varepsilon}} \lesssim \|V(0)\|_{H^{\frac{\beta}{2}+\varepsilon}} + \|\zeta(t)e\|_{X^{\frac{\beta}{2}+\varepsilon, \frac{1}{2}+\varepsilon}} + \left\|D^\beta(|V|^2 V)\right\|_{X^{\frac{\beta}{2}+\varepsilon, -\frac{1}{2}+\varepsilon}(I)}$$

$$\lesssim \|V(0)\|_{H^{\frac{\beta}{2}+\varepsilon}} + \|\zeta(t)e\|_{X^{\frac{\beta}{2}+\varepsilon, \frac{1}{2}+\varepsilon}} + \|V\|_{X^{\frac{\beta}{2}+\varepsilon, \frac{1}{2}+\varepsilon}(I)}^3.$$

Now it reveals by the continuity argument with sufficiently small ε that $\|V\|_{X^{\frac{\beta}{2}+\varepsilon, \frac{1}{2}+\varepsilon}(I)} \lesssim \varepsilon$.

Let $w := u - V$. Then w satisfies the equation

$$i\partial_t w - \partial_x^2 w = D^\beta \left(|w|^2 w + 2|w|^2 V + 2w|V|^2 + w^2 \bar{V} + \bar{w} V^2 \right) - E, \quad w(0) = u(0) - V(0).$$

which is written in integral form as

$$w(t) = U(t)w(0) - e(t)$$

$$- i \int_0^t U(t - t') \left(D^\beta \left(|w|^2 w + 2|w|^2 V + 2w|V|^2 + w^2 \bar{V} + \bar{w} V^2 \right) \right)(t') \, dt'.$$

We have by taking again $X^{\frac{\beta}{2}+\varepsilon,\frac{1}{2}+\varepsilon}(I)$ norm on both sides of the above equation that

$$
\begin{aligned}
\|w\|_{X^{\frac{\beta}{2}+\varepsilon,\frac{1}{2}+\varepsilon}} &\lesssim \|u(0)-V(0)\|_{H^{\frac{\beta}{2}+\varepsilon}} + \|\zeta(t)e\|_{X^{\frac{\beta}{2}+\varepsilon,\frac{1}{2}+\varepsilon}} \\
&\quad + \left\||w|^2 w + 2|w|^2 V + 2w|V|^2 + w^2\bar{V} + \bar{w}V^2\right\|_{X^{\frac{\beta}{2}+\varepsilon,-\frac{1}{2}+\varepsilon}(I)} \\
&\lesssim \|u(0)-V(0)\|_{H^{\frac{\beta}{2}+\varepsilon}} + \|\zeta(t)e\|_{X^{\frac{\beta}{2}+\varepsilon,\frac{1}{2}+\varepsilon}} \\
&\quad + \|w\|_{X^{\frac{\beta}{2}+\varepsilon,\frac{1}{2}+\varepsilon}(I)} \left(\|w\|_{X^{\frac{\beta}{2}+\varepsilon,\frac{1}{2}+\varepsilon}(I)} + \|V\|_{X^{\frac{\beta}{2}+\varepsilon,\frac{1}{2}+\varepsilon}(I)}\right)^2.
\end{aligned}
$$

Finally the desired bound is deduced by the continuity argument with respect to time provided ε is sufficiently small. $\qquad\square$

Lemma 3.4. *If e is a solution of equation $\mathrm{i}\partial_t e - \partial_x^2 e = E$ with $e(0)=0$, and ζ is the smooth time cut-off function given above, the $V(t,x)$ in E satisfy (30) and (29), then*

$$
\|\zeta(t)e\|_{X^{\frac{\beta}{2}+\varepsilon,\frac{1}{2}+\varepsilon}} \lesssim \varepsilon N^{-1+\frac{3\beta}{2}+\varepsilon}.
$$

Proof. Using the Plancherel theorem we have

$$
\begin{aligned}
\|\zeta(t)e_1\|_{X^{\frac{\beta}{2}+\varepsilon,\frac{1}{2}+\varepsilon}} &\lesssim \|\zeta(t)E_1\|_{X^{\frac{\beta}{2}+\varepsilon,-\frac{1}{2}+\varepsilon}} \\
&= \left\|\langle\tau-\xi^2\rangle^{-\frac{1}{2}+\varepsilon}\langle\xi\rangle^{\frac{\beta}{2}+\varepsilon}\widehat{\zeta(t)E_1}\right\|_{L^2_\tau L^2_\xi} \\
&\leq \left\|\langle\xi\rangle^{\frac{\beta}{2}+\varepsilon}\widehat{\zeta(t)E_1}\right\|_{L^2_\tau L^2_\xi} \\
&= \left\|\zeta(t)\langle\xi\rangle^{\frac{\beta}{2}+\varepsilon}\widehat{E_1}(t,\xi)\right\|_{L^2_t L^2_\xi} \\
&\leq \left\|\langle\xi\rangle^{\frac{\beta}{2}+\varepsilon}\widehat{E_1}(t,\xi)\right\|_{L^\infty_t L^2_x([0,1]\times\mathbb{R})}.
\end{aligned}
$$

A direct computation leads to that

$$
\widehat{\zeta(t)E_1}(\tau,\xi) = |\xi|^\beta \widehat{\zeta|v|^2 v}(\tau+N^2-2N\xi, \xi-N).
$$

Let $P_{\lambda,\mu}$ be the Littlewood-Paley projection with dyadic numbers λ, μ. The fact that $\zeta(t)$ is compactly supported yields

$$
\left\|P_{\lambda,\mu}\widehat{\zeta|v|^2 v}(\tau,\xi)\right\|_{L^2_\tau L^2_\xi} \lesssim \frac{\varepsilon}{\langle\lambda\rangle^K \langle\mu\rangle^K}
$$

by choosing K large enough, and so

$$
\left\|P_{\lambda,\mu}\widehat{\zeta|v|^2 v}(\tau+N^2-2N\xi, \xi-3N)\right\|_{L^2_\tau L^2_\xi} \lesssim \frac{\varepsilon}{\langle\lambda+N^2-2N\xi\rangle^K \langle\mu-N\rangle^K}.
$$

Rewriting $\|\zeta(t)E_1\|_{X^{\frac{\beta}{2}+\varepsilon,-\frac{1}{2}+\varepsilon}}$ by dyadic decomposition,

$$
\begin{aligned}
\|\zeta(t)E_1\|_{X^{\frac{\beta}{2}+\varepsilon,-\frac{1}{2}+\varepsilon}}^2 &\lesssim \sum_{\lambda,\mu\geq 1,\text{dyadic}} \langle\lambda-\mu^2\rangle^{-1+2\varepsilon}\langle\mu\rangle^{3\beta} \\
&\qquad\times \left\|P_{\lambda,\mu}\widehat{\zeta|v|^2 v}(\tau+N^2-2N\xi, \xi-3N)\right\|_{L^2_\tau L^2_\xi} \\
&\lesssim \sum_{\lambda,\mu\geq 1,\text{dyadic}} \langle\lambda-\mu^2\rangle^{-1+2\varepsilon}\langle\mu\rangle^{3\beta} \frac{\varepsilon^2}{\langle\lambda+N^2-2N\xi\rangle^{2K}\langle\mu-N\rangle^{2K}} \\
&\lesssim \varepsilon^2 N^{-2+3\beta+2\varepsilon}.
\end{aligned}
$$

The estimate of $\left\|\zeta(t)E_2\right\|^2_{X^{\frac{\beta}{2},-\frac{1}{2}+\varepsilon}}$ is similar to $\left\|\zeta(t)E_1\right\|^2_{X^{\frac{\beta}{2},-\frac{1}{2}+\varepsilon}}$. We omit it. $\qquad\square$

Now we are ready to prove our ill-posedness result.

Proof of Theorem 1.2. Let $0 < \delta \ll \varepsilon \ll 1$ and $T > 0$ be given. From Lemma 3.2 we have two global solution v_1, v_2 with initial data ϕ_1, ϕ_2, respectively, such that

$$\|\phi_1\|_{H^s}, \|\phi_2\|_{H^s} \lesssim \varepsilon,$$

$$\|\phi_1 - \phi_2\|_{H^s} \lesssim \delta,$$

$$\sup_{0 \le t \le T} \|v_1(t) - v_2(t)\|_{H^s} \gtrsim \varepsilon. \qquad (31)$$

$$\sup_{0 \le t \le \infty} \|v_1(t)\|_{H^5}, \|v_2(t)\|_{H^5} \lesssim \varepsilon.$$

Define V_1, V_2 by

$$V_j(t,x) = e^{iNx}e^{iN^2t}v_j(s,y), j = 1, 2.$$

And let u_1, u_2 be smooth solutions of (28) with initial data $V_1(0,x), V_2(0,x)$, respectively. Let $\lambda \gg 1$ be a large parameter to be chosen later. For $j = 1, 2$, set

$$u_j^\lambda := \lambda u_j(\lambda^2 t, \lambda x), V_j^\lambda := \lambda V_j(\lambda^2 t, \lambda x).$$

Thus we have

$$u_j^\lambda(0,x) = e^{iN\lambda x}V_j^\lambda(0,x) = \lambda e^{iN\lambda x}v_j(0, \lambda x).$$

Lemma 3.1 with $M = N\lambda$, $\tau = \lambda^{-1}$ implies that if $s \ge 0$,

$$\left\|u_j^\lambda(0)\right\|_{H^s} \lesssim \lambda^{s+\frac{1}{2}}N^s \left\|v_j(0)\right\|_{H^s},$$

while for $s < 0$, for sufficiently large k we obtain

$$\left\|u_j^\lambda(0)\right\|_{H^s} \lesssim \lambda^{s+\frac{1}{2}}N^s \left\|v_j(0)\right\|_{H^k}.$$

We choose $\lambda = N^{\frac{-s}{s+\frac{1}{2}}}$, then

$$\left\|u_j^\lambda(0)\right\|_{H^s} \lesssim \varepsilon, \quad \left\|u_1^\lambda(0) - u_2^\lambda(0)\right\|_{H^s} \lesssim \delta.$$

Rescaling gives

$$\left\|u_j^\lambda(t) - V_j^\lambda(t)\right\|_{H^s} \lesssim \lambda^{\max(s,0)+\frac{1}{2}} \left\|u_j(\lambda^2 t) - V_j(\lambda^2 t)\right\|_{H^s}$$
$$\lesssim \lambda^{\max(s,0)+\frac{1}{2}} \left\|u_j(\lambda^2 t) - V_j(\lambda^2 t)\right\|_{H^{\frac{\beta}{2}+\varepsilon}}.$$

Induction argument on time interval up to $\frac{\log N}{\lambda^2}$ yields

$$\left\|u_j(\lambda^2 t) - V_j(\lambda^2 t)\right\|_{H^{\frac{\beta}{2}+\varepsilon}} \lesssim \varepsilon N^{-1+\frac{3\beta}{2}+\varepsilon},$$

whenever $0 < t \ll \frac{\log N}{\lambda^2}$. Hence we have

$$\left\|u_j^\lambda(t) - V_j^\lambda(t)\right\|_{H^s} \lesssim \lambda^{\max(s,0)+\frac{1}{2}}\varepsilon N^{-1+\frac{3\beta}{2}+\varepsilon}.$$

From the hypothesis $s > \frac{3\beta-2}{2(3-3\beta)}$, it follows that

$$\left\|u_j^\lambda(t) - V_j^\lambda(t)\right\|_{H^s} \lesssim \varepsilon. \qquad (32)$$

Applying Lemma 3.1, we have

$$\left\|u_j^\lambda(t)\right\|_{H^s} \le \left\|u_j^\lambda(t) - V_j^\lambda(t)\right\|_{H^s} + \left\|V_j^\lambda(t)\right\|_{H^s} \lesssim \varepsilon + \left\|v_j(\lambda^2 t)\right\|_{H^s} \lesssim \varepsilon. \qquad (33)$$

By the ill-posedness result (31), we can find a time $t_0 > 0$ such that $\|v_1(t_0) - v_2(t_0)\|_{L^2} \gtrsim \varepsilon$. Using Lemma 3.1 we obtain

$$\left\| V_1\left(\frac{t_0}{\lambda^2}\right) - V_2\left(\frac{t_0}{\lambda^2}\right) \right\|_{H^s} \gtrsim \lambda^{s+\frac{1}{2}}\varepsilon N^s \sim \varepsilon. \tag{34}$$

Combining (32), (33) and (34), a triangle inequality shows

$$\left\| u_1\left(\frac{t_0}{\lambda^2}\right) - u_2\left(\frac{t_0}{\lambda^2}\right) \right\|_{H^s} \gtrsim \varepsilon,$$

for $t_0 \ll \log N$. Choosing λ large enough that $\frac{t_0}{\lambda^2} < T$, we get (6). This completes the proof. $\qquad\square$

4. **Standing wave.** In this section we study the existence of standing waves of (1).

The following result is direct consequence of the Gagliardo-Nirenberg inequality.

Lemma 4.1. *Let* $-\frac{1}{2} < \beta < \frac{3}{2}$ *and* $0 \le q \le \frac{q^*}{2} - 1$, *where*

$$q^* = \begin{cases} \frac{2}{\beta - 1}, & \beta > 1, \\ \infty^-, & \beta \le 1, \end{cases}$$

where ∞^- *is any number* $q_1 < \infty$. *Then there is a constant* $C > 0$ *such that for any* $g \in \mathscr{X}$,

$$\|g\|_{L^{2q+2}(\mathbb{R})} \le C \left\| D^{-\frac{\beta}{2}} g \right\|_{L^2(\mathbb{R})}^{1 - \frac{1}{2}(\beta + \frac{q}{q+1})} \left\| D^{-\frac{\beta}{2}} g_x \right\|_{L^2(\mathbb{R})}^{\frac{\beta}{2} + \frac{q}{2(q+1)}}. \tag{35}$$

As a consequence, it follows that $\mathscr{X}$ *is continuously embedded in* $L^{2q+2}(\mathbb{R})$.

Proof. If $\beta \ge 0$, then (35) is obtained from the following Gagliardo-Nirenberg inequality

$$\|D^{\frac{\beta}{2}} g\|_{L^{2q+2}(\mathbb{R})} \le C \|g\|_{L^2(\mathbb{R})}^{1 - \frac{1}{2}(\beta + \frac{q}{q+1})} \|g_x\|_{L^2(\mathbb{R})}^{\frac{\beta}{2} + \frac{q}{2(q+1)}}. \tag{36}$$

If $-\frac{1}{2} < \beta \le 0$, then we have again from the Gagliardo-Nirenberg inequality with $p > 1$ that

$$\|g\|_{L^{2q+2}(\mathbb{R})} \le C \|g\|_{L^p(\mathbb{R})}^{1-\theta} \left\| D^{1-\frac{\beta}{2}} g \right\|_{L^2(\mathbb{R})}^{\theta},$$

where $\theta = \frac{2 - \frac{p}{q+1}}{2 + (1-\beta)p} \in (0,1)$. Finally inequality (35) is followed from

$$\|g\|_{L^p(\mathbb{R})} \le C \|D^{-\frac{\beta}{2}} g\|_{L^2(\mathbb{R})},$$

with $p = \frac{2}{\beta+1}$. $\qquad\square$

Lemma 4.2. *Assume that* $\omega > 0$. *Equation* (8) *possesses no nontrivial solution* $\varphi \in \mathscr{X} \cap L^4(\mathbb{R})$ *if* $\beta \ge \frac{3}{2}$ *or* $\beta \le -\frac{1}{2}$ *holds.*

Proof. Let φ be a nontrivial solution of (8). First we multiply (8) by φ and integrate over $\mathbb{R}$ to get

$$\omega \|D^{-\frac{\beta}{2}} \varphi\|_{L^2(\mathbb{R})}^2 + \|D^{1-\frac{\beta}{2}} \varphi\|_{L^2(\mathbb{R})}^2 = \|\varphi\|_{L^4(\mathbb{R})}^4. \tag{37}$$

Next, we multiply (8) by $x\varphi_x$ and integrate over $\mathbb{R}$, and use the identity (see [14, Lemma 3] and [15, Theorem 4.1])

$$\int_{\mathbb{R}} x\varphi_x D^\alpha \varphi \, \mathrm{d}x = \frac{\alpha - 1}{2} \|D^{\frac{\alpha}{2}} \varphi\|_{L^2(\mathbb{R})}^2$$

to deduce that

$$\frac{1}{2}\|\varphi\|_{L^4(\mathbb{R})}^4 + (1-\beta)\|D^{1-\frac{\beta}{2}}\varphi\|_{L^2(\mathbb{R})}^2 = \omega(\beta+1)\|D^{-\frac{\beta}{2}}\varphi\|_{L^2(\mathbb{R})}^2. \tag{38}$$

Combining (37) and (38), we derive

$$\|D^{1-\frac{\beta}{2}}\varphi\|_{L^2(\mathbb{R})}^2 = \frac{2\beta+1}{4}\|\varphi\|_{L^4(\mathbb{R})}^4$$

$$\omega\|D^{-\frac{\beta}{2}}\varphi\|_{L^2(\mathbb{R})}^2 = \frac{3-2\beta}{4}\|\varphi\|_{L^4(\mathbb{R})}^4. \tag{39}$$

And the proof is deduced form (39). □

Theorem 4.3. *Let* $-\frac{1}{2} < \beta < \frac{3}{2}$. *If* $\varphi \in \mathscr{X}$ *be a nontrivial solution of* (8), *then* $\varphi \in L^1(\mathbb{R}) \cap H^\infty(\mathbb{R})$ *and*

$$|\varphi(x)| = O(e^{-\sqrt{\omega}|x|}), \tag{40}$$

at infinity if $\beta \geq 0$ *and* $|\varphi(x)| = O(|x|^{-1-\beta})$ *if* $\beta \leq 0$.

Proof. One can observe that any nontrivial solution $\varphi \in \mathscr{X}$ satisfies

$$\varphi = K * \varphi^3, \tag{41}$$

where

$$\hat{K}(\xi) = \frac{|\xi|^\beta}{\omega + |\xi|^2}.$$

We note that $K = D^\beta K_0$, where $K_0(x) = (\omega + D^2)^{-1}$. It is known that

$$K_0(x) \cong \frac{1}{\sqrt{\omega}}e^{-\sqrt{\omega}|x|}.$$

It is easy to see that $\|K_0\|_{L^q(\mathbb{R})} = \omega^{-\frac{1}{2}(1+\frac{1}{q})}$. It is known from [7] that

$$K(x) = \frac{\pi}{2}\csc(\frac{\pi}{2}(1+\beta))\cosh(\sqrt{\omega}x)$$
$$+ \frac{1}{2}\Gamma(1+\beta)\cos(\frac{\pi}{2}(1+\beta))$$
$$\times \left(e^{-\sqrt{\omega}|x|-\beta\pi i}\gamma(-\beta, -\sqrt{\omega}|x|) - e^{\sqrt{\omega}|x|}\gamma(-\beta, \sqrt{\omega}|x|),\right)$$

where $\gamma(\cdot, \cdot)$ is the lower incomplete gamma function. Furthermore,

$$D^\beta K_0 \in L^q(\mathbb{R}) \tag{42}$$

for all $1 \leq q \leq \infty$ and $\beta \geq 0$ with $\beta - 1/q < 1$ by an interpolation. To show the regularity of the solutions of (8), we claim that $\varphi \in L^\infty(\mathbb{R})$. Actually, it is obvious from the Sobolev inequality for th case $\beta \leq 1$. In the case $\beta > 1$, we can use (41) and apply the Young inequality and (42) to get $\varphi \in L^r(\mathbb{R})$ when $\frac{2}{r} \geq 5\beta - 7$; so that $\varphi \in L^\infty(\mathbb{R})$ if $\beta < \frac{7}{5}$. If we repeat this process once again, we conclude $\varphi \in L^r(\mathbb{R})$ when $\frac{2}{r} \geq 7\beta - 11$; so that $\varphi \in L^\infty(\mathbb{R})$ for any $\beta < \frac{3}{2}$.

Now, (42) shows that $\varphi \in H^{2(1-\frac{\beta}{2})}(\mathbb{R})$, so if $\beta < 1$, then $\varphi \in H^1(\mathbb{R})$. We proceed to find from (41) that

$$\|D^{3-\beta}\varphi\|_{L^2(\mathbb{R})} \lesssim \|\varphi^2\varphi_x\|_{L^2(\mathbb{R})} \lesssim \|\varphi_x\|_{L^2(\mathbb{R})} < +\infty.$$

So $\varphi \in H^{3-\beta}(\mathbb{R})$. In the case $\beta > 1$, let $L > 0$ be an integer such that $1/(L+1) \leq 1 - \frac{\beta}{2} < 1/L < 1$. Then by an iteration argument as done for the case $\beta < 1$, we can deduce that $\varphi \in H^{3-\beta}(\mathbb{R})$. Repeating the above argument show that $\varphi \in H^k(\mathbb{R})$ for any $k \geq 1$.

The decay analysis follows again from the properties of the kernel associated with (8). We show that $D^\beta K_0(x) = O(\exp(-\sqrt{\omega}|x|))$. Indeed, by using the equivalence form of D^β [9], we have

$$|D^\beta K_0(x)| = \left| \text{p.v.} \int_{\mathbb{R}} \frac{e^{-\sqrt{\omega}|x|} - e^{-\sqrt{\omega}|y|}}{|x - y|^{1+\beta}} dy \right|$$

$$\leq e^{-\sqrt{\omega}|x|} \left| \text{p.v.} \int_{\mathbb{R}} \frac{1 - e^{\sqrt{\omega}(|x|-|y|)}}{|x - y|^{1+\beta}} dy \right|$$

$$\leq e^{-\sqrt{\omega}|x|} \left(\text{p.v.} \int_{|y| \leq 1} \frac{|1 - e^{\sqrt{\omega}(|x|-|x-y|)}|}{|y|^{1+\beta}} dy \right.$$

$$\left. + \int_{|y| \geq 1} \frac{|1 - e^{\sqrt{\omega}(|x|-|x-y|)}|}{|y|^{1+\beta}} dy \right)$$

$$\leq e^{-\sqrt{\omega}|x|} \left(\text{p.v.} \int_{|y| \leq 1} \frac{|1 - e^{-\sqrt{\omega}|y|}|}{|y|^{1+\beta}} dy + \int_{|y| \geq 1} \frac{|1 - e^{-\sqrt{\omega}|y|}|}{|y|^{1+\beta}} dy \right).$$

The second integral of the above expression is easily bounded. The first integral is also bounded by using $|1 - \exp(-\sqrt{\omega}|y|)| \leq y^2$.

Next we consider the case $\beta \leq 0$. Without loss of generality, we can assume that $\omega = 1$. we have

$$|D^\beta K_0(x) - |x|^{-1-\beta}| = \left| \int_{\mathbb{R}} e^{-|y|} \left(\frac{1}{|x - y|^{1+\beta}} - \frac{1}{|x|^{1+\beta}} \right) dy \right|$$

Note that when $|y| \leq 2|x|$, one has

$$\left| \frac{1}{|x - y|^{1+\beta}} - \frac{1}{|x|^{1+\beta}} \right| \lesssim \frac{|y|}{|x|^{2+\beta}}$$

and thus

$$\left| \int_{|y| \leq 2|x|} e^{-|y|} \left(\frac{1}{|x - y|^{1+\beta}} - \frac{1}{|x|^{1+\beta}} \right) dy \right| \lesssim \frac{1}{|x|^{2+\beta}} \left| \int_{\mathbb{R}} e^{-|y|} |y| dy \right|$$

$$\lesssim \frac{1}{|x|^{2+\beta}} \left(1 - (1 + 2|x|)e^{-2|x|} \right).$$

On the other hand, we have in the case $|y| \geq 2|x|$ that

$$\left| \frac{1}{|x - y|^{1+\beta}} - \frac{1}{|x|^{1+\beta}} \right| \lesssim \frac{1}{|x|^{1+\beta}},$$

from which we obtain that

$$\left| \int_{|y| \geq 2|x|} e^{-|y|} \left(\frac{1}{|x - y|^{1+\beta}} - \frac{1}{|x|^{1+\beta}} \right) dy \right| \lesssim \frac{1}{|x|^{1+\beta}} e^{-2|x|}.$$

Since $e^{-|x|}$ is a bounded positive solution of $-u'' + u = 0$, then there exist $\bar{r}, \bar{s}, \underline{r}, \underline{s} \in \mathbb{R}$ such that $u_{\bar{r},\bar{s}}$ and $u_{\underline{r},\underline{s}}$ are the subsolution and supersolution of this equation, respectively, where

$$u_{r,s}(x) = |x|^{-1-\beta} + r|x|^{-3-\beta} + se^{-|x|}.$$

We conclude from the comparison principle and the above inequalities that $D^\beta K_0(x) \leq |x|^{-1-\beta}$. Finally, we obtain the result by following the ideas of [3]. $\square$

Remark 1. It is interesting to know by using the Bessel function J that

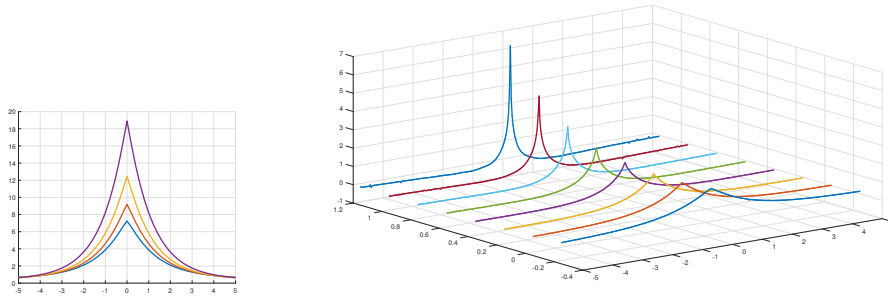$$\max_{\mathbb{R}} K(x) = K(0) = \frac{\pi}{2\sqrt{2}\Gamma(1/2)\sin(\pi(\beta+1)/4)}.$$

FIGURE 1. The graphs of K with negative values of β are shown in the left and in the right figures with both signs.

Define the functional

$$I(u) = \omega\|D^{-\frac{\beta}{2}}u\|^2_{L^2(\mathbb{R})} + \|D^{-\frac{\beta}{2}}u_x\|^2_{L^2(\mathbb{R})}$$

and consider the minimization problem

$$M_\lambda = \inf\{I(u),\ u \in \mathscr{X},\ \|u\|^4_{L^4(\mathbb{R})} = \lambda\} \tag{43}$$

for some $\lambda > 0$. Then if $\varphi \in \mathscr{X}$ achieves the minimum, there there exists a Lagrange multiplier $\theta \in \mathbb{R}$ such that

$$\omega D^{-\beta}\varphi + D^{2(1-\frac{\beta}{2})}\varphi = \theta|\varphi|^2\varphi.$$

An appropriate scaling of φ satisfies (8). By the homogeneity, these solutions also achieve the minimum

$$m = m(\omega,\beta) = \inf_{u \in \mathscr{X}\setminus\{0\}} \frac{I(u)}{\|u\|^2_{L^4(\mathbb{R})}}.$$

It is easy to see that

$$M_\lambda = m\sqrt{\lambda} \tag{44}$$

We show that

$$G(\omega,\beta) = \{\varphi \in \mathscr{X},\ I(\varphi) = m^2 = \|\varphi\|^4_{L^4(\mathbb{R})}\}$$

is not empty. Let $\{\psi_n\}$ be a minimizing sequence. That is, $I(\psi_n) \to M_\lambda$ and $\|\psi_n\|^4_{L^4(\mathbb{R})} \to \lambda$ as $n \to \infty$. To show that $G(\omega,\beta) \neq \emptyset$, we can apply the concentration-compactness principle [17]. We observe first from (44) that the following strict subadditivity condition holds:

$$M_\lambda < M_\rho + M_{\lambda-\rho} \tag{45}$$

for any $\rho \in (0,\lambda)$. Next, we have from Lemma 4.1 and the fact $\|u\|^4_{L^4(\mathbb{R})} = \lambda > 0$ that $\|u\|^2_{\mathscr{X}} \simeq I(u) \geq C > 0$, where $C = C(\lambda,\beta,\omega)$. Thus, for the minimizing sequence $\{\psi_n\}$, the coercivity of I implies that $\{\psi_n\}$ is bounded in $\mathscr{X}$. So, if we define $\mu_n = |D^{-\beta}\psi_n|^2 + |D^{1-\beta}\psi_n|^2$, then we can assume up to a subsequence after normalizing that $\int_\mathbb{R} \mu_n dx = L > 0$. Similar to [17], the coercivity of I, Lemma 4.1 and (45) rule out dichotomy and evanescence cases. Hence, $\{\mu_n\}$ is compact, and

there exists $\{x_n\} \subset \mathbb{R}$ such $\varphi_n(x) = \psi_n(x + x_n)$ has a subsequence φ_n converges weakly to some $\varphi \in \mathscr{X}$. We now have from the weak lower semicontinuity of I over $\mathscr{X}$ that $I(\varphi) \leq \lim_{n \to \infty} I(\varphi_n) = M_\lambda$. In addition, weak convergence in $\mathscr{X}$, compactness of μ_n, and Lemma 4.1 again imply that $\{\varphi_n\}$ converges strongly to φ in $L^4(\mathbb{R})$. Therefore, $\|\varphi\|^4_{L^4(\mathbb{R})} = \lambda$. This means that $I(\varphi) \geq M_\lambda$, and then $\varphi \in G(\omega, \beta)$. Finally, in the case $\beta \leq 0$, since the kernel K is positive, then by the symmetric decreasing rearrangement we can show the existence of an even, strictly positive, decreasing solution of (8). This summarizes in the following theorem.

Theorem 4.4. *Let $\beta \in (-1/2, 3/2)$ and $\omega > 0$. For any minimizing sequence $\{\psi_n\}$ of (43), there exist the sequence $\{x_n\} \subset \mathbb{R}$ and $\varphi \in \mathscr{X}$ such that, $\psi_n(\cdot + x_n)$ converges weakly in $\mathscr{X}$, up to a subsequence, to φ. Moreover, φ attains the minimum $I(\varphi) = M_\lambda$ with $\|\varphi\|^4_{L^4(\mathbb{R})} = \lambda$. Furthermore, if $\beta \leq 0$, there exists an even, strictly positive, decreasing solution of (8).*

Remark 2. It is worth noting that because of the structure of the dispersion of (1) in the case $\beta > 0$, the standing waves of (8) should be sign-changing. We cannot show this claim, but our numerical computations confirm it. See Figures 2.
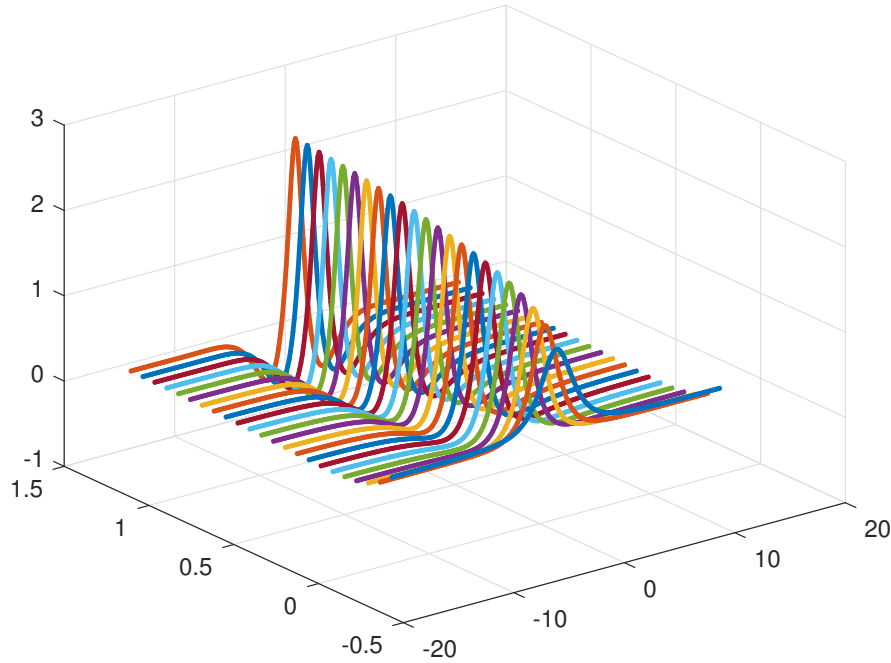

FIGURE 2. Plots of ground states of (1) with $\omega = 1$ and various values of β.

Next, we show that the minima obtained in Theorem 4.4 are precisely the ground states of (8).

Theorem 4.5. *The following assertions are, up to a change of scale, equivalent.*

(i) φ is a ground state,

(ii) φ minimizes I subject to the constraint $\|\varphi\|_{L^4(\mathbb{R})}^4 = \lambda_0 = 4M_1^2$.

Proof. The proof can be obtained by modification of one of Theorem 1.2 in [8], so we omit the details. $\qquad\square$

Next aim is to find best constant for inequality (35). It is standard that this constant related to an equation similar to (8) with the power-law nonlinearity. As in the present paper we have considered (1), we investigate the existence of the best constant of

$$\|u\|_{L^4(\mathbb{R})} \le C_{\text{best}} \left\| D^{-\frac{\beta}{2}} u \right\|_{L^2(\mathbb{R})}^{\frac{3}{4}-\frac{\beta}{2}} \left\| D^{1-\frac{\beta}{2}} u \right\|_{L^2(\mathbb{R})}^{\frac{\beta}{2}+\frac{1}{4}}. \tag{46}$$

However we should remark that our argument is valid for the nonlinearity $|u|^{2q}u$. It can be seen that the best constant C_{best} in (35) is obtained via

$$C_{\text{best}}^{-4} = \inf_{u \in \mathscr{X}\backslash\{0\}} \frac{\left\| D^{-\frac{\beta}{2}} u \right\|_{L^2(\mathbb{R})}^{3-2\beta} \left\| D^{1-\frac{\beta}{2}} u \right\|_{L^2(\mathbb{R})}^{2\beta+1}}{\|u\|_{L^4(\mathbb{R})}^4} \tag{47}$$

Theorem 4.6. *There exist A, ϖ and y in $\mathbb{R}$ such that the best constant C_{best} in (35) is attained at $\tilde{\varphi} \in \mathscr{X}$, where $\tilde{\varphi}(x) = A\varphi(\varpi x + y)$ and φ is a ground state of (8) with $\omega = 1$. Moreover, there holds that*

$$C_{\text{best}}^4 = (4-\theta)^{\frac{2\beta-3}{2}} \theta^{-\frac{1+\beta}{2}} d^{-1}, \tag{48}$$

where $\theta = 1 + 2\beta > 0$ and

$$d = \inf\{S(u),\ u \in \mathscr{X}\backslash\{0\},\ S'(u) = 0\},$$

recalling S from (9).

To prove Theorem 4.6, we show a profile decomposition by mimicking the proof lines of [10, 13]. This also enables us to give another approach to study the existence of solutions of (8).

Theorem 4.7. *Let $\{v_n\}_n$ be a bounded sequence in $\mathscr{X}$. Then there exist a subsequence of $\{v_n\}_n$ (still denoted by the same), a family $\{x_n^j\}_j$ of sequences in $\mathbb{R}$ and a sequence $\{V^j\}_j$ of $\mathscr{X}$-functions such that*

(i) *for every $k \ne j$,*

$$\lim_{n\to\infty} |x_n^k - x_n^j| = +\infty, \tag{49}$$

(ii) *for every $\ell \ge 1$ and every $x \in \mathbb{R}$*

$$v_n(x) = \sum_{j=1}^{\ell} V^j(x - x_n^j) + v_n^\ell(x), \tag{50}$$

with

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \|v_n^\ell\|_{L^p(\mathbb{R})} = 0 \tag{51}$$

for every $p \in (\frac{2}{\beta+1}, q^*)$.

Moreover, we have as $n \to \infty$ that

$$\|v_n\|_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})} = \|v_n^\ell\|^2_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})} + \sum_{j=1}^\ell \|V^j\|^2_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})} + o_n(1) \tag{52}$$

$$\|v_n\|^2_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})} = \|v_n^\ell\|^2_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})} + \sum_{j=1}^\ell \|V^j\|^2_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})} + o_n(1). \tag{53}$$

Proof. Since $\mathscr{X}$ is a Hilbert space, we denote $\mu(v_n)$ is the set of functions obtained as weak limits of subsequences of the translated $v_n(x + x_n)$ with $\{x_n\}_n$ in $\mathscr{X}$. Denote

$$\eta(v_n) = \sup_{V \in \mu(v_n)} \|V\|_{\mathscr{X}}.$$

It is obvious that

$$\eta(v_n) \leq \limsup_{n \to \infty} \|v_n\|_{\mathscr{X}}.$$

Next, we shall prove that there exist a $\{V^j\}_j$ of $\mu(v_n)$ and a family $\{x_n^j\}$ of sequences of $\mathbb{R}$ such that (49) holds, and up to a subsequence, we can write for every $\ell \geq 1$ and every $x \in \mathbb{R}$ that

$$v_n(x) = v_n^\ell(x) + \sum_{j=1}^\ell V^j(x - x_n^j)$$

such that $\lim_{\ell \to \infty} \eta(v_n^\ell) = 0$. Moreover, (52) and (53) hold. In fact, if $\eta(v_n) = 0$, we can take $V^j = 0$ for all j and the proof is finished, otherwise, we choose $V^1 \in \mu(v_n)$ such that $\|V^1\|_{\mathscr{X}} \geq \frac{1}{2}\eta(v_n) > 0$. There exists, from the definition of $\mu(v_n)$, a subsequence $\{x_n^1\}$ of $\mathbb{R}$ such that up to a subsequence, we have that $v_n(\cdot + x_n^1)$ converges weakly to V^1 in $\mathscr{X}$. By setting $v_n^1(x) = v_n(x) - V^1(x - x_n^1)$, we obtain that $v_n^1(x + x_n^1)$ converges weakly to zero in $\mathscr{X}$, and then

$$\|v_n\|_{\mathscr{X}} = \|v_n^1\|^2_{\mathscr{X}} + \|V^1\|^2_{\mathscr{X}} + o_n(1).$$

Now, by replacing $\{v_n\}_n$ by $\{v_n^1\}$ and repeating the same process, we find $V^2 \in \mu(v_n^1)$ and $\{x_n^2\} \subset \mathbb{R}$ such that $\|V^2\|_{\mathscr{X}} \geq \frac{1}{2}\eta(v_n^1) > 0$, $v_n^1(x + x_n^2)$ and $v_n^2(x) = v_n^1(x) - V^2(x - x_n^2)$ converge weakly to V^2 and zero in $\mathscr{X}$, respectively. Moreover,

$$\|v_n^1\|_{\mathscr{X}} = \|v_n^2\|^2_{\mathscr{X}} + \|V^2\|^2_{\mathscr{X}} + o_n(1).$$

Furthermore, $|x_n^1 - x_n^2|$ tends to infinity as $n \to \infty$. In fact, if it is not true, then the facts $v_n^1(\cdot + x_n^1) \rightharpoonup 0$ and

$$v_n^1(x + x_n^2) = v_n^1((x + x_n^2 - x_n^1) + x_n^1)$$

lead us to $V^2 = 0$, and consequently the contradiction $\eta(v_n^1) = 0$. By an iteration procedure with orthogonal extraction, we are able to construct the families $\{x_n^j\}_j \subset \mathbb{R}$ and $\{V^j\}_j \subset \mathscr{X}$ satisfying our claims above. In addition, we have from the convergence of the series $\sum_j \|V^j\|^2_{\mathscr{X}}$ that $\lim_{j \to \infty} \|V^j\|_{\mathscr{X}} = 0$. Hence, $\eta(v_n^j)$ tends to zero as $j \to \infty$. Finally we show (51). Let $R > 1$ and $\nu_R \in \mathcal{S}$ such that $\hat{\nu}_R(\xi) = 1$ if $R^{-1} \leq |\xi| \leq R$ and $\mathrm{supp}(\hat{\nu}_R) = [(2R)^{-1}, 2R]$. Then, we can write

$$v_n^\ell = \nu_R * v_n^\ell + (\delta_0 - \nu_R) * v_n^\ell,$$

where δ_0 is the Dirac function. Thus, we have for any $p \in (\beta_0, q^*)$ that

$$\|(\delta_0 - \nu_R) * v_n^\ell\|_{L^p(\mathbb{R})} \le \|(\delta_0 - \nu_R) * v_n^\ell\|_{\dot{H}^s(\mathbb{R})}$$

$$\le \left(\int_{|\xi| \le R^{-1}} |\xi|^{2s} |\hat{v}_n^\ell(\xi)|^2 \mathrm{d}\xi\right)^{1/2} + \left(\int_{|\xi| \ge R} |\xi|^{2s} |\hat{v}_n^\ell(\xi)|^2 \mathrm{d}\xi\right)^{1/2}$$

$$\lesssim R^{-s-\frac{\beta}{2}} \|v_n^\ell\|_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})} + R^{s-1+\frac{\beta}{2}} \|v_n^\ell\|_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})},$$

where $\beta_0 = \max\{2, \frac{2}{\beta+1}\}$ and $s = \frac{p-2}{2p}$. On the other hand, the Hölder inequality and the Sobolev embedding imply that

$$\|\nu_R * v_n^\ell\|_{L^p(\mathbb{R})} \lesssim \|v_n^\ell\|_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})}^{\frac{\beta_0}{p}} \|\nu_R * v_n^\ell\|_{L^\infty(\mathbb{R})}^{1-\frac{\beta_0}{p}}.$$

Now we observe from the definition of $\mu(n_n^\ell)$ that

$$\limsup_{n \to \infty} \|\nu_R * v_n^\ell\|_{L^\infty(\mathbb{R})} = \sup_{x_n} \limsup_{n \to \infty} |\nu_R * v_n^\ell(x_n)|$$

and

$$\limsup_{n \to \infty} \|\nu_R * v_n^\ell\|_{L^\infty(\mathbb{R})} \le \sup\left\{\left|\int_{\mathbb{R}} \nu_R(-x) v(x) \mathrm{d}x\right|, \ v \in \mu(v_n^\ell)\right\}.$$

But, we have from the Plancherel formula that

$$\left|\int_{\mathbb{R}} \nu_R(-x) v(x) \mathrm{d}x\right| \lesssim \left\||\xi|^{\frac{\beta}{2}} \hat{\nu}_R\right\|_{L^2(\mathbb{R})} \left\||\xi|^{-\frac{\beta}{2}} \hat{v}\right\|_{L^2(\mathbb{R})}$$

$$\lesssim R^{\frac{1}{2}+\frac{\beta}{2}} \|\hat{\nu}_R\|_{\dot{H}^{\frac{\beta}{2}}(\mathbb{R})} \|v\|_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})} \le R^{\frac{1}{2}+\frac{\beta}{2}} \eta(v_n^\ell).$$

Hence we obtain for every $\ell \ge 1$ that

$$\limsup_{n \to \infty} \|v_n^\ell\|_{L^p(\mathbb{R})} \le R^{-s-\frac{\beta}{2}} \|v_n^\ell\|_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})} + R^{s-1+\frac{\beta}{2}} \|v_n^\ell\|_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})}$$

$$+ \|v_n^\ell\|_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})} \left(R^{\frac{1}{2}+\frac{\beta}{2}} \eta(v_n^\ell)\right)^{1-\frac{\beta_0}{p}}.$$

By choosing $R = \left(\eta(v_n^\ell)\right)^{\epsilon-\frac{1}{2}-\frac{\beta}{2}}$ for some $\epsilon > 0$ small enough, we have from the the uniform boundedness $\{v_n^\ell\}_\ell$ in $\mathcal{X}$ and the fact $\lim_{\ell \to \infty} \eta(v_n^\ell) = 0$ that

$$\limsup_{n \to \infty} \|v_n^\ell\|_{L^p(\mathbb{R})} = 0,$$

as $n \to \infty$. And the proof is now complete. $\qquad\square$

Proof of Theorem 4.6. One first can observe easily that the best constant in (47) is invariant under any scaling. Then, given $u \in \mathcal{X} \setminus \{0\}$, we can find $a, b \in \mathbb{R}$ such that $u_{a,b} = au(bx)$ satisfies $\|u_{a,b}\|_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})} = \|u_{a,b}\|_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})} = 1$. More precisely, one has

$$a = \frac{\|u\|_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})}^{\frac{1}{2}+\frac{\beta}{2}}}{\|u\|_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})}^{\frac{3}{2}+\frac{\beta}{2}}} \qquad b = \frac{\|u\|_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})}}{\|u\|_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})}}.$$

Now suppose that $\{v_n\} \subset \mathcal{X}$ is the minimizing sequence of (47). We can also normalize it as above, and then it is uniformly bounded in $\mathcal{X}$. By Theorem 4.7 we can find a sequence $\{V^j\} \subset \mathcal{X}$ and $\{x_n^j\}_j \subset \mathbb{R}$ such that, up to a subsequence,

(49)-(53) hold. Hence, we deduce from (51) and the facts $\sum_{j=1}^{\ell} \|V^j\|^2_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})} \le 1$ and $\sum_{j=1}^{\ell} \|V^j\|^2_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})} \le 1$ that

$$
\begin{aligned}
C^4_{\text{best}} = \lim_{n\to\infty} \|v_n\|^4_{L^4(\mathbb{R})} &\le \limsup_{n\to\infty} \left(\|\sum_{j=1}^{\ell} V^j(\cdot - x_n^j)\|_{L^4(\mathbb{R})} + \|v_n^{\ell}\|_{L^4(\mathbb{R})} \right)^4 \\
&\le \limsup_{n\to\infty} \|\sum_{j=1}^{\ell} V^j(\cdot - x_n^j)\|^4_{L^4(\mathbb{R})}.
\end{aligned}
\tag{54}
$$

The relation (49) shows that $C^4_{\text{best}} \le \sum_{j=1}^{\ell} \|V^j\|^4_{L^4(\mathbb{R})}$. The definition of C_{best} and (35) imply that

$$
1 \le C^{-4}_{\text{best}} \sum_{j=1}^{\ell} \|V^j\|^4_{L^4(\mathbb{R})} \le \sup_j \|V^j\|^{\theta}_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})} \sum_{j=1}^{\ell} \|V^j\|^{4-\theta}_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})}.
$$

Hence, we obtain from the convergence of $\sum_j \|V^j\|^2_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})}$ that there is $j_0 \in \mathbb{N}$ such that $\|V^{j_0}\|_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})} = \sup_j \|V^j\|_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})}$. Therefore, $1 \le \|V^{j_0}\|_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})}$, and consequently V^{j_0} is only nonzero element of the sequence $\{V^j\}$. Thereby, $\|V^{j_0}\|_{\dot{H}^{-\frac{\beta}{2}}(\mathbb{R})} = \|V^{j_0}\|_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})} = 1$ and $C_{\text{best}} = \|V^{j_0}\|_{L^4(\mathbb{R})}$. That is, V^{j_0} is a minimizer of (47) and its Fréchet derivative is zero. This means that V^{j_0} satisfies

$$
\theta C^4_{\text{best}} D^{2(1-\frac{\beta}{2})} V^{j_0} + (4-\theta) D^{-\beta} V^{j_0} - 4(V^{j_0})^3 = 0.
$$

By setting $V^{j_0}(x) = A\varphi(\varpi x + y)$, where $y \in \mathbb{R}$,

$$
A^2 = \frac{C^4}{4}(4-\theta)^{1-\frac{\beta}{2}} \theta^{\frac{\beta}{2}}, \qquad \varpi^2 = \frac{4}{\theta} - 1,
$$

one observe that φ satisfies (8) with $\omega = 1$. Finally, we have from (39) that

$$
C^4_{\text{best}} = 4(4-\theta)^{\frac{2\beta-1}{2}} \theta^{-\frac{1+\beta}{2}} \mathbb{F}^{-1}(\varphi).
$$

Finally, it is obvious that $d \le S(\varphi)$ (see Theorem 4.5). On the other hand, if $S'(u) = 0$ for some $u \in \mathscr{X} \setminus \{0\}$, then u is a nontrivial solution of (8). This means that $S(\varphi) \le S(u)$ and thereby $S(\varphi) \le d$. This fact combined with (39) lead us to (48). $\qquad\square$

As an application of inequality (46), we show the uniform bound of solutions of (1).

We recall the following calculus result that is helpful.

Lemma 4.8 ([1]). *Let $I := [0, T) \subset \mathbb{R}$ be a non-degenerated interval. Let $q > 1$, $a > 0$, $b > 0$, be real constants. Define $\vartheta = (bq)^{-1/(q-1)}$ and $f(r) = a - r + br^q$ for $r \ge 0$. Let $G(t)$ be a continuous nonnegative function on I. If $G(0) < \vartheta$, $a < (1 - 1/q)\vartheta$ and $f \circ G \ge 0$, then $G(t) < \vartheta$, for any $t \in I$.*

Theorem 4.9. *Let $u_0 \in H^s(\mathbb{R})$, $s > 2$, and $u \in C([0, T); H^s(\mathbb{R}))$ be the solution of (1), associated with the initial value u_0. Then $u(t)$ is uniformly bounded in $\mathscr{X}$, for $t \in [0, T)$, if $\varsigma = 1$, or $\varsigma = -1$ and one of the following conditions hold:*

  (i) $1 > 2\beta$;

(ii) $1 = 2\beta$ *and*

$$\mathbb{F}(u_0) < 2^{\frac{\beta}{1+2\beta}}\mathbb{F}(\varphi); \tag{55}$$

(iii) $1 < 2\beta$ *and* u_0 *satisfies*

$$\|u_0\|^{2(\theta-2)}_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})}\mathbb{F}^{4-\theta}(u_0) < C_{1,\theta}\|\varphi\|^{2(\theta-2)}_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})}\mathbb{F}^{4-\theta}(\varphi), \quad \mathbb{E}(u_0) > 0, \tag{56}$$

and

$$\mathbb{E}^{\theta-2}(u_0)\mathbb{F}^{4-\theta}(u_0) < C_{2,\theta}\mathbb{E}^{\theta-2}(\varphi)\mathbb{F}^{4-\theta}(\varphi) \tag{57}$$

where

$$C_{1,\theta} = (4-\theta)^{4-\theta}\theta^{\theta-2}\left(\frac{3-2\beta}{\frac{1}{4}+\frac{\beta}{2}}\right)^{\theta-2}, \quad C_{2,\theta} = (\theta-2)^{\theta-2}(4-\theta)^{4-\theta}\left(\frac{3-2\beta}{2\beta-1}\right)^{\theta-2},$$

φ *is a ground state solution of* (8) *with* $\omega = 1$. *Moreover, we have, in this case, the following bound for the solutions*

$$\|u(t)\|^{2(\theta-2)}_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})}\mathbb{F}^{4-\theta}(u_0) < C_{1,\theta}\|\varphi\|^{2(\theta-2)}_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})}\mathbb{F}^{4-\theta}(\varphi). \tag{58}$$

Proof. We consider only the case $\varsigma = -1$. The case $\varsigma = 1$ will be more simpler.

Let $u \in C([0,T);H^s(\mathbb{R}))$ be the solution of (1) with the initial data $u_0 \in H^s(\mathbb{R}^2)$, $s > 2$. Then by using the invariants $\mathbb{E}$ and $\mathbb{F}$, we have

$$\begin{aligned}
2\mathbb{E}(u_0) &= \|u(t)\|^2_{\dot{H}^{1-\frac{\beta}{2}}} - \frac{1}{2}\int_{\mathbb{R}}u^4(t)\mathrm{d}x \\
&\geq \|u(t)\|^2_{\dot{H}^{1-\frac{\beta}{2}}} - \frac{C^4_{\mathrm{best}}}{2}\mathbb{F}^{\frac{4-\theta}{2}}(u_0)\|u(t)\|^\theta_{\dot{H}^{1-\frac{\beta}{2}}},
\end{aligned} \tag{59}$$

where $\theta = 2\beta + 1$. If $1 > 2\beta$, then (59) immediately implies that $\|u(t)\|_{\dot{H}^{1-\frac{\beta}{2}}}$ (thereby $\|u\|_{\mathscr{X}}$) is uniformly bounded for all $t \in [0,T)$. If $1 = 2\beta$, then we have the uniform bound provided

$$2 > C^4_{\mathrm{best}}\mathbb{F}^{\frac{4-\theta}{2}}(u_0). \tag{60}$$

Using (48) we see that (60) is equivalent to (55).

By (59) and Lemma 4.8, we can define $G(t) = \|u(t)\|^2_{\dot{H}^{1-\frac{\beta}{2}}(\mathbb{R})}$ and $f(r) = a - r + br^{\frac{\theta}{2}}$, where

$$a = 2\mathbb{E}(u_0) \quad \text{and} \quad b = \frac{C^4_{\mathrm{best}}}{2}\mathbb{F}^{\frac{4-\theta}{2}}(u_0).$$

It is deduced from Theorem 1.1 that G is continuous. Furthermore, we have from (59) that $f \circ G \geq 0$. Hence, the proof of theorem will be complete if we show that $G(0) < \vartheta$ and $a < (1-2/\theta)\vartheta$, where $\vartheta^{\theta-2} = \left(\frac{2}{b\theta}\right)^2$. Now it is easy to check by using (35) that $G(0) < \vartheta$ is equivalent to (56). In addition, we obtain from (39) that

$$\mathbb{E}(\varphi) = \frac{2\beta-1}{2(3-2\beta)}\mathbb{F}(\varphi).$$

Therefore, $a < (1-1/q)\vartheta$ is equivalent to (57). Thus, we get from Lemma 4.8 that $G(t) < \vartheta$, and equivalently (58). Hence, it is concluded from $\mathbb{F}(u(t)) = \mathbb{F}(u_0)$ for all $t \in [0,T)$ that $u(t)$ is uniformly bounded in $\mathscr{X}$ for all $[0,T)$. $\qquad\square$

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