# Exact Formulas of the Transition Probabilities of the Multi-Species Asymmetric Simple Exclusion Process 

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#### Abstract

We find the formulas of the transition probabilities of the $N$-particle multispecies asymmetric simple exclusion processes (ASEP), and show that the transition probabilities are written as a determinant when the order of particles in the final state is the same as the order of particles in the initial state.


Key words: ASEP; multi-species ASEP; integrable probability
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## 1 Introduction

The exact formulas of the transition probabilities of the exactly solvable models may be a good starting point to study interesting distributions and their asymptotic behaviours $[2,6,7,10$, $14,15,18]$. In the multi-species versions of these models, particles in the system may belong to different classes and there is a hierarchy of these classes. In this paper, we consider the multi-species version of the asymmetric simple exclusion processes (ASEP). (See [3, 5] for the multi-species version of the totally asymmetric zero range processes.) We consider $N$-particle systems which may consist of up to $N$ species, labelled $1, \ldots, N$. The rules for the multi-species ASEP are as follows: a particle at $x \in \mathbb{Z}$ waits an exponential random time with rate 1 and then chooses $x+1$ with probability $p$ or $x-1$ with probability $q=1-p$ to jump. If the particle at $x$ belongs to species $l$ and the chosen site to jump is already occupied by another particle belonging to species $l^{\prime} \geq l$, then the jump is prohibited, but if $l^{\prime}<l$, the particle belonging to $l$ jumps to the chosen site by interchanging sites with the particle belonging to $l^{\prime}$. According to this rule, we may view an empty site as a "particle" labelled 0 .

A state of the process is denoted by a pair $(X, \pi)$ where $X=\left(x_{1}, \ldots, x_{N}\right) \in \mathbb{Z}^{N}$ with $x_{1}<\cdots<x_{N}$ and $\pi=\pi_{1} \pi_{2} \cdots \pi_{N}$ is a permutation of a multi-set $\mathcal{M}=\left[i_{1}, \ldots, i_{N}\right]$ with elements taken from $\{1, \ldots, N\}$. Each $x_{i}$ represents the position of the $i$ th leftmost particle and $\pi_{i}$ represents the species the $i$ th leftmost particle belongs to.

The purposes of this paper are to provide the explicit formula of the probability $P_{(Y, \nu)}(X, \pi ; t)$ that the system is in $(X, \pi)$ at time $t$, given an initial state $(Y, \nu)$ for each $\mathcal{M}$, and to show that the formula of $P_{(Y, \nu)}(X, \pi ; t)$ is in the form of a determinant in the case that all particles move only in one direction and $\nu=\pi$. If $\mathcal{M}=[i, i, \ldots, i]$, then the permutation of $[i, i, \ldots, i]$ is uniquely $i i \cdots i$, so the system is the ASEP. If $\mathcal{M}=[1,2, \ldots, N]$, then all $N$ particles in the system belong to all different species. The exact formulas of the transition probabilities of the ASEP, written $P_{Y}(X ; t)$, were obtained by Tracy and Widom [15], generalizing Schütz's formulas [14] for the totally asymmetric simple exclusion process (TASEP). Tracy and Widom's
formula for the transition probabilities of the ASEP is an $N$-fold contour integral

$$
\begin{equation*}
P_{Y}(X ; t)=f_{c} \cdots f_{c} \sum_{\sigma \in S_{N}} A_{\sigma} \prod_{i=1}^{N}\left(\xi_{\sigma(i)}^{x_{i}-y_{\sigma(i)}-1} \mathrm{e}^{\varepsilon\left(\xi_{i}\right) t}\right) \mathrm{d} \xi_{1} \cdots \mathrm{~d} \xi_{N} \tag{1.1}
\end{equation*}
$$

where the notation $f_{c}$ implies $\frac{1}{2 \pi \mathrm{i}} \int_{c}$ and the contour $c$ is a circle centered at the origin with sufficiently small radius. The sum inside the integrals in (1.1) is over all permutations $\sigma$ in the symmetric group $S_{N}$ and

$$
\begin{equation*}
A_{\sigma}=\prod_{(\beta, \alpha)} S_{\beta \alpha} \tag{1.2}
\end{equation*}
$$

where

$$
S_{\beta \alpha}=-\frac{p+q \xi_{\alpha} \xi_{\beta}-\xi_{\beta}}{p+q \xi_{\alpha} \xi_{\beta}-\xi_{\alpha}}
$$

and

$$
\varepsilon\left(\xi_{i}\right)=\frac{p}{\xi_{i}}+q \xi_{i}-1
$$

The product in (1.2) is over all inversions $(\beta, \alpha)$ with $\beta>\alpha$ in $\sigma$, and if $\sigma$ is the identity permutation, written Id, then we define $A_{\sigma}=1$. Tracy and Widom have shown that there is a formula analogous to (1.1) for the transition probabilities of the multi-species ASEP, written as

$$
P_{(Y, \nu)}(X, \pi ; t)=f_{c} \cdots f_{c} \sum_{\sigma \in S_{n}} A_{\sigma}^{\nu, \pi} \prod_{i=1}^{N}\left(\xi_{\sigma(i)}^{x_{i}-y_{\sigma(i)}-1} \mathrm{e}^{\varepsilon\left(\xi_{i}\right) t}\right) \mathrm{d} \xi_{1} \cdots \mathrm{~d} \xi_{N}
$$

in our notation [17]. But, the formula of $A_{\sigma}^{\nu, \pi}$ was not given explicitly except a special case in [17, p. 458]. In this paper, we provide a method to find the formula of $A_{\sigma}^{\nu, \pi}$. This is elaborated in Section 2. In Section 3, we find a determinantal expression for transition probabilities.

## 2 Transition probabilities

We first review the basic concepts used in $[1,8,9]$ to extend the results for the TASEP with second class particles there to the multi-species ASEP. The state space of the multi-species ASEP with $N$ particles is countable, so we may view $P_{(Y, \nu)}(X, \pi ; t)$ as a matrix element of an infinite matrix, denoted $\mathbf{P}(t)$, which is a member of a probability semigroup $\{\mathbf{P}(t): t \geq 0\}$. We assume that the rows are labelled by $(X, \pi)$ and the columns are labelled by $(Y, \nu)$, following a rule that if $\pi \prec \pi^{\prime}$ where $\prec$ is the lexicographical order, then the row $\left(X, \pi^{\prime}\right)$ is below the row $(X, \pi)$ for a given $X$, and similarly, the column $\left(Y, \nu^{\prime}\right)$ is to the right of the column $(Y, \nu)$ if $\nu \prec \nu^{\prime}$ for a given $Y$. For fixed $X$ and $Y$, let $\mathbf{P}_{Y}(X ; t)$ be a sub-matrix of $\mathbf{P}(t)$, consisting of the rows labelled $(X, \cdot)$ and the columns labelled $(Y, \cdot)$ of $\mathbf{P}(t)$. Then, $\mathbf{P}_{Y}(X ; t)$ is the $N^{N} \times N^{N}$ matrix in the form of

$$
\mathbf{P}_{Y}(X ; t)=\left[\begin{array}{cccc}
P_{(Y, 1 \cdots 1)}(X, 1 \cdots 11 ; t) & P_{(Y, 1 \cdots 12)}(X, 1 \cdots 11 ; t) & \cdots & P_{(Y, N \cdots N)}(X, 1 \cdots 11 ; t) \\
P_{(Y, 1 \cdots 1)}(X, 1 \cdots 12 ; t) & P_{(Y, 1 \cdots 12)}(X, 1 \cdots 12 ; t) & \cdots & P_{(Y, N \cdots N)}(X, 1 \cdots 12 ; t) \\
\vdots & \vdots & \ddots & \vdots \\
P_{(Y, 1 \cdots 1)}(X, N \cdots N ; t) & P_{(Y, \cdots 12)}(X, N \cdots N ; t) & & P_{(Y, N \cdots N)}(X, N \cdots N ; t)
\end{array}\right] .
$$

The labels of the rows and the columns of this matrix are the permutations of the multi-sets of cardinality $N$ with elements taken from $\{1, \ldots, N\}$ in the lexicographical order from the top to the bottom and from the left to the right, respectively.

The semigroup $\{\mathbf{P}(t): t \geq 0\}$ is uniform, and $\mathbf{P}(t)$ is the unique solution to the forward equation which is an element-wise matrix differential equation

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \mathbf{P}(t)=\mathbf{G P}(t)
$$

subject to the initial condition $\mathbf{P}(0)=\mathbf{I}_{\infty}$ where $\mathbf{I}_{\infty}$ is the infinite identity matrix and $\mathbf{G}$ is the generator. (We use the notation $\mathbf{I}_{n}$ for $n \times n$ identity matrix and $\mathbf{0}_{n}$ for $n \times n$ zero matrix.) The initial condition $\mathbf{P}(0)=\mathbf{I}_{\infty}$ implies that the sub-matrices $\mathbf{P}_{Y}(X ; t)$ satisfy

$$
\mathbf{P}_{Y}(X ; 0)= \begin{cases}\mathbf{I}_{N^{N}} & \text { if } X=Y \\ \mathbf{0}_{N^{N}} & \text { otherwise }\end{cases}
$$

By the simple extension of Section 2.1 in [9] for the TASEP with second class particles to the ASEP with second class particles, we can obtain a matrix which generalizes (2.11) in [9],

$$
\widetilde{\mathbf{R}}_{\beta \alpha}=\left[\begin{array}{cccc}
S_{\beta \alpha} & 0 & 0 & 0 \\
0 & P_{\beta \alpha} & p T_{\beta \alpha} & 0 \\
0 & q T_{\beta \alpha} & Q_{\beta \alpha} & 0 \\
0 & 0 & 0 & S_{\beta \alpha}
\end{array}\right]
$$

where

$$
\begin{aligned}
S_{\beta \alpha} & =-\frac{p+q \xi_{\alpha} \xi_{\beta}-\xi_{\beta}}{p+q \xi_{\alpha} \xi_{\beta}-\xi_{\alpha}}, & P_{\beta \alpha} & =\frac{\left(p-q \xi_{\alpha}\right)\left(\xi_{\beta}-1\right)}{p+q \xi_{\alpha} \xi_{\beta}-\xi_{\alpha}} \\
T_{\beta \alpha} & =\frac{\xi_{\beta}-\xi_{\alpha}}{p+q \xi_{\alpha} \xi_{\beta}-\xi_{\alpha}}, & Q_{\beta \alpha} & =\frac{\left(p-q \xi_{\beta}\right)\left(\xi_{\alpha}-1\right)}{p+q \xi_{\alpha} \xi_{\beta}-\xi_{\alpha}}
\end{aligned}
$$

Notice that the denominators of the nonzero elements in $\widetilde{\mathbf{R}}_{\beta \alpha}$ have the same form as the denominator of $S_{\beta \alpha}$ in $[15,16]$. This fact makes it easy to prove the initial condition later. The matrix $\widetilde{\mathbf{R}}_{\beta \alpha}$ is a building block for $N$-particle system. In $N$-particle ASEP with multi-species, a matrix corresponding to $\mathbf{B}$ in (2.6) of [9] which is for a two-particle TASEP is an $N^{2} \times N^{2}$ matrix $\mathbf{B}$ with

$$
[\mathbf{B}]_{i j, k l}= \begin{cases}1 & \text { if } i j=k l \text { with } i=j, \\ p & \text { if either } i j=k l \text { or } i j=l k \text { with } i<j, \\ q & \text { if either } i j=k l \text { or } i j=l k \text { with } i>j, \\ 0 & \text { for all other cases },\end{cases}
$$

where $i j$ and $k l$ are labels for rows and columns, represented by $11,12, \ldots, N N$. In the same way as $\widetilde{\mathbf{R}}_{\beta \alpha}$ is obtained, we define an $N^{2} \times N^{2}$ matrix $\mathbf{R}_{\beta \alpha}$ by

$$
\mathbf{R}_{\beta \alpha}=-\left[\left(p+q \xi_{\alpha} \xi_{\beta}\right) \mathbf{I}_{N^{2}}-\xi_{\alpha} \mathbf{B}\right]^{-1}\left[\left(p+q \xi_{\alpha} \xi_{\beta}\right) \mathbf{I}_{N^{2}}-\xi_{\beta} \mathbf{B}\right]
$$

where $\alpha, \beta=1, \ldots, N$ and $\alpha \neq \beta$. Here, $\mathbf{I}_{N^{2}}$ is the $N^{2} \times N^{2}$ identity matrix. The entries of $\mathbf{R}_{\beta \alpha}$ are given by

$$
\left[\mathbf{R}_{\beta \alpha}\right]_{i j, k l}= \begin{cases}S_{\beta \alpha} & \text { if } i j=k l \text { with } i=j, \\ P_{\beta \alpha} & \text { if } i j=k l \text { with } i<j, \\ Q_{\beta \alpha} & \text { if } i j=k l \text { with } i>j, \\ p T_{\beta \alpha} & \text { if } i j=l k \text { with } i<j, \\ q T_{\beta \alpha} & \text { if } i j=l k \text { with } i>j, \\ 0 & \text { for all other cases. }\end{cases}
$$

If $\mathbf{U}(X ; t)=\mathbf{U}\left(x_{1}, \ldots, x_{N} ; t\right)$ be an $N^{N} \times N^{N}$ matrix whose entries are functions on $\mathbb{Z}^{N} \times[0, \infty)$ which satisfies the differential equation

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathbf{U}(X ; t)=\sum_{i=1}^{N} & {\left[p \mathbf{U}\left(x_{1}, \ldots, x_{i-1}, x_{i}-1, x_{i+1}, \ldots, x_{N} ; t\right)\right.} \\
& \left.+q \mathbf{U}\left(x_{1}, \ldots, x_{i-1}, x_{i}+1, x_{i+1}, \ldots, x_{N} ; t\right)\right]-N \mathbf{U}\left(x_{1}, \ldots, x_{N} ; t\right) \tag{2.1}
\end{align*}
$$

and the initial condition

$$
\mathbf{U}\left(x_{1}, \ldots, x_{N} ; 0\right)= \begin{cases}\mathbf{I}_{N^{N}} & \text { if }\left(x_{1}, \ldots, x_{N}\right)=\left(y_{1}, \ldots, y_{N}\right) \text { and } x_{1}<\cdots<x_{N}  \tag{2.2}\\ \mathbf{0}_{N^{N}} & \text { if }\left(x_{1}, \ldots, x_{N}\right) \neq\left(y_{1}, \ldots, y_{N}\right) \text { and } x_{1}<\cdots<x_{N}\end{cases}
$$

for a given initial positions of particles $\left(y_{1}, \ldots, y_{N}\right)$ with $y_{1}<\cdots<y_{N}$, and the boundary condition

$$
\begin{align*}
& p \mathbf{U}\left(x_{1}, \ldots, x_{i-1}, x_{i}, x_{i}, x_{i+2}, \ldots, x_{N} ; t\right)+q \mathbf{U}\left(x_{1}, \ldots, x_{i-1}, x_{i}+1, x_{i}+1, x_{i+2}, \ldots, x_{N} ; t\right) \\
& \quad=\left(\mathbf{I}_{N}^{\otimes(i-1)} \otimes \mathbf{B} \otimes \mathbf{I}_{N}^{\otimes(N-i-1)}\right) \mathbf{U}\left(x_{1}, \ldots, x_{i-1}, x_{i}, x_{i}+1, x_{i+2}, \ldots, x_{N} ; t\right) \tag{2.3}
\end{align*}
$$

for all $i=1, \ldots, N-1$, then we may assert that the restriction of $\mathbf{U}\left(x_{1}, \ldots, x_{N} ; t\right)$ on $\left\{\left(x_{1}, \ldots, x_{N}\right)\right.$ $\left.\in \mathbb{Z}^{N}: x_{1}<\cdots<x_{N}\right\}$ is $\mathbf{P}_{Y}(X ; t)$. The solution of (2.1) by the Bethe ansatz is given

$$
\begin{equation*}
\sum_{\sigma \in S_{N}} \mathbf{A}_{\sigma} \prod_{i=1}^{N}\left(\xi_{\sigma(i)}^{x_{i}-y_{\sigma(i)}-1} \mathrm{e}^{\varepsilon\left(\xi_{i}\right) t}\right) \tag{2.4}
\end{equation*}
$$

where

$$
\varepsilon\left(\xi_{i}\right)=\frac{p}{\xi_{i}}+q \xi_{i}-1
$$

for $\xi_{i} \in \mathbb{C} \backslash\{0\}$, where $\mathbf{A}_{\sigma}$ is an $N^{N} \times N^{N}$ matrix whose entries are independent of $x_{1}, \ldots, x_{N}$ and $t$. Next, we will construct the matrix $\mathbf{A}_{\sigma}$ so that (2.4) satisfies the boundary condition (2.3).

Step 1. Let $T_{i}$ be a simple transposition which interchanges the $i$ th entry and the $(i+1) s t$ entry and leaves the other entries fixed, that is,

$$
T_{i} \sigma=\sigma^{\prime}
$$

where $\sigma=\sigma(1) \cdots \sigma(i) \sigma(i+1) \cdots \sigma(N)$ and $\sigma^{\prime}=\sigma(1) \cdots \sigma(i-1) \sigma(i+1) \sigma(i) \sigma(i+2) \cdots \sigma(N)$. Since $T_{1}, \ldots, T_{N-1}$ generate $S_{N}$, there is a finite sequence $a_{1}, \ldots, a_{n}$ where each $a_{i}$ belongs to $\{1,2, \ldots, N-1\}$ such that

$$
\sigma=T_{a_{n}} \cdots T_{a_{1}}
$$

for any given $\sigma \in S_{N}$.
Step 2. For each $1 \leq l \leq N-1$, we define

$$
\begin{equation*}
\mathbf{T}_{l}(\beta, \alpha):=\mathbf{I}_{N}^{\otimes(l-1)} \otimes \mathbf{R}_{\beta \alpha} \otimes \mathbf{I}_{N}^{\otimes(N-l-1)} \tag{2.5}
\end{equation*}
$$

Step 3. Let us denote $T_{a_{k}} \cdots T_{a_{1}}$ by $\sigma^{(k)}$ and let $\sigma^{(0)}$ be the identity permutation. Define

$$
\begin{equation*}
\mathbf{A}_{\sigma}:=\mathbf{T}_{a_{n}}\left(\sigma^{(n-1)}\left(a_{n}+1\right), \sigma^{(n-1)}\left(a_{n}\right)\right) \cdots \mathbf{T}_{a_{1}}\left(\sigma^{(0)}\left(a_{1}+1\right), \sigma^{(0)}\left(a_{1}\right)\right) \tag{2.6}
\end{equation*}
$$

Lemma 2.1. If $\mathbf{A}_{\sigma}$ is given as in (2.6), then (2.4) satisfies (2.3) for each $i$.

Proof. Substituting (2.4) into (2.3) for $i$,

$$
\begin{align*}
& \sum_{\sigma \in S_{N}}\left(p \mathbf{A}_{\sigma} \xi_{\sigma(1)}^{x_{1}} \cdots \xi_{\sigma(i-1)}^{x_{i-1}} \xi_{\sigma(i)}^{x_{i}} \xi_{\sigma(i+1)}^{x_{i}} \xi_{\sigma(i+2)}^{x_{i+2}} \cdots \xi_{\sigma(N)}^{x_{N}}\right. \\
& \quad+q \mathbf{A}_{\sigma} \xi_{\sigma(1)}^{x_{1}} \cdots \xi_{\sigma(i-1)}^{x_{i-1}} \xi_{\sigma(i)}^{x_{i}+1} \xi_{\sigma(i+1)}^{x_{i}+1} \xi_{\sigma(i+2)}^{x_{i+2}} \cdots \xi_{\sigma(N)}^{x_{N}}  \tag{2.7}\\
& \left.\quad-\left(\mathbf{I}_{N}^{\otimes(i-1)} \otimes \mathbf{B} \otimes \mathbf{I}_{N}^{\otimes(N-i-1)}\right) \mathbf{A}_{\sigma} \xi_{\sigma(1)}^{x_{1}} \cdots \xi_{\sigma(i-1)}^{x_{i-1}} \xi_{\sigma(i)}^{x_{i}} \xi_{\sigma(i+1)}^{x_{i}+1} \xi_{\sigma(i+2)}^{x_{i+2}} \cdots \xi_{\sigma(N)}^{x_{N}}\right)=\mathbf{0}_{N^{N}}
\end{align*}
$$

If $\sigma^{\prime}$ is an even permutation, then $T_{i} \sigma^{\prime}$ is an odd permutation, and vice-versa. So, if we express (2.7) as a sum over the alternating group $A_{N}$,

$$
\begin{align*}
& \sum_{\sigma^{\prime} \in A_{N}}\left(\left[\mathbf{I}_{N^{N}}\left(p+q \xi_{\sigma^{\prime}(i)} \xi_{\sigma^{\prime}(i+1)}\right)-\left(\mathbf{I}_{N}^{\otimes(i-1)} \otimes \mathbf{B} \otimes \mathbf{I}_{N}^{\otimes(N-i-1)}\right) \xi_{\sigma^{\prime}(i+1)}\right] \mathbf{A}_{\sigma^{\prime}}\right. \\
& \left.\quad+\left[\mathbf{I}_{N^{N}}\left(p+q \xi_{\sigma^{\prime}(i+1)} \xi_{\sigma^{\prime}(i)}\right)-\left(\mathbf{I}_{N}^{\otimes(i-1)} \otimes \mathbf{B} \otimes \mathbf{I}_{N}^{\otimes(N-i-1)}\right) \xi_{\sigma^{\prime}(i)}\right] \mathbf{A}_{T_{i} \sigma^{\prime}}\right) \\
& \quad \times \xi_{\sigma^{\prime}(1)}^{x_{1}} \cdots \xi_{\sigma^{\prime}(i-1)}^{x_{i-1}} \xi_{\sigma^{\prime}(i)}^{x_{i}} \xi_{\sigma^{\prime}(i+1)}^{x_{i}} \xi_{\sigma^{\prime}(i+2)}^{x_{i+2}} \cdots \xi_{\sigma^{\prime}(N)}^{x_{N}}=\mathbf{0}_{N^{N}} . \tag{2.8}
\end{align*}
$$

A sufficient condition for (2.8) is that for each $\sigma \in S_{N}$,

$$
\begin{align*}
\mathbf{A}_{T_{i} \sigma}= & -\left[\mathbf{I}_{N^{N}}\left(p+q \xi_{\sigma(i+1)} \xi_{\sigma(i)}\right)-\left(\mathbf{I}_{N}^{\otimes(i-1)} \otimes \mathbf{B} \otimes \mathbf{I}_{N}^{\otimes(N-i-1)}\right) \xi_{\sigma(i)}\right]^{-1} \\
& \times\left[\mathbf{I}_{N^{N}}\left(p+q \xi_{\sigma(i)} \xi_{\sigma(i+1)}\right)-\left(\mathbf{I}_{N}^{\otimes(i-1)} \otimes \mathbf{B} \otimes \mathbf{I}_{N}^{\otimes(N-i-1)}\right) \xi_{\sigma(i+1)}\right] \mathbf{A}_{\sigma} \\
= & -\left(\mathbf{I}_{N^{(i-1)}} \otimes\left[\mathbf{I}_{N^{2}}\left(p+q \xi_{\sigma(i+1)} \xi_{\sigma(i)}\right)-\mathbf{B} \xi_{\sigma(i)}\right]^{-1} \otimes \mathbf{I}_{N^{(N-i-1)}}\right) \\
& \times\left(\mathbf{I}_{N^{(i-1)}} \otimes\left[\mathbf{I}_{N^{2}}\left(p+q \xi_{\sigma(i+1)} \xi_{\sigma(i)}\right)-\mathbf{B} \xi_{\sigma(i+1)}\right] \otimes \mathbf{I}_{N^{(N-i-1)}}\right) \mathbf{A}_{\sigma} \\
= & \left(\mathbf{I}_{N^{(i-1)}} \otimes \mathbf{R}_{\sigma(i+1) \sigma(i)} \otimes \mathbf{I}_{N^{(N-i-1)}}\right) \mathbf{A}_{\sigma} . \tag{2.9}
\end{align*}
$$

Now, it remains to show that if $\mathbf{A}_{\sigma}$ is given by (2.6), then (2.9) is satisfied. This is immediately obtained because

$$
\begin{aligned}
\mathbf{A}_{T_{i} \sigma} & =\mathbf{T}_{i} \mathbf{T}_{a_{n}} \cdots \mathbf{T}_{a_{1}}=\mathbf{T}_{i}(\sigma(i+1), \sigma(i)) \mathbf{A}_{\sigma} \\
& =\left(\mathbf{I}_{N^{(i-1)}} \otimes \mathbf{R}_{\sigma(i+1) \sigma(i)} \otimes \mathbf{I}_{N^{(N-i-1)}}\right) \mathbf{A}_{\sigma}
\end{aligned}
$$

by (2.5).
Remark 2.2. The consistency relations

$$
\begin{align*}
& \mathbf{T}_{i}(\beta, \alpha) \mathbf{T}_{j}(\delta, \gamma)=\mathbf{T}_{j}(\delta, \gamma) \mathbf{T}_{i}(\beta, \alpha) \quad \text { if }|i-j| \geq 2, \\
& \mathbf{T}_{i}(\gamma, \beta) \mathbf{T}_{j}(\gamma, \alpha) \mathbf{T}_{i}(\beta, \alpha)=\mathbf{T}_{j}(\beta, \alpha) \mathbf{T}_{i}(\gamma, \alpha) \mathbf{T}_{j}(\gamma, \beta) \quad \text { if }|i-j|=1, \\
& \mathbf{T}_{i}(\beta, \alpha) \mathbf{T}_{i}(\alpha, \beta)=\mathbf{I}_{N^{N}} \tag{2.10}
\end{align*}
$$

can be directly verified by the matrix multiplication. In particular, (2.10) with $p=1$ was verified in [1].

Now, we choose $\mathbf{A}_{1 \cdots N}=\mathbf{I}_{N^{N}}$ and take a multiple-fold contour integral of each element of the matrix (2.6) with respect to $\xi_{1}, \ldots, \xi_{N}$ over sufficiently small circles centered at the origin, denoting the matrix by

$$
\begin{equation*}
f_{c} \cdots f_{c} \sum_{\sigma \in S_{N}} \mathbf{A}_{\sigma} \prod_{i=1}^{N}\left(\xi_{\sigma(i)}^{x_{i}-y_{\sigma(i)}-1} \mathrm{e}^{\varepsilon\left(\xi_{i}\right) t}\right) \mathrm{d} \xi_{1} \cdots \mathrm{~d} \xi_{N} \tag{2.11}
\end{equation*}
$$

Finally, it should be proved that (2.11) satisfies the initial condition (2.2), but its proof is essentially the same as the proof for the ASEP in [16, 17] because the poles arising in the integrands are the same as the poles in the ASEP formulas. So, we omit the proof and conclude that

$$
\begin{equation*}
\mathbf{P}_{Y}(X ; t)=f_{c} \cdots f_{c} \sum_{\sigma \in S_{N}} \mathbf{A}_{\sigma} \prod_{i=1}^{N}\left(\xi_{\sigma(i)}^{x_{i}-y_{\sigma(i)}-1} \mathrm{e}^{\varepsilon\left(\xi_{i}\right) t}\right) \mathrm{d} \xi_{1} \cdots \mathrm{~d} \xi_{N} . \tag{2.12}
\end{equation*}
$$

## 3 Determinantal formulas

In this section we focus on the diagonal of the matrix (2.12) for the totally asymmetric case $p=1$, that is, the transition probabilities $P_{(Y, \pi)}(X, \pi ; t)$. In the case of the totally asymmetric model ( $p=1$ ), if the order of particles at $t=0$ is represented by a permutation $\pi$ of a given multi-set and the order of particles at time $t=t_{0}>0$ is also given by $\pi$, then the order of particles at each time $0 \leq t \leq t_{0}$ must be $\pi$. If all particles belong to the same species, then $P_{(Y, i \cdots i)}(X, i \cdots i ; t)$ is simply the TASEP's transition probability. Schütz found its determinantal formula [14]. For the TASEP with second class particles (two-species TASEP), Chatterjee and Schütz found the determinantal formulas for the transition probabilities when there is no change in order of particles [1]. We will extend Chatterjee and Schütz's result to the multi-species TASEP. Let

$$
F_{n}(x ; t)=f_{c} \xi^{x-1}(1-\xi)^{-n} \mathrm{e}^{(1 / \xi-1) t} \mathrm{~d} \xi
$$

For a given permutation $\pi=\pi_{1} \cdots \pi_{N}$ of a multi-set, let $\#_{i j}$ be the number of pairs of places $(i, i+1)$ between the $i$ th place and $j$ th place such that $\pi_{i}>\pi_{i+1}$. For example, if $\pi=214311$, then $\#_{12}=1, \#_{14}=2$, and $\#_{15}=\#_{16}=3$. Obviously, $\#_{i j}=\#_{j i}$. Let

$$
n(i, j)=\operatorname{sgn}(i-j) \times\left(|i-j|-\#_{i j}\right) .
$$

If we denote the diagonal elements of $\mathbf{U}\left(x_{1}, \ldots, x_{N} ; t\right)$ by $U_{\pi, \pi}\left(x_{1}, \ldots, x_{N} ; t\right)$ where $\pi=\pi_{1} \cdots \pi_{N}$, then the equations for the diagonal of (2.1), (2.2) and (2.3) are

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t} U_{\pi, \pi}\left(x_{1}, \ldots, x_{N} ; t\right)= & \sum_{i=1}^{N} U_{\pi, \pi}\left(x_{1}, \ldots, x_{i-1}, x_{i}-1, x_{i+1}, \ldots, x_{N} ; t\right) \\
& -N U_{\pi, \pi}\left(x_{1}, \ldots, x_{N} ; t\right)  \tag{3.1}\\
U_{\pi, \pi}\left(x_{1}, \ldots, x_{N} ; 0\right)= & \prod_{i=1}^{N} \delta_{x_{i} y_{i}} \quad \text { when } \quad x_{1}<\cdots<x_{N} \text { for given } y_{1}<\cdots<y_{N} \tag{3.2}
\end{align*}
$$

and

$$
\begin{align*}
& U_{\pi, \pi}\left(x_{1}, \ldots, x_{i}, x_{i}, x_{i+2}, \ldots, x_{N} ; t\right) \\
& \quad= \begin{cases}U_{\pi, \pi}\left(x_{1}, \ldots, x_{i}, x_{i}+1, x_{i+2}, \ldots, x_{N} ; t\right), & \text { if } \pi_{i} \leq \pi_{i+1}, \\
0 & \text { if } \pi_{i}>\pi_{i+1}\end{cases} \tag{3.3}
\end{align*}
$$

The following theorem gives the solution to (3.1), (3.2), and (3.3).
Theorem 3.1. For given $Y=\left(y_{1}, \ldots, y_{N}\right)$ with $y_{1}<\cdots<y_{N}$ and $\pi$, define

$$
G\left(x_{1}, \ldots, x_{N} ; t\right)=\left[\begin{array}{cccc}
F_{n(1,1)}\left(x_{1}-y_{1} ; t\right) & F_{n(1,2)}\left(x_{1}-y_{2} ; t\right) & \cdots & F_{n(1, N)}\left(x_{1}-y_{N} ; t\right) \\
F_{n(2,1)}\left(x_{2}-y_{1} ; t\right) & F_{n(2,2)}\left(x_{2}-y_{2} ; t\right) & \cdots & F_{n(2, N)}\left(x_{2}-y_{N} ; t\right) \\
\vdots & \vdots & \ddots & \vdots \\
F_{n(N, 1)}\left(x_{N}-y_{1} ; t\right) & F_{n(N, 2)}\left(x_{N}-y_{2} ; t\right) & \cdots & F_{(N, N)}\left(x_{N}-y_{N} ; t\right)
\end{array}\right] .
$$

Then,

$$
P_{(Y, \pi)}\left(x_{1}, \ldots, x_{N}, \pi ; t\right)=\operatorname{det} G\left(x_{1}, \ldots, x_{N} ; t\right) .
$$

Proof. (i) To show that $\operatorname{det} G\left(x_{1}, \ldots, x_{N} ; t\right)$ satisfies (3.1). Since

$$
\frac{\mathrm{d}}{\mathrm{~d} t} F_{n(i, j)}\left(x_{i}-y_{j} ; t\right)=-F_{n(i, j)}\left(x_{i}-y_{j} ; t\right)+F_{n(i, j)}\left(x_{i}-1-y_{j} ; t\right),
$$

we have

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} t} \operatorname{det} G\left(x_{1}, \ldots, x_{N} ; t\right) \\
&=\left|\begin{array}{cccc}
\frac{\mathrm{d}}{\mathrm{~d} t} F_{n(1,1)}\left(x_{1}-y_{1} ; t\right) & \frac{\mathrm{d}}{\mathrm{~d} t} F_{n(1,2)}\left(x_{1}-y_{2} ; t\right) & \cdots & \frac{\mathrm{d}}{\mathrm{~d} t} F_{n(1, N)}\left(x_{1}-y_{N} ; t\right) \\
F_{n(2,1)}\left(x_{2}-y_{1} ; t\right) & F_{n(2,2)}\left(x_{2}-y_{2} ; t\right) & \cdots & F_{n(2, N)}\left(x_{2}-y_{N} ; t\right) \\
\vdots & \vdots & \ddots & \vdots \\
F_{n(N, 1)}\left(x_{N}-y_{1} ; t\right) & F_{n(N, 2)}\left(x_{N}-y_{2} ; t\right) & \cdots & F_{(N, N)}\left(x_{N}-y_{N} ; t\right)
\end{array}\right|+\cdots \\
&+\left|\begin{array}{cccc}
F_{n(1,1)}\left(x_{1}-y_{1} ; t\right) & F_{n(1,2)}\left(x_{1}-y_{2} ; t\right) & \cdots & F_{n(1, N)}\left(x_{1}-y_{N} ; t\right) \\
F_{n(2,1)}\left(x_{2}-y_{1} ; t\right) & F_{n(2,2)}\left(x_{2}-y_{2} ; t\right) & \cdots & F_{n(2, N)}\left(x_{2}-y_{N} ; t\right) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\mathrm{d}}{\mathrm{~d} t} F_{n(N, 1)}\left(x_{N}-y_{1} ; t\right) & \frac{\mathrm{d}}{\mathrm{~d} t} F_{n(N, 2)}\left(x_{N}-y_{2} ; t\right) & \cdots & \frac{\mathrm{d}}{\mathrm{~d} t} F_{(N, N)}\left(x_{N}-y_{N} ; t\right)
\end{array}\right| \\
&=-N \operatorname{det} G\left(x_{1}, \ldots, x_{N} ; t\right)+\sum_{i=1}^{N} \operatorname{det} G\left(x_{1}, \ldots, x_{i-1}, x_{i}-1, x_{i+1}, \ldots, x_{N} ; t\right) .
\end{aligned}
$$

(ii) To show that $\operatorname{det} G\left(x_{1}, \ldots, x_{N} ; t\right)$ satisfies (3.3): Suppose that $\pi_{i}>\pi_{i+1}$. First, we will show that $n(i, j)=n(i+1, j)$. If $j>i+1$, then

$$
n(i, j)=(-1) \times\left(j-i-\#_{i j}\right)=(-1) \times[j-(i+1)-(\underbrace{\#_{i j}-1}_{\#_{(i+1) j}})]=n(i+1, j)
$$

If $j<i-1$, then

$$
n(i, j)=1 \times\left(i-j-\#_{i j}\right)=1 \times[i+1-j-(\underbrace{\#_{i j}+1}_{\#_{(i+1) j}})]=n(i+1, j) .
$$

If $i=j$ or $j=i \pm 1$, then

$$
n(i, j)=0=n(i+1, j) .
$$

Hence, the $i$ th row and the $(i+1)$ th row of $G\left(x_{1}, \ldots, x_{i}, x_{i}, x_{i+2}, \ldots, x_{N} ; t\right)$ are the same and

$$
\operatorname{det} G\left(x_{1}, \ldots, x_{i}, x_{i}, x_{i+2}, \ldots, x_{N} ; t\right)=0
$$

Next, suppose that $\pi_{i} \leq \pi_{i+1}$, so $\#_{i j}=\#_{(i+1) j}$. We will show that $n(i, j)=n(i+1, j)-1$. If $j>i+1$, then

$$
n(i+1, j)=(-1) \times\left(j-i-1-\#_{(i+1) j}\right)=(-1) \times\left(j-i-\#_{i j}\right)+1=n(i, j)+1 .
$$

If $j<i-1$, then

$$
n(i+1, j)=1 \times\left(i+1-j-\#_{(i+1) j}\right)=1 \times\left(i-j-\#_{i j}\right)+1=n(i, j)+1
$$

If $i=j$, then $n(i+1, j)=1=n(i, i)+1$. If $j=i+1$, then $n(i+1, j)=0=n(i, j)+1$. If $j=i-1$, then

$$
\begin{align*}
n(i+1, j) & =(+1) \times\left(2-\#_{(i-1)(i+1)}\right)=(+1) \times\left(2-\#_{(i-1) i}\right) \\
& =(+1) \times\left(1-\#_{(i-1) i}\right)+1=n(i, i-1)+1=n(i, j)+1 . \tag{3.4}
\end{align*}
$$

Also, using the identity

$$
F_{n}(x+1 ; t)=F_{n}(x ; t)-F_{n-1}(x ; t)
$$

we obtain

$$
\begin{array}{|ccc|}
\vdots & \vdots & \vdots  \tag{3.5}\\
F_{n(i, 1)}\left(x_{i}-y_{1} ; t\right) & \cdots & F_{n(i, N)}\left(x_{i}-y_{N} ; t\right) \\
F_{n(i+1,1)}\left(x_{i}+1-y_{1} ; t\right) & \cdots & F_{n(i+1, N)}\left(x_{i}+1-y_{N} ; t\right) \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
=\left\lvert\, \begin{array}{ccc}
F_{n(i, 1)}\left(x_{i}-y_{1} ; t\right) & \cdots & F_{n(i, N)}\left(x_{i}-y_{N} ; t\right) \\
F_{n(i+1,1)}\left(x_{i}-y_{1} ; t\right) & \cdots & F_{n(i+1, N)}\left(x_{i}-y_{N} ; t\right) \\
\vdots & \vdots & \vdots \\
\vdots & & \vdots
\end{array}\right. \\
-\left|\begin{array}{ccc}
F_{n(i, 1)}\left(x_{i}-y_{1} ; t\right) & \cdots & F_{n(i, N)}\left(x_{i}-y_{N} ; t\right) \\
F_{n(i+1,1)-1}\left(x_{i}-y_{1} ; t\right) & \cdots & F_{n(i+1, N)-1}\left(x_{i}-y_{N} ; t\right) \\
\vdots & \vdots & \vdots \\
& &
\end{array}\right|
\end{array}
$$

where the second term on the right-hand side of (3.5) is zero by (3.4). Hence,

$$
\operatorname{det} G\left(x_{1}, \ldots, x_{i}, x_{i}+1, x_{i+2}, \ldots, x_{N} ; t\right)=\operatorname{det} G\left(x_{1}, \ldots, x_{i}, x_{i}, x_{i+2}, \ldots, x_{N} ; t\right)
$$

(iii) To show that $\operatorname{det} G\left(x_{1}, \ldots, x_{N}\right)$ satisfies (3.2): Since $F_{n(i, i)}\left(x_{i}-y_{i} ; 0\right)=\delta_{x_{i} y_{i}}$ and $F_{n(i, j)}\left(x_{i}-y_{j} ; 0\right)=0$ for $i>j$ because $x_{i}-y_{j} \geq 1$ when $x_{1}<\cdots<x_{N}$ for given $y_{1}<\cdots<y_{N}$,

$$
\operatorname{det} G\left(x_{1}, \ldots, x_{N} ; 0\right)=\prod_{i=1}^{N} \delta_{x_{i} y_{i}}
$$

when $x_{1}<\cdots<x_{N}$ for given $y_{1}<\cdots<y_{N}$.
When $\pi=\left(\begin{array}{llll}1 & 1 & \cdots & 1\end{array}\right)$, that is, for the TASEP with single species, $F_{n(i, j)}\left(x_{i}-y_{j} ; t\right)=$ $F_{i-j}\left(x_{i}-y_{j} ; t\right)$ has some interesting properties, for example, [11, equation (4)], which are needed to find one-point distributions and multi-point distributions [11, 12, 13]. At this moment, it is not clear if $F_{n(i, j)}\left(x_{i}-y_{j} ; t\right)$ has similar properties for general $\pi$. It would be interesting to see if there is a $\pi$ other than $(11 \cdots 1)$ whose $F_{n(i, j)}\left(x_{i}-y_{j} ; t\right)$ has nice properties to enable us to find the one-point and the multi-point distributions. Also, it is notable that Kuan recently obtained a determinantal expression for a multi-point distribution in a different context (that is, of the maximum species number among all particles to the left of a point (see [4] for detailed definition) in the inhomogeneous multi-species TASEP [4]. But, the determinantal formulas in this paper are for the transition probabilities $(\pi \rightarrow \pi)$ for all possible cases that particles belong to some species.

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## References

[1] Chatterjee S., Schütz G.M., Determinant representation for some transition probabilities in the TASEP with second class particles, J. Stat. Phys. 140 (2010), 900-916, arXiv:1003.5815.
[2] Korhonen M., Lee E., The transition probability and the probability for the left-most particle's position of the $q$-totally asymmetric zero range process, J. Math. Phys. 55 (2014), 013301, 15 pages, arXiv:1308.4769.
[3] Kuan J., Probability distributions of multi-species $q$-TAZRP and ASEP as double cosets of parabolic subgroups, Ann. Henri Poincaré 20 (2019), 1149-1173, arXiv:1801.02313.
[4] Kuan J., Determinantal expressions in multi-species TASEP, SIGMA 16 (2020), 133, 6 pages, arXiv:2007.02913.
[5] Kuniba A., Mangazeev V.V., Maruyama S., Okado M., Stochastic $R$ matrix for $U_{q}\left(A_{n}^{(1)}\right)$, Nuclear Phys. B 913 (2016), 248-277, arXiv:1604.08304.
[6] Lee E., Distribution of a particle's position in the ASEP with the alternating initial condition, J. Stat. Phys. 140 (2010), 635-647, arXiv:1004.1470.
[7] Lee E., The current distribution of the multiparticle hopping asymmetric diffusion model, J. Stat. Phys. 149 (2012), 50-72, arXiv:1203.0501.
[8] Lee E., Some conditional probabilities in the TASEP with second class particles, J. Math. Phys. 58 (2017), 123301, 11 pages, arXiv:1707.02539.
[9] Lee E., On the TASEP with second class particles, SIGMA 14 (2018), 006, 17 pages, arXiv:1705.10544.
[10] Lee E., Wang D., Distributions of a particle's position and their asymptotics in the $q$-deformed totally asymmetric zero range process with site dependent jumping rates, Stochastic Process. Appl. 129 (2019), 1795-1828, arXiv:1703.08839.
[11] Nagao T., Sasamoto T., Asymmetric simple exclusion process and modified random matrix ensembles, Nuclear Phys. B 699 (2004), 487-502, arXiv:cond-mat/0405321.
[12] Rákos A., Schütz G.M., Bethe ansatz and current distribution for the TASEP with particle-dependent hopping rates, Markov Process. Related Fields 12 (2006), 323-334, arXiv:cond-mat/0506525.
[13] Sasamoto T., Spatial correlations of the 1D KPZ surface on a flat substrate, J. Phys. A: Math. Gen. 38 (2005), L549-L556, arXiv:cond-mat/0504417.
[14] Schütz G.M., Exact solution of the master equation for the asymmetric exclusion process, J. Stat. Phys. 88 (1997), 427-445, arXiv:cond-mat/9701019.
[15] Tracy C.A., Widom H., Integral formulas for the asymmetric simple exclusion process, Comm. Math. Phys. 279 (2008), 815-844, arXiv:0704.2633.
[16] Tracy C.A., Widom H., Erratum to: Integral formulas for the asymmetric simple exclusion process, Comm. Math. Phys. 304 (2011), 875-878.
[17] Tracy C.A., Widom H., On the asymmetric simple exclusion process with multiple species, J. Stat. Phys. 150 (2013), 457-470, arXiv:1105.4906.
[18] Wang D., Waugh D., The transition probability of the $q$-TAZRP ( $q$-bosons) with inhomogeneous jump rates, SIGMA 12 (2016), 037, 16 pages, arXiv:1512.01612.

