# GLOBAL ANALYTIC SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER EQUATION 

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#### Abstract

We prove the existence of global analytic solutions to the nonlinear Schrödinger equation in one dimension for a certain type of analytic initial data in $L^{2}$.


## 1. Introduction

The nonlinear Schrödinger equation is the equation

$$
\begin{equation*}
i u_{t}+\Delta u=|u|^{p-1} u \tag{1}
\end{equation*}
$$

where $u: \mathbb{R}^{1+d} \rightarrow \mathbb{C}$. This equation has been studied extensively for data in the Sobolev spaces $H^{s}$. For a detailed discussion the $H^{s}$ theory for this equation, see chapter 3 of [19] and the many references therein. Recently, there has been much interest in developing a theory of analytic solutions to partial differential equations of all types, and many results exist in this direction. For a brief sampling of results, see [1, 2, 3, 8, $, 7,15,6,13,10,9,18, ~ 17, ~ 4, ~ 5] . ~$

In the case of equation (1), there exist several results regarding the existence of global analytic solutions. An early result in this direction is that of Hayashi 11, who studied the cubic case in dimensions $d \geq 2$ for small initial data. In the same year, Hayashi and Saitoh [12] obtained a similar result, but using milder smallness assumptions on the data. This was later generalized by Nakamitsu [16] to $p-1=2 \kappa$, where

$$
\frac{d}{2}-1 \leq \frac{1}{\kappa} \leq \frac{d}{2}
$$

but again requiring smallness assumptions on the initial data. For the cubic case, these smallness assumptions were later removed by Tesfahun in [20], who considered the problem in dimensions $d=1,2,3$.

In the present work, we will extend these results to the one-dimensional case where $p$ can be any odd number. In particular, we will prove the following theorem:

[^0]Theorem 1. Let $p$ be an odd natural number, and let $f \in L^{2}(\mathbb{R})$. Suppose that $f$ admits a holomorphic extension $\tilde{f}$ on the set

$$
S_{\sigma_{0}}=\left\{x+i y \in \mathbb{C}:|y|<\sigma_{0}\right\},
$$

and that

$$
\sup _{|y|<\sigma_{0}}\|\tilde{f}(\cdot+i y)\|_{L_{x}^{2}}<\infty
$$

Then for any $T>0$, the Cauchy problem

$$
\begin{align*}
i u_{t}+\Delta u & =|u|^{p-1} u, \\
u(x, 0) & =f(x) . \tag{2}
\end{align*}
$$

has a unique solution $u \in C\left([0, T] ; L^{2}\right)$. Moreover, this solution is the restriction to the real line of a function $\tilde{u}$ which is holomorphic on the set $S_{\sigma}$, where

$$
\sigma<\min \left\{\sigma_{0}, C T^{-1-\epsilon}\right\}
$$

for some constant $C>0$ and any $\epsilon>0$. Thus, the analyticity of $u$ persists for all time.

For the proof, we first construct local solutions by a standard fixedpoint argument. The procedure is standard, but for completeness, it will be shown in section 3, In section 4, we then show that the local solutions can be extended to arbitrarily large time intervals, if we allow the radius of analyticity $\sigma$ to decay. The proof uses a bootstrap argument and an almost conserved quantity, which we control by using the parameter $\sigma$. We begin our discussion by introducing the necessary tools in section 2,

## 2. Preliminaries

An important tool in our construction of analytic solutions to (2) are the Gevrey spaces $G^{\sigma}(\mathbb{R})$, which are defined by the norm

$$
\|f\|_{G^{\sigma}}=\left\|e^{\sigma|\xi|} \hat{f}(\xi)\right\|_{L_{\xi}^{2}},
$$

where $\hat{f}$ denotes the spatial Fourier transform, $\langle x\rangle=\left(1+|x|^{2}\right)^{1 / 2}$, and $\sigma>0$. The importance of the Gevrey spaces comes from the following Paley-Wiener theorem, for which a proof can be found in [14]:

Theorem 2. Let $\sigma>0$. Then, the following are equivalent:
(1) $f \in G^{\sigma}(\mathbb{R})$;
(2) $f$ is the restriction to the real line of a function $\tilde{f}$ which is holomorphic in the strip

$$
S_{\sigma}=\{x+i y: x, y \in \mathbb{R},|y|<\sigma\}
$$

and satisfies

$$
\sup _{|y|<\sigma}\|\tilde{f}(x+i y)\|_{L_{x}^{2}}<\infty .
$$

Remark 1. It should be noted that there is no assumption in this theorem that the function $f$ must be real-valued. This is important, as initial data and solutions to equation (11) are complex-valued.

In addition to the spaces $G^{\sigma}$, we will also make use of the hybrid Gevrey-Sobolev spaces $G^{\sigma, s}(\mathbb{R})$ defined by the norm

$$
\|f\|_{G^{\sigma, s}}=\left\|e^{\sigma|\xi|}\langle\xi\rangle^{s} \hat{f}(\xi)\right\|_{L_{\xi}^{2}} .
$$

It is a simple matter to see that these spaces satisfy the embeddings

$$
\begin{equation*}
G^{\sigma^{\prime}, s^{\prime}} \hookrightarrow G^{\sigma, s} \tag{3}
\end{equation*}
$$

for $\sigma \leq \sigma^{\prime}$ and $s, s^{\prime} \in \mathbb{R}$, which follow from the inequalities

$$
\|f\|_{G^{\sigma, s}} \lesssim\|f\|_{G^{\sigma^{\prime}, s^{\prime}}} .
$$

Note that $G^{0, s}=H^{s}$, so that for $\sigma=0$ the inequality becomes

$$
\begin{equation*}
\|f\|_{H^{s}} \lesssim\|f\|_{G^{\sigma^{\prime}, s^{\prime}}} \tag{4}
\end{equation*}
$$

and the associated embedding is

$$
G^{\sigma^{\prime}, s^{\prime}} \hookrightarrow H^{s}
$$

Gevrey-Sobolev spaces also obey the following generalization to the standard alegbra property of Sobolev spaces.

Lemma 3. If $s>1 / 2$ and $\sigma \geq 0$, then the space $G^{\sigma, s}(\mathbb{R})$ is an algebra, and

$$
\|u v\|_{G^{\sigma, s}} \lesssim\|u\|_{G^{\sigma, s}}\|v\|_{G^{\sigma, s}}
$$

Proof. By definition, we have

$$
\|u v\|_{G^{\sigma, s}}=\left\|e^{\sigma|\xi|}\langle\xi\rangle^{s} \widehat{u v}(\xi)\right\|_{L_{\xi}^{2}} .
$$

Observe that

$$
\widehat{u v}(\xi)=\int_{\mathbb{R}} \hat{u}(\xi-\eta) \hat{v}(\eta) d \eta
$$

By the triangle inequality, we also have that

$$
\begin{aligned}
e^{\sigma|\xi|} & \leq e^{\sigma|\xi-\eta|} e^{\sigma|\eta|} \\
\langle\xi\rangle^{s} & \lesssim\langle\xi-\eta\rangle^{s}+\langle\eta\rangle^{s} .
\end{aligned}
$$

It follows from these observations that

$$
\begin{aligned}
\|u v\|_{G^{\sigma, s}} \lesssim & \left\|\int_{\mathbb{R}}\left[e^{\sigma|\xi-\eta|}\langle\xi-\eta\rangle^{s}|\hat{u}(\xi-\eta)|\right]\left[e^{\sigma|\eta|}|\hat{v}(\eta)|\right] d \eta\right\|_{L_{\xi}^{2}} \\
& +\left\|\int_{\mathbb{R}}\left[e^{\sigma|\xi-\eta|}|\hat{u}(\xi-\eta)|\right]\left[e^{\sigma|\eta|}\langle\eta\rangle^{s}|\hat{v}(\eta)|\right] d \eta\right\|_{L_{\xi}^{2}} .
\end{aligned}
$$

Applying Young's inequality to this, we obtain

$$
\begin{aligned}
\|u v\|_{G^{\sigma, s}} & \lesssim \\
& \left\|e^{\sigma|\xi|}\langle\xi\rangle^{s}|\hat{u}(\xi)|\right\|_{L_{\xi}^{2}}\left\|e^{\sigma|\xi|}|\hat{v}(\xi)|\right\|_{L_{\xi}^{1}} \\
& \lesssim\|u\|_{G^{\sigma, s}}\left\||\xi| \hat{u}(\xi)\left|\left\|_{G^{\sigma, s}}\right\| e^{\sigma|\xi|}\langle\xi\rangle^{s}\right| \hat{v}(\xi) \mid\right\|_{L_{\xi}^{2}}
\end{aligned}
$$

Here, we have used the fact that

$$
\int_{\mathbb{R}} f d x \leq\left(\int_{\mathbb{R}}\langle x\rangle^{-2 s} d x\right)^{1 / 2}\left(\int_{\mathbb{R}}\langle x\rangle^{2 s}|f(x)|^{2} d x\right)^{1 / 2}
$$

and that the first integral on the right converges for $s>1 / 2$.
With all these facts in mind, we will use the following strategy to prove Theorem [1:

- The assumptions on $f$ imply that $f \in G^{\sigma_{0}}$. By the embedding in equation (3), $f \in G^{\sigma^{\prime}, s^{\prime}}$ for any $\sigma^{\prime}<\sigma_{0}$ and $s^{\prime} \in \mathbb{R}$. We use this fact to construct local solutions in $G^{\sigma^{\prime}, s^{\prime}}$ for $s^{\prime}>1 / 2$.
- By a standard argument, it suffices to show that the $G^{\sigma^{\prime}, s^{\prime}}$ norm of the solution $u$ remains finite in the interval $[0, T]$ for the solution to exist up to time $T>0$. As will be shown in section 4. this will require that we choose $\sigma^{\prime}$ sufficiently small, and $s^{\prime}=1$.
- Once it is known that the $G^{\sigma^{\prime}, 1}$ norm remains finite, the embedding (4) will imply that the $L^{2}$ norm remains bounded up to time $T$. Thus, by the standard $L^{2}$ theory, the solutions may be continued up to time $T$ in $L^{2}$. Moreover since $u(t) \in G^{\sigma^{\prime}, 1}$, it will also be analytic.


## 3. Local Well-Posedness

To begin, let us first recall some basic facts about the Schrödinger equation. Recall that the Cauchy problem

$$
\begin{gathered}
i u_{t}+\Delta u=F \\
u(x, 0)=f
\end{gathered}
$$

can be rewritten in integral form using the Duhamel formula

$$
u(x, t)=e^{i t \Delta} f-i \int_{0}^{t} e^{i(t-\tau) \Delta} F(\tau) d \tau
$$

Applying this to equation (2), we have

$$
\begin{equation*}
u(x, t)=e^{i t \Delta} f-i \int_{0}^{t} e^{i(t-\tau) \Delta}|u(\tau)|^{p-1} u(\tau) d \tau \tag{5}
\end{equation*}
$$

A strong solution to (22) is a solution to the integral equation (5). With this definition in mind, we may state our local result in a precise form.
Proposition 4. Let $p$ be an odd natural number, $\sigma^{\prime} \geq 0$, and let $s^{\prime}>$ $1 / 2$. Then the Cauchy problem (2) is locally well-posed in $G^{\sigma^{\prime}, s^{\prime}}(\mathbb{R})$. That is, for any $f \in G^{\sigma^{\prime}, s^{\prime}}$, there exists a time $\delta=\delta(\|f\|)>0$ such that the Cauchy problem (21) has a unique strong solution

$$
u \in C\left([0, \delta) ; G^{\sigma^{\prime}, s^{\prime}}\right) .
$$

Furthermore, the solution map $f \mapsto u$ is Lipschitz continuous from $G^{\sigma^{\prime}, s^{\prime}}$ to $C\left([0, \delta) ; G^{\sigma^{\prime}, s^{\prime}}\right)$.

Proof. Fix $f \in G^{\sigma^{\prime}, s^{\prime}}$, and define an operator $\Phi$ on $G^{\sigma^{\prime}, s^{\prime}}$ by

$$
\Phi(u)=e^{i t \Delta} f-i \int_{0}^{t} e^{i(t-\tau) \Delta}|u(x, \tau)|^{p-1} u(x, \tau) d \tau
$$

Since the operator $e^{i t \Delta}$ is unitary, it is easy to see that this integral formula implies the inequality

$$
\begin{equation*}
\|\Phi(u)(t)\|_{G^{\sigma^{\prime}, s^{\prime}}} \leq\|f\|_{G^{\sigma^{\prime}, s^{\prime}}}+\int_{0}^{t}\left\||u(x, \tau)|^{p-1} u(x, \tau)\right\|_{G^{\sigma^{\prime}, s^{\prime}}} d \tau \tag{6}
\end{equation*}
$$

By Lemma 3 we have, for $s^{\prime}>1 / 2$,

$$
\|\Phi(u)(t)\|_{G^{\sigma^{\prime}, s^{\prime}}} \leq\|f\|_{G^{\sigma^{\prime}, s^{\prime}}}+C \int_{0}^{t}\|u(\tau)\|_{G^{\sigma^{\prime}, s^{\prime}}}^{p} d \tau
$$

for some generic constant $C>0$. Taking the supremum over $t \in[0, \delta)$ gives us

$$
\begin{equation*}
\|\Phi(u)\|_{L^{\infty} G^{\sigma^{\prime}, s^{\prime}}} \leq\|f\|_{G^{\sigma^{\prime}, s^{\prime}}}+C \delta\|u\|_{L^{\infty} G^{\sigma, s^{\prime}}}^{p} . \tag{7}
\end{equation*}
$$

It follows that $\Phi$ maps $C\left([0, \delta) ; G^{\sigma^{\prime}, s^{\prime}}\right)$ to itself.
Next, we show that $\Phi$ is a contraction. The existence of a unique fixed point will follow from the Contraction Mapping Principle. Let $u, v \in C\left([0, \delta) ; G^{\sigma^{\prime}, s^{\prime}}\right)$ such that

$$
\|u\|_{L^{\infty} G^{\sigma^{\prime}, s^{\prime}}} \leq\|f\|_{G^{\sigma^{\prime}, s^{\prime}}} \text { and }\|v\|_{L^{\infty} G^{\sigma^{\prime}, s^{\prime}}} \leq\|f\|_{G^{\sigma^{\prime}, s^{\prime}}}
$$

By applying equation (6), it is easy to see that

$$
\begin{aligned}
\|\Phi(u)-\Phi(v)\|_{L^{\infty} G^{\sigma^{\prime}, s^{\prime}}} & \leq C \delta\left(\|u\|_{L^{\infty} G_{G^{\sigma^{\prime}, s^{\prime}}}}^{p-1}+\|v\|_{L^{\infty} G^{\sigma^{\prime}, s^{\prime}}}^{p-1}\right)\|u-v\|_{L^{\infty} G^{\sigma^{\prime}, s^{\prime}}} \\
& \leq 2 C \delta\|f\|_{G^{\sigma^{\sigma^{\prime}, s^{\prime}}}}^{p-1}\|u-v\|_{L^{\infty} G^{\sigma^{\prime}, s^{\prime}}}
\end{aligned}
$$

If

$$
\delta<\frac{1}{2 C\|f\|_{G^{\sigma^{\prime}, s^{\prime}}}^{p-1}}
$$

then

$$
\|\Phi(u)-\Phi(v)\|_{L^{\infty} G^{\sigma^{\prime}, s^{\prime}}}<\|u-v\|_{L^{\infty} G_{\sigma^{\sigma^{\prime}, s^{\prime}}}}
$$

Thus $\Phi$ is a contraction. The existence of a unique fixed point for $u$ follows from the Contraction Mapping Principle. This fixed point satisfies equation (5), and so is a strong solution to the Cauchy problem (2).

Finally, we must show that the solution map $f \mapsto u$ is continuous from $G^{\sigma^{\prime}, s^{\prime}}$ to $L^{\infty} G^{\sigma^{\prime}, s^{\prime}}$. Suppose for two initial conditions $f, g \in G^{\sigma^{\prime}, s^{\prime}}$, respectively with

$$
\|f\|_{G^{\sigma^{\prime}, s^{\prime}}} \leq R \quad \text { and } \quad\|g\|_{G^{\sigma^{\prime}, s^{\prime}}} \leq R
$$

we have the corresponding solutions $u$ and $v$, respectively. As in the computations above, we may apply equation (6) and Lemma3to obtain

$$
\begin{aligned}
\|u-v\|_{L^{\infty} G_{G^{\sigma^{\prime}, s^{\prime}}} \leq} & \|f-g\|_{G^{\sigma^{\prime}, s^{\prime}}}+\int_{0}^{t}\left\||u|^{p-1} u-|v|^{p-1} v\right\|_{G^{\sigma^{\prime}, s^{\prime}}} d \tau \\
\leq & \|f-g\|_{G^{\sigma^{\prime}, s^{\prime}}} \\
& +C \delta\left(\|u\|_{L^{\infty} G^{\sigma^{\prime}, s^{\prime}}}^{p-1}+\|v\|_{L^{\infty} G^{\sigma^{\prime}, s^{\prime}}}^{p-1}\right)\|u-v\|_{L^{\infty} G^{\sigma^{\prime}, s^{\prime}}} .
\end{aligned}
$$

From equation (7) and the choice of $\delta$, it follows that

$$
\|u-v\|_{L^{\infty} G^{\sigma^{\prime}, s^{\prime}}} \leq\|f-g\|_{G^{\sigma^{\prime}, s^{\prime}}}+2 C \delta R^{p-1}\|u-v\|_{L^{\infty} G_{G^{\sigma^{\prime}, s^{\prime}}}} .
$$

If we now further make the assumption that $\delta$ also satisfies

$$
2 C \delta R^{p-1}<1
$$

then we may conclude that

$$
\|u-v\|_{L^{\infty} G_{\sigma^{\sigma^{\prime}, s^{\prime}}}} \leq \frac{\|f-g\|_{G^{\sigma^{\prime}, s^{\prime}}}}{1-2 C \delta R^{p-1}} .
$$

Continuity of the solution map follows. This completes the proof of Proposition 4.

## 4. Global Existence

In this section, we will prove the following proposition, from which the second conclusion of Theorem 1 will follow:

Proposition 5. Let u be the local solution to the Cauchy problem (2), and let $T>0$ be arbitrary. Then there exists $\sigma>0$ such that

$$
\sup _{t \in[0, T]}\|u(\cdot, t)\|_{G^{\sigma, 1}(\mathbb{R})}<C
$$

for some constant $C>0$.
To prove this, we first state a preliminary lemma.
Lemma 6. Let $n \in \mathbb{N}, n \geq 2$, and let $\eta_{1}, \ldots, \eta_{n} \in \mathbb{R}$. Then

$$
e^{\sigma \sum_{j=1}^{n}\left|\eta_{j}\right|}-e^{\sigma\left|\sum_{j=1}^{n} \eta_{j}\right|} \leq \sum_{k=1}^{n}\left(2 \sigma \min \left(\left|\sum_{j \neq k} \eta_{j}\right|,\left|\eta_{k}\right|\right)\right)^{\theta} e^{\sigma \sum_{j=1}^{n}\left|\eta_{j}\right|}
$$

for any $\theta \in[0,1]$.
Proof. By strong induction on $n$. The case $n=2$ was shown in [17. Thus, we may assume the result holds for $n \leq m$. Consider the case $n=m+1$. We may write

$$
\begin{aligned}
e^{\sigma \sum_{j=1}^{m+1}\left|\eta_{j}\right|}-e^{\sigma\left|\sum_{j=1}^{m+1} \eta_{j}\right|}= & e^{\sigma\left|\eta_{m+1}\right|}\left[e^{\sigma \sum_{j=1}^{m}\left|\eta_{j}\right|}-e^{\sigma\left|\sum_{j=1}^{m} \eta_{j}\right|}\right] \\
& +e^{\sigma\left|\eta_{m+1}\right|} e^{\sigma\left|\sum_{j=1}^{m} \eta_{j}\right|}-e^{\sigma\left|\sum_{j=1}^{m+1} \eta_{j}\right|} .
\end{aligned}
$$

Applying the inductive hypothesis to the first line above, we have

$$
e^{\sigma \sum_{j=1}^{m}\left|\eta_{j}\right|}-e^{\sigma\left|\sum_{j=1}^{m} \eta_{j}\right|} \leq \sum_{k=1}^{m}\left(2 \sigma \min \left(\left|\sum_{j \neq k} \eta_{j}\right|,\left|\eta_{k}\right|\right)\right)^{\theta} e^{\sigma \sum_{j=1}^{m}\left|\eta_{j}\right|}
$$

Applying the inductive hypothesis and the triangle inequality to the second line, we have

$$
\begin{aligned}
& e^{\sigma\left|\eta_{m+1}\right|} e^{\sigma\left|\sum_{j=1}^{m} \eta_{j}\right|}-e^{\sigma\left|\sum_{j=1}^{m+1} \eta_{j}\right|} \\
& \quad \leq\left(2 \sigma \min \left(\left|\sum_{j=1}^{m} \eta_{j}\right|,\left|\eta_{m+1}\right|\right)\right)^{\theta} e^{\sigma \sum_{j=1}^{m+1}\left|\eta_{j}\right|}
\end{aligned}
$$

The desired result follows.
Proof of Proposition 5. Recall that

$$
\|f\|_{H^{s}} \sim\|f\|_{H^{s-1}}+\|\nabla f\|_{H^{s-1}}
$$

for $f \in H^{s}$ (see [19], Appendix A). By the commutativity of Fourier multipliers, this implies that

$$
\|u\|_{G^{\sigma, 1}} \sim\|u\|_{G^{\sigma}}+\|\nabla u\|_{G^{\sigma}} \sim\left(\|u\|_{G^{\sigma}}^{2}+\|\nabla u\|_{G^{\sigma}}^{2}\right)^{1 / 2}
$$

for $u \in G^{\sigma, 1}$. Thus, it suffices to estimate the norms above to obtain the desired result.

Let $\Lambda$ be the pseudodifferential operator defined by the Fourier multiplier

$$
\widehat{\Lambda u}=e^{\sigma|\xi|} \hat{u}(\xi, t),
$$

and define

$$
U(x, t)=\Lambda u .
$$

We observe that

$$
\|u\|_{G^{\sigma}}=\|U\|_{L^{2}} \quad \text { and } \quad\|\nabla u\|_{G^{\sigma}}=\|\nabla U\|_{L^{2}} .
$$

Moreover, it is easy to see that $U$ and $\bar{U}$ satisfy the equations

$$
U_{t}=i \Delta U-i \Lambda\left(|u|^{p-1} u\right)
$$

and

$$
\bar{U}_{t}=-i \Delta \bar{U}+i \Lambda\left(|u|^{p-1} \bar{u}\right) .
$$

Next, define a quantity $S(t)$ by

$$
S(t)=\int_{\mathbb{R}}|U(x, t)|^{2}+|\nabla U|^{2}+\frac{2}{p+1}|U|^{p+1} d x .
$$

Observe that in the case $\sigma=0$, this quantity would be conserved for equation (11). By the Fundamental Theorem of Calculus, we have that

$$
S(t)=S(0)+\int_{0}^{t} \frac{d S(\tau)}{d t} d \tau
$$

A lengthy computation will show that

$$
\begin{aligned}
\frac{d S}{d t}= & i \int_{\mathbb{R}} \overline{\nabla U} \cdot \nabla N(u)-\nabla N(\bar{u}) \cdot \nabla U d x \\
& +i \int_{\mathbb{R}} \bar{U} N(u)-N(\bar{u}) U d x \\
& +i \int_{\mathbb{R}}|U|^{p-1} \bar{U} N(u)-N(\bar{u})|U|^{p-1} U d x
\end{aligned}
$$

where

$$
N(u)=|U|^{p-1} U-\Lambda\left(|u|^{p-1} u\right) .
$$

Applying Hölder's inequality, we see that

$$
\begin{align*}
S(t) \leq & S(0) \\
& +2 \int_{0}^{t}\|\nabla U(\tau)\|_{L^{2}}\|\nabla N(u)(\tau)\|_{L^{2}} d \tau  \tag{8}\\
& +2 \int_{0}^{t}\|U(\tau)\|_{L^{2}}\|N(u)(\tau)\|_{L^{2}} d \tau \\
& +2 \int_{0}^{t}\left\|\left.U(\tau)\right|^{p-1} U(\tau)\right\|_{L^{2}}\|N(u(\tau))\|_{L^{2}} d \tau
\end{align*}
$$

Note that

$$
\left\||U(\tau)|^{p-1} U(\tau)\right\|_{L^{2}}=\|U(\tau)\|_{L^{2 p}}^{p}
$$

We estimate this using the Gagliardo-Nirenberg inequality, giving us

$$
\|U(\tau)\|_{L^{2 p}} \lesssim\|U(\tau)\|_{L^{2}}^{1-\alpha}\|\nabla U\|_{L^{2}}^{\alpha}
$$

with

$$
\alpha=\frac{1}{2}\left(1-\frac{1}{p}\right) .
$$

Thus, equation (8) becomes

$$
\begin{align*}
S(t) \leq & S(0)+2 \int_{0}^{t}\|U(\tau)\|_{L^{2}}\|N(u)(\tau)\|_{L^{2}} d \tau \\
& +2 \int_{0}^{t}\|\nabla U(\tau)\|_{L^{2}}\|\nabla N(u)(\tau)\|_{L^{2}} d \tau  \tag{9}\\
& +2 C \int_{0}^{t}\left(\|U(\tau)\|_{L^{2}}^{1-\alpha}\|\nabla U\|_{L^{2}}^{\alpha}\right)^{p}\|N(u(\tau))\|_{L^{2}} d \tau
\end{align*}
$$

Next, we must estimate each of the terms involving the nonlinear operator $N(u)$. By Plancherel's theorem, it suffices to consider

$$
\|\widehat{N(u)}\|_{L_{\xi}^{2}} \quad \text { and } \quad\|\widehat{\nabla N(u)}\|_{L_{\xi}^{2}} .
$$

For this, we first rewrite $N(u)$ as

$$
N(u)=\left|e^{\sigma|\nabla|} u\right|^{p-1}\left(e^{\sigma|\nabla|} u\right)-\Lambda\left(|u|^{p-1} u\right) .
$$

We then take the spatial Fourier transform, which we write as the convolution integral

$$
\widehat{N(u)}=\int_{H}\left(e^{\sum_{j=1}^{2 k+1} \sigma\left|\eta_{j}\right|}-e^{\sigma|\xi|}\right)\left[\prod_{j=1}^{k} \hat{\bar{u}}\left(\eta_{2 j-1}\right) \hat{u}\left(\eta_{2 j}\right)\right] \hat{u}\left(\eta_{2 k+1}\right) d \eta
$$

where $H$ denotes the hyperplane $\xi=\eta_{1}+\cdots+\eta_{2 k+1}$ and

$$
d \eta=d \eta_{1} \cdots d \eta_{2 k+1}
$$

To estimate $|\widehat{N(u)}|$, we apply Lemma 6 to obtain

$$
\begin{align*}
|\widehat{N(u)}| & \leq C \sigma^{\theta} \sum_{m=1}^{2 k+1} \int_{H}\left(\min \left\{\left|\sum_{m \neq j} \eta_{j}\right|,\left|\eta_{m}\right|\right\}\right)^{\theta} G(\eta) d \eta  \tag{10}\\
& \leq C \sigma^{\theta} \sum_{m=1}^{2 k+1} \int_{H}\left|\eta_{m}\right|^{\theta} G(\eta) d \eta
\end{align*}
$$

where $C>0$ is a generic constant which may be different in each line, $\eta=\left(\eta_{1}, \ldots, \eta_{2 k+1}\right)$, and

$$
\begin{aligned}
G(\eta) & =\left[\prod_{j=1}^{k} \mid e^{\left.\sigma\left|\eta_{2 j-1}\right| \hat{\bar{u}}\left(\eta_{2 j-1}\right)| | e^{\sigma\left|\eta_{2 j}\right|} \hat{u}\left(\eta_{2 j-1}\right) \mid\right]\left|e^{\sigma \mid \eta_{2 k+1}} \hat{u}\left(\eta_{2 k+1}\right)\right|}\right. \\
& =\left[\prod_{j=1}^{k}\left|\hat{\bar{U}}\left(\eta_{2 j-1}\right)\right|\left|\hat{U}\left(\eta_{2 j-1}\right)\right|\right]\left|\hat{U}\left(\eta_{2 k+1}\right)\right| .
\end{aligned}
$$

If we define $\hat{v}_{j}=\left|\hat{U}\left(\eta_{j}\right)\right|$ or $\left|\hat{\bar{U}}\left(\eta_{j}\right)\right|$, as appropriate, then we can rewrite each of the integrals in equation (10) as a convolution of the form

$$
\hat{v}_{1} * \ldots * \hat{v}_{j-1} * \widehat{|\nabla|^{\theta} v_{j}} * \hat{v}_{j+1} * \cdots * \hat{v}_{2 k+1} .
$$

Thus we have that

$$
\begin{aligned}
\|\widehat{N(u)}\|_{L^{2}} & \lesssim \sigma^{\theta} \sum_{m=1}^{2 k+1}\left\|\hat{v}_{1} * \ldots * \hat{v}_{m-1} * \mid \widehat{\left.\nabla\right|^{\theta} v_{m}} * \hat{v}_{m+1} * \cdots * \hat{v}_{2 k+1}\right\|_{L^{2}} \\
& \lesssim \sigma^{\theta} \sum_{m=1}^{2 k+1}\left\|v_{1} \ldots v_{m-1}\left(|\nabla|^{\theta} v_{m}\right) v_{m+1} \cdots v_{2 k+1}\right\|_{L^{2}} \\
& \lesssim \sigma^{\theta} \sum_{m=1}^{2 k+1}\left(\prod_{j \neq m}\left\|v_{j}\right\|_{L^{\infty}}\right)\left\||\nabla|^{\theta} v_{m}\right\|_{L^{2}} \\
& \lesssim \sigma^{\theta} \sum_{m=1}^{2 k+1}\left(\prod_{j \neq m}\left\|v_{j}\right\|_{H^{1}}\right)\left\|v_{m}\right\|_{H^{1}} \\
& \lesssim \sigma^{\theta} \prod_{m=1}^{2 k+1}\left\|v_{j}\right\|_{H^{1}} \\
& \lesssim \sigma^{\theta}\left(\|U\|_{L^{2}}^{2}+\|\nabla U\|_{L^{2}}^{2}\right)^{p / 2} .
\end{aligned}
$$

It follows that

$$
\begin{equation*}
\|N(u)\|_{L^{2}} \lesssim \sigma^{\theta}\left(\|U\|_{L^{2}}^{2}+\|\nabla U\|_{L^{2}}^{2}\right)^{p / 2} \tag{11}
\end{equation*}
$$

For the $\nabla N(u)$ terms, we observe that $|\widehat{\nabla N(u)}|=|\xi||\widehat{N(u)}|$. Analogously, the problem of estimating the norm of $\widehat{\nabla N(u)}$ can be reduced by Lemma 6 to estimating a sum of terms of the form

$$
\begin{aligned}
& \sigma^{\theta} \int_{H}|\xi|\left(\min \left\{\left|\sum_{j \neq \ell} \eta_{j}\right|,\left|\eta_{\ell}\right|\right\}\right)^{\theta} \times \\
& \quad \times\left[\prod_{j=1}^{k}\left|e^{\sigma\left|\eta_{2 j}\right|} \hat{\bar{u}}\left(\eta_{2 j}\right)\right|\left|e^{\sigma\left|\eta_{2 j-1}\right|} \hat{u}\left(\eta_{2 j-1}\right)\right|\right]\left|e^{\sigma\left|\eta_{2 k+1}\right|} \hat{u}\left(\eta_{2 k+1}\right)\right| d \eta
\end{aligned}
$$

To estimate these integrals, we recall that $\xi=\eta_{1}+\cdots+\eta_{2 k+1}$, so that

$$
|\xi| \leq\left|\sum_{j \neq \ell} \eta_{j}\right|+\left|\eta_{\ell}\right|
$$

In the case where

$$
\left|\sum_{j \neq \ell} \eta_{j}\right| \geq\left|\eta_{\ell}\right|
$$

we obtain that

$$
|\xi| \leq 2\left|\sum_{j \neq \ell} \eta_{j}\right| \leq 2 \sum_{j \neq \ell}\left|\eta_{j}\right|
$$

It follows that $|\widehat{\nabla N(u)}|$ can be estimated by a sum of terms of the form

$$
\sigma^{\theta} \int_{H}\left|\eta_{\ell}\right|^{\theta}\left|\eta_{n}\right|\left[\prod_{j=1}^{k}\left|\hat{\bar{U}}\left(\eta_{2 j}\right)\right|\left|\hat{U}\left(\eta_{2 j-1}\right)\right|\right]\left|\hat{U}\left(\eta_{2 k+1}\right)\right| d \eta
$$

where $\ell \neq n$. The case $\left|\sum_{j \neq \ell} \eta_{j}\right| \leq\left|\eta_{\ell}\right|$ is similar. For both of these cases, we observe that these integrals can be written in convolution form as

$$
\hat{v}_{1} * \cdots * \hat{v}_{j-1} *\left(\widehat{|\nabla|^{\theta} v_{j}}\right) * \hat{v}_{j+1} * \cdots * v_{2 k} *\left(\widehat{\nabla v_{2 k+1}}\right)
$$

We estimate these terms by

$$
\begin{aligned}
\| v_{1} \cdots v_{j-1}\left(|\nabla|^{\theta} v_{j}\right) v_{j+1} & \cdots v_{2 k} \nabla v_{2 k+1} \|_{L_{x}^{2}} \\
& \lesssim\left\||\nabla| v_{2 k+1}\right\|_{L^{2}}\left\|\left.\nabla\right|^{\theta} v_{j}\right\|_{L^{\infty}} \prod_{\ell \neq j, 2 k+1}\left\|v_{\ell}\right\|_{L^{\infty}} \\
& \lesssim \prod_{j=1}^{2 k+1}\left\|v_{j}\right\|_{H^{1}} \\
& \lesssim\left(\|U\|_{L^{2}}^{2}+\|\nabla U\|_{L^{2}}^{2}\right)^{p / 2}
\end{aligned}
$$

We remark that we have once again used the Gagliardo-Nirenberg inequality to estimate

$$
\left\||\nabla|^{\theta} v_{j}\right\|_{L^{\infty}} \lesssim\left\||\nabla|^{\theta} v_{j}\right\|_{L^{2}}^{\beta}\left\||\nabla|^{\theta} v_{j}\right\|_{\dot{H}^{1-\theta}}^{1-\beta} \lesssim\left\|v_{j}\right\|_{H^{1}}
$$

which requires that

$$
\frac{1}{2}=\beta(1-\theta)
$$

Thus, it is necessary that $0 \leq \theta<1$. We may now conclude that

$$
\begin{equation*}
\|\nabla N(u)\|_{L^{2}} \lesssim \sigma^{\theta}\left(\|U\|_{L^{2}}^{2}+\|\nabla U\|_{L^{2}}^{2}\right)^{p / 2} \tag{12}
\end{equation*}
$$

If we now combine equations (9), (11), and (12), we obtain that

$$
\begin{equation*}
S(t) \leq S(0)+C \sigma^{\theta} \int_{0}^{t} S^{\frac{p}{2}}(\tau)\left(2 S^{1 / 2}(\tau)+S^{p / 2}(\tau)\right) d \tau \tag{13}
\end{equation*}
$$

Our next step is to show that this quantity remains bounded for $t \in[0, T]$. We apply a simple bootstrap argument. Let $\mathrm{H}(t)$ and $\mathrm{C}(t)$ be the statements

- $\mathrm{H}(t): S(\tau) \leq 4 S(0)$ for $0 \leq \tau \leq t$.
- $\mathrm{C}(t): S(\tau) \leq 2 S(0)$ for $0 \leq \tau \leq t$.

To close the bootstrap, we must prove the following four statements:
(a) $\mathrm{H}(t) \Rightarrow \mathrm{C}(t)$;
(b) $\mathrm{C}(t) \Rightarrow \mathrm{H}\left(t^{\prime}\right)$ for all $t^{\prime}$ in a neighborhood of $t$;
(c) If $t_{1}, t_{2}, \ldots$ is a sequence in $[0, T]$ such that $t_{n} \rightarrow t \in[0, T]$, with $\mathrm{C}\left(t_{n}\right)$ true for all $t_{n}$, then $\mathrm{C}(t)$ is also true;
(d) $\mathrm{H}(t)$ is true for at least one $t \in[0, T]$.

Proof of (a). Assuming the statement $\mathrm{H}(t)$, equation (13) gives us the estimate

$$
S(t) \leq S(0)+C \sigma^{\theta}(4 S(0))^{\frac{p}{2}}\left(2(4 S(0))^{1 / 2}+(4 S(0))^{p / 2}\right) t
$$

Taking the supremum, this gives us

$$
\sup _{t \in[0, T]} S(t) \leq S(0)+C \sigma^{\theta}(4 S(0))^{\frac{p}{2}}\left(2(4 S(0))^{1 / 2}+(4 S(0))^{p / 2}\right) T
$$

Choose $\sigma$ so that

$$
\sigma \leq\left[C(4 S(0))^{\frac{p}{2}}\left(2(4 S(0))^{1 / 2}+(4 S(0))^{p / 2}\right) T\right]^{-1 / \theta},
$$

then the conclusion $\mathrm{C}(t)$ follows. Note that this requires that $\theta>0$.
Proof of (b). Assume $S(\tau) \leq 2 S(0)$ for all $\tau$ with $0 \leq \tau \leq t$. We may then apply the local existence theory to construct solutions which exist
on an interval $[t, t+\delta) \subset[0, T]$ for some small $\delta>0$. In particular, we can do this so that

$$
\sup _{\tau \in[t, t+\delta)} S(\tau) \leq 4 S(0)
$$

Since

$$
\sup _{\tau \in(t-\delta, t]} S(\tau) \leq 2 S(0)
$$

by assumption, the statement $\mathrm{H}\left(t^{\prime}\right)$ holds for all $t^{\prime} \in(t-\delta, t+\delta)$.
Proof of (c). The statement in (c) follows immediately from the fact that our solutions are constructed so that the norm $\|u(t)\|_{G^{\sigma, 1}}$ defines a continuous function in time.

Proof of (d). $\mathrm{H}(0)$ holds by assumption.
Based on the above, we may close the bootstrap, and it follows that $\mathrm{C}(t)$ holds for all $t \in[0, T]$.

To conclude the proof of Theorem 1, we summarize what we have accomplished: our assumptions on $f$ imply that $f \in G^{\sigma_{0}}$. By the Gevrey embedding (3), $f \in G^{\sigma, 1}$ for $\sigma$ given by

$$
\sigma<\min \left\{\sigma_{0},\left[C(4 S(0))^{\frac{p}{2}}\left(2(4 S(0))^{1 / 2}+(4 S(0))^{p / 2}\right) T\right]^{-1-\epsilon}\right\}
$$

Using the local theory of section 3, we can construct a solution $u$ up to some small time $\delta>0$. By the global theory of section 4, the norm of $u$ remains bounded, so we may continue our solution past time $\delta$ to the desired time $T$.

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