Masters Thesis

The probability distribution of the species-1 particle in the two-species ASEP with initial configuration 22...21

School of Sciences and Humanities
Nazarbayev University

Zhanibek Tokebayev
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Supervisor: Prof. Eunghyun Lee
Second Reader: Prof. Zhenisbek Assylbekov
School of Sciences and Humanities
Department of Mathematics
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Abstract

In this paper, we investigate the behaviour of the unique right-most species-1 particle in the 2-species ASEP. We start with the introduction, where we first introduce the model itself, which was introduced by Frank Spitzer in the paper [4]. Then we give a brief outline of the research done in the field, and the results obtained by Schütz (in the paper [3]), Tracy and Widom (in the papers [5], [6]), Lee (in the papers [1], [2]), and Raimbekov (in the paper [2]). We then proceed to the preliminary section, where we explain the necessary background needed to understand the thesis. After that, we start experimenting with some special cases of the general N-particle problem; this will naturally lead us to the main results section. The main result of the thesis is the integral formula for the probability distribution of the unique species-1 particle which is right-most at time $t = 0$ in a 2-species ASEP.

Contents

Chapter 1

Introduction

The theory of Asymmetric Simple Exclusion Processes (ASEP) has been studied extensively since 1970, when Frank Spitzer first introduced them. In the paper [3], Gunter Schütz constructed a solution for the (single species) TASEP using two different ways: the constructive Bethe Ansatz technique, and as a determinant of a certain matrix. He also obtained the solutions to the ASEP by Bethe Ansatz in the number of particles $N = 1$ and $N = 2$.

Then, in the paper [5], Craig Tracy and Harold Widom obtained the solutions to the forward equation for the ASEP with general N. They then obtained probability distributions of each particle x_m separately, $m \leq N$. Moreover, they obtained these probabilities in 2 different forms: in terms of contour integrals with sufficiently small contours C_r, such that all poles of the integrand lie outside the contours C_r, and in terms of contour integrals with sufficiently large contours C_R, such that all poles of the integrand lie inside the contours C_R.

Next, in another paper [6], Craig Tracy and Harold Widom deduced the form of the solutions to the forward equation for the multi-species ASEP with general N. However, in their solution, the leading coefficients were not given explicitly. Eunghyun Lee in [1] has constructed the algorithm to obtain those coefficients. Moreover, in the TASEP case, he expressed the solution as the determinant of a certain matrix. Further, in the paper [2] Eunghyun Lee and Temirlan Raimbekov came up with those coefficients explicitly in the special case when the model is a 2-species ASEP with the initial configurations 2...21 and 1...12.

In this paper, our goal is to obtain the variant of what Tracy and Widom obtained in [5]: we want to find the distribution of the right-most particle position x_N which belongs to the species 1, given that all other particles belong to species-2.

Chapter 2

Preliminary

2.1 What is a multi-species ASEP?

First of all, as was said before, ASEP stands for Asymmetric Simple Exclusion Process. ASEP were first introduced by Frank Spitzer in the paper [4].

Consider a continuous-time process of N different particles moving randomly on $\mathbb{Z}$, belonging to species $1, 2, ..., k$, with the time being a random variable that is distributed exponentially, with parameter $\lambda = 1$; since there are several different species present, we call this process a **multi-species process**. In the special case when there is only one species, the process is called a **single-species process**.

Each particle is equipped with an exponential random clock; when the exponential random clock ticks, the particle moves either to the left with probability q, or to the right with probability $p = 1 - q$; since generally $q \neq p \neq \frac{1}{2}$, we call the process an **asymmetric process**. In the special case when $p = 1$, and $q = 0$, we call the process a **Totally Asymmetric Simple Exclusion Process (TASEP)**; the geometric interpretation of TASEP is that particles can move **only** to the right. Next, there is a restriction on how many particles can occupy a single spot in $\mathbb{Z}$: there can be at most 1 particle occupying a single spot; that is why the process is called an **exclusion process**. Now, suppose that 2 particles stand next to each other and one is supposed to jump onto the other; then in this situation the difference in species comes into play. If the particle that is about to jump belongs to higher or the same species, then the particles swap their positions; if the particle that is about to jump belongs to lower species, then there is no motion, and the Poisson clock begins anew. In other words, the species-2 particles are in some sense superior to the species-1 particle.

Given the above, the state space for this system (multi-species ASEP) can be represented by the set $S = S' \times I$, where $S' \subset \mathbb{Z}^N$, $S = \{(x_1, ..., x_N) \in \mathbb{Z}^N : x_1 < x_2 < ... < x_N\}$, and I is the set of all

N-tuples of indices $1, ..., k$; by abuse of notation, to simplify the notation, we will denote the tuples by strings: $(n_1, n_2, ..., n_N) = n_1 n_2 ... n_N = \pi$.

2.2 Finding transition probability in a single-species ASEP

To analyze ASEP, the first step is to construct the forward Kolmogorov equation.

2.2.1 N=1.

For simplicity, let us first consider the $N = 1$ case. Suppose that we have 1 particle that starts moving randomly on $\mathbb{Z}$ at time $t = 0$, such that at time $t = 0$ the position of the particle is at $y \in \mathbb{Z}$. Then, we denote the probability that at time t the particle is at $x \in \mathbb{Z}$ by $P_y(x;t)$. To simplify the notation, we write $P_y(x;t) = P(x;t)$. Then, our forward equation takes the form:

$$\frac{d}{dt}P(x;t) = pP(x-1;t) + qP(x+1;t) - P(x;t).$$

This differential equation is supplied with a very natural initial condition:

$$P(x;0) = P_y(x;0) = \delta_{xy}.$$

The meaning of this initial condition is that, at time $t = 0$, the probability that a particle is at y is 1, and is zero otherwise.

To solve this equation, we apply the separation of variables technique: suppose that $P(x;t) = P(x)T(t)$, where $x \in \mathbb{Z}$, $t \in \mathbb{R}^{\geq 0}$. Then, we get:

$$\begin{aligned} P(x)\frac{d}{dt}T(t) &= pP(x-1)T(t) + qP(x+1)T(t) - P(x)T(t) \\ &= \left[pP(x-1) + qP(x+1) - P(x)\right]T(t) \end{aligned}$$

It is readily seen that $T(t) = Ce^{\epsilon t}$. Then, we cancel $Ce^{\epsilon t}$ from both sides, thus obtaining a difference equation:

$$P(x) \cdot \epsilon = pP(x-1) + qP(x+1) - P(x).$$

Such equations are solved by a substitution: $P(x) = \xi^{x-1}$. Using this substitution, we get:

$$\epsilon = p\frac{P(x-1)}{P(x)} + q\frac{P(x+1)}{P(x)} - \frac{P(x)}{P(x)} = \frac{p}{\xi} + q\xi - 1 = \epsilon(\xi).$$

Therefore, we finally get our most general solution:

$$P(x;t) = P(x)T(t) = \xi^{x-1}\cdot Ce^{\epsilon t} = C\xi^{x-1}e^{\epsilon t} = C(\xi)\xi^{x-1}e^{\epsilon t}.$$

Note that the constant C is really a function of ξ. Now, we need to construct a specific solution that satisfies our particular initial condition $P_y(x;0) = \delta_{xy}$; for this, choose $C(\xi) = \xi^{-y}$, and construct the contour integral:

$$P(x;t) = \frac{1}{2\pi i}\oint_c \xi^{-y}\cdot\xi^{x-1}e^{\epsilon t}\,d\xi = \frac{1}{2\pi i}\oint_c \xi^{x-y-1}e^{\left(\frac{p}{\xi}+q\xi-1\right)t}\,d\xi.$$

where the contour c is a circle around 0. Then, it is not hard to see that the integrand satisfies the initial condition: let $t = 0$; then, there are 2 cases: either $x = y$, or $x \neq y$.

- $x = y$: in this case, our integral is:

$$P(x;0) = \frac{1}{2\pi i}\oint_c \xi^{x-y-1}e^{\left(\frac{p}{\xi}+q\xi-1\right)\cdot 0}\,d\xi = \frac{1}{2\pi i}\oint_c \xi^{-1}\cdot 1\,d\xi = \frac{1}{2\pi i}\cdot 2\pi i = 1.$$

- $x \neq y$: in this case, our integral is:

$$P(x;0) = \frac{1}{2\pi i}\oint_c \xi^{x-y-1}e^{\left(\frac{p}{\xi}+q\xi-1\right)\cdot 0}\,d\xi = \frac{1}{2\pi i}\oint_c \xi^{n}\cdot 1\,d\xi = 0.$$

  where $n \neq -1$, since $x \neq y$.

Therefore, indeed, $P(x;t)$ satisfies the initial condition $P(x;0) = \delta_{xy}$, and hence is our desired solution.

2.2.2 N=2.

Next, suppose that we have 2 particles that start moving randomly on $\mathbb{Z}$ at time $t = 0$, such that at time $t = 0$ the position of the left-most particle is at $y_1 \in \mathbb{Z}$, and the position of the right-most particle is at $y_2 \in \mathbb{Z}$, $y_1 < y_2$. Then, we denote the probability that at time t the left-most particle is at $x_1 \in \mathbb{Z}$, and the right-most particle is at $x_2 \in \mathbb{Z}$ by $P_{(y_1,y_2)}(x_1, x_2; t) = P_{\mathbf{y}}(x_1, x_2; t)$. To simplify the notation, we write $P_{\mathbf{y}}(x_1, x_2; t) = P(x_1, x_2; t)$. Then, our forward equation can now take 2 different forms, depending on the mutual position of particles:

Case 1: the particles are far from each other ($x_2 > x_1 + 1$):

$$\frac{d}{dt}P(x_1, x_2; t) = pP(x_1 - 1, x_2; t) + pP(x_1, x_2 - 1; t) + qP(x_1 + 1, x_2; t) + qP(x_1, x_2 + 1; t) - 2P(x_1, x_2; t).$$

Case 2: the particles are next to each other ($x_2 = x_1 + 1 = x + 1$)

$$\frac{d}{dt}P(x, x+1; t) = pP(x-1, x+1; t) + qP(x, x+2; t) - P(x, x+1; t).$$

These equations are supplied with the initial condition:

$$P(x_1, x_2; 0) = P_{\mathbf{y}}(x_1, x_2; 0) = \delta_{\mathbf{xy}} = \delta_{x_1 y_1} \cdot \delta_{x_2 y_2}.$$

To simplify our analysis, we would like to have a single differential equation. To do this, we would need to define a function $u(x_1, x_2; t)$ to be the solution to the forward equations above, but defined on an extended domain: $u : \overline{\Omega} \times [0; \infty) \longrightarrow [0; 1]$, where $\overline{\Omega} = \{(x, y) \in \mathbb{Z}^2 : \quad x \le y\} \supset \Omega$, so that we have: $u \upharpoonright_{\Omega \times [0;\infty)} = P$. The reason for this extension is that we will need the term $u(x, x; t)$, which takes the input that doesn't belong to the state space of our Markov process.

And so, to combine our 2 cases of the forward equation into a single equation, equalize $\frac{d}{dt}u(x_1, x_2; t)$ from case 1 subject to $x_2 = x_1 + 1 = x + 1$ with $\frac{d}{dt}u(x, x+1; t)$ from case 2:

$$\begin{aligned} pu(x-1, x+1; t) + pu(x, x; t) + qu(x+1, x+1; t) + qu(x, x+2; t) - 2u(x, x+1; t) \\ = pu(x-1, x+1; t) + qu(x, x+2; t) - u(x, x+1; t). \end{aligned}$$

This leads us to the boundary condition:

$$pu(x, x; t) + qu(x+1, x+1; t) - u(x, x+1; t) = 0.$$

And so, we finally get a single differential equation:

$$\begin{aligned} \frac{d}{dt}u(x_1, x_2; t) = pu(x_1 - 1, x_2; t) + pu(x_1, x_2 - 1; t) \\ + qu(x_1 + 1, x_2; t) + qu(x_1, x_2 + 1; t) - 2u(x_1, x_2; t). \end{aligned}$$

As before, we solve this by separation of variables: $u(x_1, x_2; t) = u(x_1, x_2) \cdot T(t)$. Then, we immediately get: $T(t) = Ae^{\epsilon t}$, so that our equation becomes:

$$\begin{aligned} u(x_1, x_2) \cdot \epsilon = pu(x_1 - 1, x_2) + pu(x_1, x_2 - 1) \\ + qu(x_1 + 1, x_2) + qu(x_1, x_2 + 1) - 2u(x_1, x_2). \end{aligned}$$

Such equations are solved by substitution: $u(x_1, x_2) = \xi_1^{x_1-1}\xi_2^{x_2-1}$. Then, we get:

$$\epsilon(\xi_1,\xi_2) = p\frac{u(x_1-1,x_2)}{u(x_1,x_2)} + p\frac{u(x_1,x_2-1)}{u(x_1,x_2)} + q\frac{u(x_1+1,x_2)}{u(x_1,x_2)} + q\frac{u(x_1,x_2+1)}{u(x_1,x_2)} - 2\frac{u(x_1,x_2)}{u(x_1,x_2)}$$
$$= \frac{p}{\xi_1} + q\xi_1 - 1 + \frac{p}{\xi_2} + q\xi_2 - 1 = \epsilon_1(\xi_1) + \epsilon_2(\xi_2).$$

And so we get a general solution:

$$u(x_1,x_2;t) = u(x_1,x_2)\cdot T(t) = \xi_1^{x_1-1}\xi_2^{x_2-1}\cdot Ae^{\epsilon(\xi_1,\xi_2)t} = A(\xi_1,\xi_2)\xi_1^{x_1-1}\xi_2^{x_2-1}e^{\epsilon(\xi_1,\xi_2)t}.$$

Now, notice that the solution is symmetric in the variables ξ_1 and ξ_2; and so, by linearity of the differential equation, we get even more general solution to the forward equation:

$$u(x_1,x_2;t) = A_{12}(\xi_1,\xi_2)\xi_1^{x_1-1}\xi_2^{x_2-1}e^{\epsilon(\xi_1,\xi_2)t} + A_{21}(\xi_1,\xi_2)\xi_2^{x_1-1}\xi_1^{x_2-1}e^{\epsilon(\xi_1,\xi_2)t}.$$

Now, we impose the boundary condition:

$$p(A_{12}(\xi_1,\xi_2)\xi_1^{x-1}\xi_2^{x-1} + A_{21}(\xi_1,\xi_2)\xi_2^{x-1}\xi_1^{x-1})e^{\epsilon(\xi_1,\xi_2)t} + q(A_{12}(\xi_1,\xi_2)\xi_1^{x}\xi_2^{x}$$
$$+A_{21}(\xi_1,\xi_2)\xi_2^{x}\xi_1^{x})e^{\epsilon(\xi_1,\xi_2)t} - (A_{12}(\xi_1,\xi_2)\xi_1^{x-1}\xi_2^{x} + A_{21}(\xi_1,\xi_2)\xi_2^{x-1}\xi_1^{x})e^{\epsilon(\xi_1,\xi_2)t} = 0.$$

This gives us the expression for A_{21}:

$$A_{21} = -\frac{p + q\xi_1\xi_2 - \xi_2}{p + q\xi_1\xi_2 - \xi_1}A_{12} =: S_{21}A_{12}.$$

And so our general solution finally becomes:

$$u(x_1,x_2;t) = (A_{12}(\xi_1,\xi_2)\xi_1^{x_1-1}\xi_2^{x_2-1} + S_{21}A_{12}(\xi_1,\xi_2)\xi_2^{x_1-1}\xi_1^{x_2-1})e^{\epsilon(\xi_1,\xi_2)t}.$$

Now, we need to satisfy the initial condition. Choose $A_{12}(\xi_1,\xi_2) = \xi_1^{-y_1}\xi_2^{-y_2}$, and construct the contour integral:

$$u(x_1,x_2;t) = \left(\frac{1}{2\pi i}\right)^2 \oint_{c_2}\oint_{c_1} \left(\xi_1^{x_1-y_1-1}\xi_2^{x_2-y_2-1} + S_{21}(\xi_1,\xi_2)\xi_2^{x_1-y_2-1}\xi_1^{x_2-y_1-1}\right)e^{\epsilon(\xi_1,\xi_2)t}\, d\xi_1 d\xi_2$$

where the contours c_1 and c_2 are around the points $\xi_1 = 0$ and $\xi_2 = 0$, and are so small that all poles of the integrand are outside them. Then, it is not difficult to verify that this expression satisfies the initial condition: let $t = 0$; then, there are 2 cases: either $\mathbf{x} = \mathbf{y}$, or $\mathbf{x} \neq \mathbf{y}$.

- $\mathbf{x} = \mathbf{y}$: in this case, we have:

$$u(x_1,x_2;0) = \left(\frac{1}{2\pi i}\right)^2 \oint_{c_2}\oint_{c_1} \left(\xi_1^{x_1-y_1-1}\xi_2^{x_2-y_2-1} + S_{21}(\xi_1,\xi_2)\xi_2^{x_1-y_2-1}\xi_1^{x_2-y_1-1}\right)e^{0}\, d\xi_1 d\xi_2$$
$$= \left(\frac{1}{2\pi i}\right)^2 \oint_{c_2}\oint_{c_1} \left(\xi_1^{-1}\xi_2^{-1} + S_{21}(\xi_1,\xi_2)\xi_2^{x_1-y_2-1}\xi_1^{x_2-y_1-1}\right) d\xi_1 d\xi_2$$
$$= \left(\frac{1}{2\pi i}\right)^2 \oint_{c_2}\oint_{c_1} \left(\xi_1^{-1}\xi_2^{-1}\right) d\xi_1 d\xi_2 + 0 = \left(\frac{1}{2\pi i}\right)^2 \cdot (2\pi i)^2 = 1.$$

- $\mathbf{x} \neq \mathbf{y}$: in this case, either $x_1 \neq y_1$, or $x_2 \neq y_2$. Then we have:

$$u(x_1,x_2;0) = \left(\frac{1}{2\pi i}\right)^2 \oint_{c_2} \oint_{c_1} \left(\xi_1^{x_1-y_1-1}\xi_2^{x_2-y_2-1} + S_{21}(\xi_1,\xi_2)\xi_2^{x_1-y_2-1}\xi_1^{x_2-y_1-1}\right) e^0 \, d\xi_1 d\xi_2$$

$$= u(x_1,x_2;0) = 0 + \left(\frac{1}{2\pi i}\right)^2 \oint_{c_2} \oint_{c_1} \left(S_{21}(\xi_1,\xi_2)\xi_2^{x_1-y_2-1}\xi_1^{x_2-y_1-1}\right) d\xi_1 d\xi_2.$$

Now, suppose that $x_1 \neq y_1$; then, either $x_1 < y_1$, or $x_1 > y_1$. If $x_1 < y_1$, then clearly $x_1 < y_2$, and so $x_1 - y_2 - 1 < -1$, and thus the integral over ξ_2 is zero. If $x_1 > y_1$, then $x_2 > y_1$, and so $x_2 - y_1 - 1 \geq 0$, and thus the integral over ξ_1 is zero.

Now, suppose that $x_2 \neq y_2$; then, either $x_2 < y_2$, or $x_2 > y_2$. If $x_2 < y_2$, then $x_1 < y_2$, and so $x_1 - y_2 - 1 < -1$, and thus the integral over ξ_2 is zero. If $x_2 > y_2$, then $x_2 > y_1$, and so $x_2 - y_1 - 1 \geq 0$, and thus the integral over ξ_1 is zero.

And so, in any case, we get: $u(x_1,x_2;0) = 0 + 0 = 0$.

Therefore, our construction indeed satisfies the initial condition $P(x_1,x_2;0) = \delta_{\mathbf{xy}} = \delta_{x_1y_1} \cdot \delta_{x_2y_2}$, and hence is our desired solution.

2.2.3 General N.

In the general case, the forward equation can take many different forms, depending on the mutual position of particles; all of them are combined into a single equation:

$$\frac{d}{dt}u(x_1,...,x_N;t) = pu(x_1-1,...,x_N;t) + ... + pu(x_1,...,x_N-1;t)$$
$$+qu(x_1+1,...,x_N;t) + ... + qu(x_1,...,x_N+1;t) - Nu(x_1,...,x_N;t).$$

by imposing the boundary conditions:

$$pu(x_1,...,x_i,x_i,...,x_N;t) + qu(x_1,...,x_i+1,x_i+1,...,x_N;t) - u(x_1,...,x_i,x_i+1,...,x_N;t) = 0.$$

$1 \leq i \leq N$. The initial condition for this equation is given by:

$$P(x_1,...,x_N;0) = \delta_{\mathbf{xy}} = \prod_{i=1}^{N} \delta_{x_iy_i}.$$

Then, it was shown in [5] that the solution to this equation is given by:

$$P_{\mathbf{y}}(\mathbf{x};t) = \sum_{\sigma \in \mathbb{S}_N} \left(\frac{1}{2\pi i}\right)^N \oint_{C_r} ... \oint_{C_r} \left(A_\sigma \prod_{i=1}^{N} \xi_{\sigma(i)}^{x_i - y_{\sigma(i)}-1}\right) e^{\sum_{i=1}^{N}\epsilon(\xi_i)t} \, d\xi_1 ... d\xi_N.$$

where $A_\sigma := \prod\{S_{\alpha\beta} : \{\alpha,\beta\}$ is an inversion in $\sigma\}$, and the term $S_{\alpha\beta}$ is defined as above: $S_{\alpha\beta} := -\frac{p+q\xi_\alpha\xi_\beta-\xi_\alpha}{p+q\xi_\alpha\xi_\beta-\xi_\beta}$.

Remark 2.2.1. The constant $\left(\frac{1}{2\pi i}\right)^N$ is unwieldy and doesn't carry any importance; therefore, further we will include each constant $\frac{1}{2\pi i}$ in the differential $d\xi_i$, $i = 1, ..., N$.

2.3 Finding transition probability in a multi-species ASEP

In this section we now review how to construct and solve the forward equation for a multi-species ASEP. The multi-species ASEP were first introduced by Tracy and Widom in the paper [6], and then developed by Lee in the paper [1].

2.3.1 N = 2

Suppose that we have 2 particles moving on $\mathbb{Z}$, such that one particle belongs to species 1, and the other particle belongs to the species 2. Then, when constructing the forward equation, we need to take much more information into account.

Case 1: the particles are far from each other ($x_2 > x_1 + 1$), and the left particle belongs to species 1, and the right particle belongs to species 2:

This case is exactly as it was in the single-species model, because differences in species start to play a role only when there are interactions:

$$\frac{d}{dt}P(x_1, x_2, 12; t) = pP(x_1 - 1, x_2, 12; t) + pP(x_1, x_2 - 1, 12; t)$$
$$+qP(x_1 + 1, x_2, 12; t) + qP(x_1, x_2 + 1, 12; t) - pP(x_1, x_2, 12; t) - pP(x_1, x_2, 12; t)$$
$$-qP(x_1, x_2, 12; t) - qP(x_1, x_2, 12; t) = pP(x_1 - 1, x_2, 12; t)$$
$$+pP(x_1, x_2 - 1, 12; t) + qP(x_1 + 1, x_2, 12; t) + qP(x_1, x_2 + 1, 12; t) - 2P(x_1, x_2, 12; t).$$

Case 2: the particles are far from each other ($x_2 > x_1 + 1$), and the left particle belongs to species 2, and the right particle belongs to species 1:

Exactly same as before:

$$\frac{d}{dt}P(x_1, x_2, 21; t) = pP(x_1 - 1, x_2, 21; t) + pP(x_1, x_2 - 1, 21; t)$$
$$+qP(x_1 + 1, x_2, 21; t) + qP(x_1, x_2 + 1, 21; t) - 2P(x_1, x_2, 21; t).$$

Case 3: the particles are next to each other ($x_2 = x_1 + 1 = x + 1$), and the left particle belongs to species 1, and the right particle belongs to species 2:

$$\frac{d}{dt}P(x, x+1, 12; t) = pP(x-1, x+1, 12; t) + pP(x, x+1, 21; t) + qP(x, x+2, 12; t)$$
$$-pP(x, x+1, 12; t) - qP(x, x+1, 12; t) - qP(x, x+1, 12; t) = pP(x-1, x+1, 12; t)$$
$$+pP(x, x+1, 21; t) + qP(x, x+2, 12; t) - P(x, x+1, 12; t) - qP(x, x+1, 12; t).$$

As it is now clear, what we have is a system of coupled linear differential equations, and so they can't be solved separately from each other.

Case 4: the particles are next to each other ($x_2 = x_1 + 1 = x + 1$), and the left particle belongs to species 2, and the right particle belongs to species 1:

$$\frac{d}{dt}P(x, x+1, 21; t) = pP(x-1, x+1, 21; t) + qP(x, x+1, 12; t) + qP(x, x+2, 21; t)$$
$$-pP(x, x+1, 21; t) - pP(x, x+1, 21; t) - qP(x, x+1, 21; t) = pP(x-1, x+1, 21; t)$$
$$+qP(x, x+1, 12; t) + qP(x, x+2, 21; t) - pP(x, x+1, 21; t) - P(x, x+1, 21; t).$$

And so, as was remarked above, we must solve these equations together; and therefore, approaching the problem as in the single-species ASEP is not feasible. To solve these equations together, we need to construct a matrix differential equation. Therefore, let us construct the following matrix:

$$\mathbf{P}_{(y_1,y_2)}(x_1, x_2; t) =$$
$$\begin{bmatrix} P_{(y_1,y_2,11)}(x_1, x_2, 11; t); & P_{(y_1,y_2,12)}(x_1, x_2, 11; t); & P_{(y_1,y_2,21)}(x_1, x_2, 11; t); & P_{(y_1,y_2,22)}(x_1, x_2, 11; t) \\ P_{(y_1,y_2,11)}(x_1, x_2, 12; t); & P_{(y_1,y_2,12)}(x_1, x_2, 12; t); & P_{(y_1,y_2,21)}(x_1, x_2, 12; t); & P_{(y_1,y_2,22)}(x_1, x_2, 12; t) \\ P_{(y_1,y_2,11)}(x_1, x_2, 21; t); & P_{(y_1,y_2,12)}(x_1, x_2, 21; t); & P_{(y_1,y_2,21)}(x_1, x_2, 21; t); & P_{(y_1,y_2,22)}(x_1, x_2, 21; t) \\ P_{(y_1,y_2,11)}(x_1, x_2, 22; t); & P_{(y_1,y_2,12)}(x_1, x_2, 22; t); & P_{(y_1,y_2,21)}(x_1, x_2, 22; t); & P_{(y_1,y_2,22)}(x_1, x_2, 22; t) \end{bmatrix}$$

First of all, notice that we don't restrict ourselves to the cases where species are different: we also include the cases where all particles belong to the same species (in particular, the matrix entries $(1,1)$ and $(2,2)$), thus reducing to the single-species ASEP in those cases.

Next, we can immediately see that the terms like $P_{(y_1,y_2,12)}(x_1, x_2, 11; t)$ are always zero: this is because the probability that one particle becomes a species-1 particle, given that at time $t = 0$ it was a species-2 particle, is always zero, at any instant of time, at any configuration (x_1, x_2). Therefore, out matrix simplifies to:

$$\mathbf{P}_{(y_1,y_2)}(x_1, x_2; t) =$$
$$\begin{bmatrix} P_{(y_1,y_2,11)}(x_1, x_2, 11; t); & 0; & 0; & 0 \\ 0; & P_{(y_1,y_2,12)}(x_1, x_2, 12; t); & P_{(y_1,y_2,21)}(x_1, x_2, 12; t); & 0 \\ 0; & P_{(y_1,y_2,12)}(x_1, x_2, 21; t); & P_{(y_1,y_2,21)}(x_1, x_2, 21; t); & 0 \\ 0; & 0; & 0; & P_{(y_1,y_2,22)}(x_1, x_2, 22; t) \end{bmatrix}$$

Now, given this matrix, we want to construct the matrix forward differential equation; for this, we need to consider cases again, but now using matrices of functions instead of plain functions. For notational simplicity, as always, we will denote $\mathbf{P}_{(y_1,y_2)}(x_1,x_2;t)$ by $\mathbf{P}(x_1,x_2;t)$.

Case 1: the particles are far from each other $(x_2 > x_1 + 1)$:

$$\frac{d}{dt}\mathbf{P}(x_1,x_2;t) = p\mathbf{P}(x_1-1,x_2;t) + p\mathbf{P}(x_1,x_2-1;t)$$
$$+q\mathbf{P}(x_1+1,x_2;t) + q\mathbf{P}(x_1,x_2+1;t) - 2\mathbf{P}(x_1,x_2;t).$$

Case 2: the particles are next to each other $(x_2 = x_1 + 1 = x + 1)$

$$\frac{d}{dt}\mathbf{P}(x,x+1;t) = p\mathbf{P}(x-1,x+1;t) + q\mathbf{P}(x,x+2;t) - \mathbf{A}\mathbf{P}(x,x+1;t).$$

where the matrix $\mathbf{A}$ is given by:

$$\mathbf{A} = \begin{bmatrix} 1; & 0; & 0; & 0 \\ 0; & 1+q; & -p; & 0 \\ 0; & -q; & 1+p; & 0 \\ 0; & 0; & 0; & 1 \end{bmatrix}$$

Now, as before, we want to combine everything into a single matrix equation with some boundary conditions. To do this, equalize the different cases with the matrix $\mathbf{U}(x_1,x_2;t)$ substituted for $\mathbf{P}(x_1,x_2;t)$:

$$p\mathbf{U}(x-1,x+1;t) + p\mathbf{U}(x,x;t) + q\mathbf{U}(x+1,x+1;t) + q\mathbf{U}(x,x+2;t)$$
$$-2\mathbf{U}(x,x+1;t) = p\mathbf{U}(x-1,x+1;t) + q\mathbf{U}(x,x+2;t) - \mathbf{A}\mathbf{U}(x,x+1;t).$$

This yields the boundary condition:

$$p\mathbf{U}(x,x;t) + q\mathbf{U}(x+1,x+1;t) = 2\mathbf{U}(x,x+1;t) - \mathbf{A}\mathbf{U}(x,x+1;t) := \mathbf{B}\mathbf{U}(x,x+1;t).$$

Where the matrix $\mathbf{B}$ is given by:

$$\mathbf{B} = \begin{bmatrix} 1; & 0; & 0; & 0 \\ 0; & p; & p; & 0 \\ 0; & q; & q; & 0 \\ 0; & 0; & 0; & 1 \end{bmatrix}$$

2.3.2 General N.

More generally, let $\mathbf{U}(\mathbf{x};t) = \mathbf{U}(x_1,\cdots,x_N;t)$ be an $N^N \times N^N$ matrix whose entries are functions on $\mathbb{Z}^N \times [0,\infty)$ such that $\mathbf{U}(\mathbf{x};t)$ satisfies the matrix differential equation:

$$\begin{aligned}\frac{d}{dt}\mathbf{U}(\mathbf{x};t) = & \sum_{i=1}^{N}\Big[p\mathbf{U}(x_1,\cdots,x_{i-1},x_i-1,x_{i+1},\cdots,x_N;t) + q\mathbf{U}(x_1,\cdots,x_{i-1},x_i+1,x_{i+1},\cdots,x_N;t)\Big] \\ & - N\mathbf{U}(x_1,\cdots,x_N;t),\end{aligned}$$

together with the initial condition

$$\mathbf{U}(x_1,\cdots,x_N;0) = \begin{cases}\mathbf{I}_{N^N} & \text{if } (x_1,\cdots,x_N) = (y_1,\cdots,y_N) \text{ and } x_1 < \cdots < x_N \\ \mathbf{0}_{N^N} & \text{if } (x_1,\cdots,x_N) \neq (y_1,\cdots,y_N) \text{ and } x_1 < \cdots < x_N\end{cases}$$

and the boundary condition

$$\begin{aligned}& p\mathbf{U}(x_1,\cdots,x_{i-1},x_i,x_i,x_{i+2},\cdots,x_N;t) + q\mathbf{U}(x_1,\cdots,x_{i-1},x_i+1,x_i+1,x_{i+2},\cdots,x_N;t) \\ & = \big(\mathbf{I}_N^{\otimes(i-1)} \otimes \mathbf{B} \otimes \mathbf{I}_N^{\otimes(N-i-1)}\big)\mathbf{U}(x_1,\cdots,x_{i-1},x_i,x_i+1,x_{i+2},\cdots,x_N;t)\end{aligned}$$

where the matrix $\mathbf{B}$ is given by:

$$\big[\mathbf{B}\big]_{ij,kl} = \begin{cases}1 & \text{if } ij = kl \text{ with } i = j; \\ p & \text{if either } ij = kl \text{ or } ij = lk \text{ with } i < j; \\ q & \text{if either } ij = kl \text{ or } ij = lk \text{ with } i > j; \\ 0 & \text{for all other cases}\end{cases}$$

Then, it was shown in [6] that the solution to this equation is given by:

$$\sum_{\sigma\in S_N}\mathbf{A}_\sigma\prod_{i=1}^{N}\Big(\xi_{\sigma(i)}^{x_i-y_{\sigma(i)}-1}e^{\varepsilon(\xi_i)t}\Big)$$

For some matrix $\mathbf{A}_\sigma$, and it was shown in [1] how to construct the solution to this equation.

2.4 Permutations, inversions, and subsets

Definition 2.4.1. A permutation σ on a set S is a bijection $\sigma : S \longrightarrow S$.

Definition 2.4.2. Consider a permutation σ on a set $\{1,2,...,N\}$. We write $\sigma = n_1 n_2 ... n_N$ to mean the permutation $\sigma : S \longrightarrow S$ such that $\sigma(i) = n_i$.

Example 2.4.3. Let σ be a permutation on $\{1,2,3\}$. Then, we write $\sigma = 123$ to mean the identity permutation, $\sigma = 132$ to mean the permutation that permutes 3 and 2, and $\sigma = 231$ is the permutation that "rotates everything clockwise".

Definition 2.4.4. Let σ be a permutation on the set $\{1,2,...,N\}$. We say that the pair $(\sigma(n_1), \sigma(n_2))$, $n_1 \in \{1,2,...,N\}$, $n_2 \in \{1,2,...,N\}$ is an inversion in σ if $n_1 < n_2$ but $\sigma(n_1) > \sigma(n_2)$.

Example 2.4.5. Let σ be acting on $\{1,2,3\}$, $\sigma = 231$. Then, there are 2 inversions in σ: $(2,1)$ and $(3,1)$.

Definition 2.4.6. Whenever we write an expression of the form $\prod\limits_{\alpha<\beta} S_{\beta\alpha}$, we imply that the product is taken over all inversions (β, α) in some permutation σ whenever there is a permutation that is clear from the context. Otherwise, if there is no permutation in the context, then the expression of the form $\prod\limits_{\alpha<\beta} S_{\beta\alpha}$ means what it should mean: the product over all pairs (α, β) with $\beta > \alpha$.

Definition 2.4.7. Let $S \subseteq \{1,2,...,N\}$ be a subset. We define $\sum(S)$ as the sum of all elements of S, $S = \{s_1, s_2, ..., s_k\}$: $\sum(S) := \sum\limits_{j=1}^{k} s_j$.

Example 2.4.8. Let S be the whole set $\{1,2,...,N\}$. Then, $\sum(S) = 1 + 2 + ... + N = \frac{N(N+1)}{2}$.

Remark 2.4.9. In general, for the subset S of cardinality k, we have the following inequality: $\sum(S) \geq \frac{k(k+1)}{2}$, because $s_i \geq i$, $\forall i$.

Chapter 3

Towards the main results

In this chapter we are experimenting with the special cases to find out what we are looking for: the probability that the rightmost particle η belonging to species-1 is at the spot x at time t, given that all other particles belong to species-2: $\mathbb{P}_Y(\eta(t) = x)$. In the paper [1] Lee discovered how to obtain the probabilities $P_Y(X, t; \pi) = P_{(y_1,...,y_N)}(x_1, ..., x_N, t; n_1 n_2 ... n_N)$ that at time t, the particle that belongs to species n_i is at x_i, $x_1 < x_2 < ... < x_N$. Therefore, by writing the event $\{\eta(t) = x\}$ as the disjoint union of the events $\{(x_1 = a_1, x_2 = a_2, ..., x_{m-1} = a_{m-1}, x_m = x, x_{m+1} = a_{m+1}, ..., x_N = a_N, 2...212...2)\}$, we immediately see that:

$$\mathbb{P}_Y(\eta(t) = x) = \sum_{i_{N-1}=1}^{\infty} ... \sum_{i_1=1}^{\infty} P_Y(x - i_{N-1} - ... - i_1, ..., x - i_1, x; 2...21)$$
$$+ \sum_{i_{N-1}=1}^{\infty} ... \sum_{i_1=1}^{\infty} P_Y(x - i_{N-1} - ... - i_2, ..., x - i_2, x, x + i_1; 2...212)$$
$$+ ... + \sum_{i_{N-1}=1}^{\infty} ... \sum_{i_1=1}^{\infty} P_Y(x, x + i_1, ..., x + i_1 + ... + i_{N-1}; 12...2).$$

In another paper [2], Lee and Raimbekov discovered how to find the terms $P_Y(x_1 = a_1, x_2 = a_2, ..., x_{m-1} = a_{m-1}, x_m = x, x_{m+1} = a_{m+1}, ..., x_N = a_N, 2...212...2)$ explicitly in the special case when we have a 2-species model with all but one particle belonging to a single species. Our goal in this chapter is to discover the main patterns of our general particle system by studying the special cases, and to lay the foundations for future work.

3.1 $N = 2$

Assume the initial state $Y = (y_1, y_2, 21)$. Then, using the results from the paper [2], we get that the transition probabilities are as follows:

$$P_Y(x_1, x_2; 21) = \int_{c_2}\int_{c_1}\Big(\xi_1^{x_1}\xi_2^{x_2} + \frac{(p - q\xi_2)(\xi_1 - 1)}{p + q\xi_1\xi_2 - \xi_1}\xi_2^{x_1}\xi_1^{x_2}\Big)\xi_1^{-y_1-1}\xi_2^{-y_2-1}e^{\left(\frac{p}{\xi_1}+q\xi_1-1+\frac{p}{\xi_2}+q\xi_2-1\right)t}d\xi_1 d\xi_2.$$

and

$$P_Y(x_1, x_2; 12) = \int_{c_2}\int_{c_1}\Big(\frac{p(\xi_2 - \xi_1)}{p + q\xi_1\xi_2 - \xi_1}\xi_2^{x_1}\xi_1^{x_2}\Big)\xi_1^{-y_1-1}\xi_2^{-y_2-1}e^{\left(\frac{p}{\xi_1}+q\xi_1-1+\frac{p}{\xi_2}+q\xi_2-1\right)t}d\xi_1 d\xi_2.$$

where c_1 and c_2 are the circles centered at the origin with radius less than 1 that do not include any singularities except at the origin. Let $\eta(t)$ be the position of the species-1 particle at time t. We want to compute

$$\mathbb{P}_Y(\eta(t) = x) = \sum_{i=1}^{\infty} P_Y(x - i, x; 21) + \sum_{i=1}^{\infty} P_Y(x, x + i; 12).$$

Let

$$W_{12}(\xi_1, \xi_2, x, t) := \xi_1^{x-y_1-1}\xi_2^{x-y_2-1}e^{\left(\frac{p}{\xi_1}+q\xi_1-1+\frac{p}{\xi_2}+q\xi_2-1\right)t}.$$

for notational simplicity.

- $\sum_{i=1}^{\infty} P_Y(x - i, x; 21)$: there are two term in the integrand. First, we compute

  $$\sum_{i=1}^{\infty}\int_{c_2}\int_{c_1}\xi_1^{-i}\ W_{12}(\xi_1, \xi_2, x, t)d\xi_1 d\xi_2.$$

  Since the radius of the contour is less than 1, the series diverges. But it is possible to deform the contour to a circle with radius larger than 1 because the only singularity of the integrand is at the origin. Let us call the circle C. Hence,

  $$\begin{aligned}\sum_{i=1}^{\infty}\int_{c_2}\int_{c_1}\xi_1^{-i}\ W_{12}(\xi_1, \xi_2, x, t)d\xi_1 d\xi_2 &= \sum_{i=1}^{\infty}\int_{c_2}\int_{C_1}\xi_1^{-i}\ W_{12}(\xi_1, \xi_2, x, t)d\xi_1 d\xi_2\\ &= \int_{c_2}\int_{C_1}\frac{1}{\xi_1 - 1}\ W_{12}(\xi_1, \xi_2, x, t)d\xi_1 d\xi_2\\ &= \int_{c_2}\int_{c_1}\frac{1}{\xi_1 - 1}\ W_{12}(\xi_1, \xi_2, x, t)d\xi_1 d\xi_2 + \text{ (Residue at } \xi_1 = 1).\end{aligned}$$

  The residue at $\xi = 1$ is

  $$\int_{c_2} W_{12}(1, \xi_2, x, t)d\xi_2 = \int_{c_2}\xi_2^{x-y_2-1}\ e^{\left(\frac{p}{\xi_2}+q\xi_2-1\right)t}d\xi_2.$$

We obtained

$$\sum_{i=1}^{\infty}\int_{c_2}\int_{c_1}\xi_1^{-i}\,W_{12}(\xi_1,\xi_2,x,t)d\xi_1 d\xi_2=\int_{c_2}\int_{c_1}\frac{1}{\xi_1-1}\,W_{12}(\xi_1,\xi_2,x,t)d\xi_1 d\xi_2+\int_{c_2}\xi_2^{x-y_2}\,e^{\left(\frac{p}{\xi_2}+q\xi_2-1\right)t}d\xi_2. \tag{3.1}$$

Now, we compute

$$\sum_{i=1}^{\infty}\int_{c_2}\int_{c_1}\frac{(p-q\xi_2)(\xi_1-1)}{p+q\xi_1\xi_2-\xi_1}\xi_2^{-i}\,W_{12}(\xi_1,\xi_2,x,t)d\xi_1 d\xi_2.$$

As before, we need to enlarge our contour c_2, but this time without approaching poles of the function $\frac{(p-q\xi_2)(\xi_1-1)}{p+q\xi_1\xi_2-\xi_1}$. We have:

$$\frac{(p-q\xi_2)(\xi_1-1)}{p+q\xi_1\xi_2-\xi_1}=\frac{(p-q\xi_2)(\xi_1-1)}{p-\xi_1(1-q\xi_2)}.$$

To avoid approaching poles, it suffices to impose: $|\xi_1(1-q\xi_2)|<|p|=p$. And so, consider:

$$|\xi_1(1-q\xi_2)|=|\xi_1|\cdot|1-q\xi_2|\le|\xi_1|\cdot(1+|q\xi_2|)=|\xi_1|\cdot(1+q|\xi_2|)<p.$$

After some calculations, this becomes:

$$|\xi_2|<\frac{1}{q}\left[\frac{p}{|\xi_1|}-1\right].$$

And so, in order for this inequality to make any sense, we must have that $\frac{p}{|\xi_1|}>1$. But this is not enough. We also need to have that $|\xi_2|>1$. This is possible only if:

$$1<|\xi_2|<\frac{1}{q}\left[\frac{p}{|\xi_1|}-1\right];\quad\Rightarrow\quad 1<\frac{1}{q}\left[\frac{p}{|\xi_1|}-1\right].$$

This becomes:

$$q+1<\frac{p}{|\xi_1|};\quad\Rightarrow\quad|\xi_1|<\frac{p}{q+1}.$$

And so, let c_1 be any sufficiently small contour, so that $\xi_1\in c_1\ \Rightarrow\ |\xi_1|<\frac{p}{q+1}$. Next, let S_2 be any contour in the annulus $1<R<\frac{1}{q}\left[\frac{p}{|\xi_1|}-1\right]$. Then, we can finally compute our integral:

$$\begin{aligned}&\sum_{i=1}^{\infty}\int_{c_1}\int_{S_2}\frac{(p-q\xi_2)(\xi_1-1)}{p+q\xi_1\xi_2-\xi_1}\xi_2^{-i}\,W_{12}(\xi_1,\xi_2,x,t)d\xi_2 d\xi_1\\&=\int_{c_1}\int_{S_2}\left(\frac{(p-q\xi_2)(\xi_1-1)}{p+q\xi_1\xi_2-\xi_1}\right)\cdot\left(\frac{1}{\xi_2-1}\right)\,W_{12}(\xi_1,\xi_2,x,t)d\xi_2 d\xi_1\\&=\int_{c_1}\int_{c_2}\left(\frac{(p-q\xi_2)(\xi_1-1)}{p+q\xi_1\xi_2-\xi_1}\right)\cdot\left(\frac{1}{\xi_2-1}\right)\,W_{12}(\xi_1,\xi_2,x,t)d\xi_2 d\xi_1+(\text{Residue at }\xi_2=1).\end{aligned}$$

The residue at $\xi_2 = 1$ is:

$$\int_{c_1} \left(\frac{(p - q \cdot 1)(\xi_1 - 1)}{p + q\xi_1 \cdot 1 - \xi_1} \right) W_{12}(\xi_1, 1, x, t)\, d\xi_1 = \int_{c_1} \left(\frac{(p - q)(\xi_1 - 1)}{p + q\xi_1 - \xi_1} \right) \xi_1^{x - y_1 - 1} e^{\left(\frac{p}{\xi_1} + q\xi_1 - 1\right)t}\, d\xi_1$$
$$= \int_{c_1} \left(\frac{(p - q)(\xi_1 - 1)}{p - p\xi_1} \right) \xi_1^{x - y_1 - 1} e^{\left(\frac{p}{\xi_1} + q\xi_1 - 1\right)t}\, d\xi_1 = \int_{c_1} \left(\frac{(p - q)(\xi_1 - 1)}{p(1 - \xi_1)} \right) \xi_1^{x - y_1 - 1} e^{\left(\frac{p}{\xi_1} + q\xi_1 - 1\right)t}\, d\xi_1$$
$$= \int_{c_1} \frac{q - p}{p} \xi_1^{x - y_1 - 1} e^{\left(\frac{p}{\xi_1} + q\xi_1 - 1\right)t}\, d\xi_1.$$

- $\sum_{i=1}^{\infty} P_Y(x, x + i; 12)$:

$$\sum_{i=1}^{\infty} P_Y(x, x + i; 12) = \sum_{i=1}^{\infty} \int_{c_2} \int_{c_1} \Big(\frac{p(\xi_2 - \xi_1)}{p + q\xi_1\xi_2 - \xi_1} \xi_1^i \Big) W_{12}(\xi_1, \xi_2, x, t) d\xi_1 d\xi_2.$$

Observe that this time, the series converges if $|\xi_1| < 1$, $\forall \xi_1 \in c_1$.

As before, consider the problematic function:

$$\frac{1}{p + q\xi_1\xi_2 - \xi_1} = \frac{1}{p - \xi_1 \cdot (1 - q\xi_2)}.$$

We want to avoid approaching singularity: $|\xi_1(1 - q\xi_2)| < |p| = p$. Then, consider:

$$|\xi_1(1 - q\xi_2)| = |\xi_1| \cdot |1 - q\xi_2| \leq |\xi_1| \cdot (1 + |q\xi_2|) = |\xi_1| \cdot (1 + q|\xi_2|) < p.$$

And so, it suffices to impose:

$$|\xi_1| < \frac{p}{(1 + q|\xi_2|)}.$$

On the other hand, we also want to have $|\xi_1| < 1$. And so, let c_1 be any contour such that $\xi_1 \in c_1 \;\Rightarrow\; |\xi_1| < \min\left\{ \frac{p}{(1 + q|\xi_2|)}; 1 \right\}$. Then, we are finally ready to compute our integral:

$$\sum_{i=1}^{\infty} P_Y(x, x + i; 12) = \sum_{i=1}^{\infty} \int_{c_2} \int_{c_1} \frac{p(\xi_2 - \xi_1)}{p + q\xi_1\xi_2 - \xi_1} \xi_1^i\, W_{12}(\xi_1, \xi_2, x, t) d\xi_1 d\xi_2$$
$$= \int_{c_2} \int_{c_1} \left(\frac{p(\xi_2 - \xi_1)}{p + q\xi_1\xi_2 - \xi_1} \right) \cdot \left(\frac{\xi_1}{1 - \xi_1} \right) W_{12}(\xi_1, \xi_2, x, t) d\xi_1 d\xi_2.$$

And so, the whole expression becomes:

$$
\begin{aligned}
\mathbb{P}_Y(\eta(t) = x) &= \sum_{i=1}^{\infty} P_Y(x-i, x; 21) + \sum_{i=1}^{\infty} P_Y(x, x+i; 12) \\
&= \int_{c_2}\int_{c_1} \frac{1}{\xi_1 - 1}\, W_{12}(\xi_1, \xi_2, x, t) d\xi_1 d\xi_2 \\
+ \int_{c_1}\int_{c_2} & \left(\frac{(p - q\xi_2)(\xi_1 - 1)}{p + q\xi_1\xi_2 - \xi_1}\right)\cdot\left(\frac{1}{\xi_2 - 1}\right) W_{12}(\xi_1, \xi_2, x, t) d\xi_2 d\xi_1 \\
+ \int_{c_2}\int_{c_1} & \left(\frac{p(\xi_2 - \xi_1)}{p + q\xi_1\xi_2 - \xi_1}\right)\cdot\left(\frac{\xi_1}{1 - \xi_1}\right) W_{12}(\xi_1, \xi_2, x, t) d\xi_1 d\xi_2 \\
+ \int_{c_2} \xi_2^{x-y_2-1}\, e^{\left(\frac{p}{\xi_2} + q\xi_2 - 1\right)t} d\xi_2 &+ \int_{c_1} \frac{q-p}{p} \xi_1^{x-y_1-1} e^{\left(\frac{p}{\xi_1} + q\xi_1 - 1\right)t}\, d\xi_1.
\end{aligned}
$$

Adding the first 3 summands together yields a nice formula:

$$
\begin{aligned}
\mathbb{P}_Y(\eta(t) = x) &= \sum_{i=1}^{\infty} P_Y(x-i, x; 21) + \sum_{i=1}^{\infty} P_Y(x, x+i; 12) \\
= \oint_{c_2}\oint_{c_1} & -\frac{(2p-1)(\xi_1 - \xi_2)(\xi_1\xi_2 - 1)}{(\xi_1 - 1)(\xi_2 - 1)(\xi_1 - p - q\xi_1\xi_2)} W_{12}(\xi_1, \xi_2, x, t)\, d\xi_1 d\xi_2 \\
+ \frac{q-p}{p} \oint_{c_1} & \xi_1^{x-y_1-1} e^{\left(\frac{p}{\xi_1} + q\xi_1 - 1\right)t}\, d\xi_1 + \oint_{c_2} \xi_2^{x-y_2-1}\, e^{\left(\frac{p}{\xi_2} + q\xi_2 - 1\right)t} d\xi_2.
\end{aligned}
$$

3.2 $N = 3$

Assume the initial state $Y = (y_1, y_2, y_3, 221)$. Then, using the results of the paper [2] we obtain the following table:

σ	$[\mathbf{A}_\sigma]_{221,221}$	$[\mathbf{A}_\sigma]_{212,221}$	$[\mathbf{A}_\sigma]_{122,221}$
123	1	0	0
132	Q_{32}	pT_{32}	0
213	S_{21}	0	0
231	$S_{21}Q_{31}$	$S_{21}pT_{31}$	0
312	$Q_{32}S_{31}$	$pT_{32}Q_{31}$	$pT_{32}pT_{31}$
321	$S_{21}Q_{31}S_{32}$	$S_{21}pT_{31}Q_{32}$	$pT_{32}pT_{31}S_{21}$

$$
\mathbb{P}_Y(\eta(t) = x) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_Y(x-i-j, x-i, x; 221) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_Y(x-i, x, x+j; 212) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_Y(x, x+i, x+i+j; 122).
$$

As before, define:

$$W_{123}(\xi_1,\xi_2,\xi_3,x,t) := \xi_1^{x-y_1-1}\xi_2^{x-y_2-1}\xi_3^{x-y_3-1}e^{\left(\frac{p}{\xi_1}+q\xi_1-1+\frac{p}{\xi_2}+q\xi_2-1+\frac{p}{\xi_3}+q\xi_3-1\right)t}.$$

Remark 3.2.1. Let $W_i(\xi,x,t) := \xi^{x-y_i-1}e^{\left(\frac{p}{\xi}+q\xi-1\right)t}$. Then, we have:

1. $W_{ijk}(\xi_1,\xi_2,\xi_3,x,t) = W_i(\xi_1,x,t)\cdot W_j(\xi_2,x,t)\cdot W_k(\xi_3,x,t)$;
2. $W_{ij}(\xi_1,\xi_2,x,t) = W_i(\xi_1,x,t)\cdot W_j(\xi_2,x,t)$.
3. $W_i(1,x,t) = 1^{x-y_i-1}e^{\left(\frac{p}{1}+q\cdot 1-1\right)t} = 1\cdot e^0 = 1$.

In particular, we have:

1. $W_{ijk}(1,\xi_2,\xi_3,x,t) = W_{jk}(\xi_2,\xi_3,x,t)$;
2. $W_{ijk}(\xi_1,1,\xi_3,x,t) = W_{ik}(\xi_1,\xi_3,x,t)$;
3. $W_{ijk}(\xi_1,\xi_2,1,x,t) = W_{ij}(\xi_1,\xi_2,x,t)$.

We also have:

- $W_{ijk}(\xi_1,\xi_2,\xi_3,x,t) = \xi_1^x\xi_2^x\xi_3^x\cdot W_{ijk}(\xi_1,\xi_2,\xi_3,0,t)$.

Definition 3.2.2.

$$S_{\beta\alpha} := -\frac{p+q\xi_\alpha\xi_\beta-\xi_\beta}{p+q\xi_\alpha\xi_\beta-\xi_\alpha};\quad T_{\beta\alpha} := \frac{\xi_\beta-\xi_\alpha}{p+q\xi_\alpha\xi_\beta-\xi_\alpha};\quad Q_{\beta\alpha} := \frac{(p-q\xi_\beta)(\xi_\alpha-1)}{p+q\xi_\alpha\xi_\beta-\xi_\alpha}.$$

Notice that all $S_{\beta\alpha}$, $T_{\beta\alpha}$, and $Q_{\beta\alpha}$ have identical denominators.

The transition probabilities are as follows:

$$P_Y(x_1,x_2,x_3;221) = \int_{c_3}\int_{c_2}\int_{c_1}\Big(\xi_1^{x_1}\xi_2^{x_2}\xi_3^{x_3} + Q_{32}\xi_1^{x_1}\xi_3^{x_2}\xi_2^{x_3} + S_{21}\xi_2^{x_1}\xi_1^{x_2}\xi_3^{x_3} + S_{21}Q_{31}\xi_2^{x_1}\xi_3^{x_2}\xi_1^{x_3}$$
$$+Q_{32}S_{31}\xi_3^{x_1}\xi_1^{x_2}\xi_2^{x_3} + S_{21}Q_{31}S_{32}\xi_3^{x_1}\xi_2^{x_2}\xi_1^{x_3}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\,d\xi_1 d\xi_2 d\xi_3.$$

$$P_Y(x_1,x_2,x_3;212) = \int_{c_3}\int_{c_2}\int_{c_1}\Big(0 + pT_{32}\xi_1^{x_1}\xi_3^{x_2}\xi_2^{x_3} + 0 + S_{21}pT_{31}\xi_2^{x_1}\xi_3^{x_2}\xi_1^{x_3}$$
$$+pT_{32}Q_{31}\xi_3^{x_1}\xi_1^{x_2}\xi_2^{x_3} + S_{21}pT_{31}Q_{32}\xi_3^{x_1}\xi_2^{x_2}\xi_1^{x_3}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\,d\xi_1 d\xi_2 d\xi_3.$$

$$P_Y(x_1,x_2,x_3;122) = \int_{c_3}\int_{c_2}\int_{c_1}\Big(pT_{32}pT_{31}\xi_3^{x_1}\xi_1^{x_2}\xi_2^{x_3} + S_{21}pT_{31}pT_{32}\xi_3^{x_1}\xi_2^{x_2}\xi_1^{x_3}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3.$$

Now, we need to put everything together. The first series is:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_Y(x-i-j,x-i,x;221) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(\xi_1^{x-i-j}\xi_2^{x-i}\xi_3^{x}$$
$$+Q_{32}\xi_1^{x-i-j}\xi_3^{x-i}\xi_2^{x} + S_{21}\xi_2^{x-i-j}\xi_1^{x-i}\xi_3^{x} + S_{21}Q_{31}\xi_2^{x-i-j}\xi_3^{x-i}\xi_1^{x}$$
$$+Q_{32}S_{31}\xi_3^{x-i-j}\xi_1^{x-i}\xi_2^{x} + S_{21}Q_{31}S_{32}\xi_3^{x-i-j}\xi_2^{x-i}\xi_1^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3.$$

The second series is:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_Y(x-i,x,x+j;212) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(pT_{32}\xi_1^{x-i}\xi_3^{x}\xi_2^{x+j} + S_{21}pT_{31}\xi_2^{x-i}\xi_3^{x}\xi_1^{x+j}$$
$$+pT_{32}Q_{31}\xi_3^{x-i}\xi_1^{x}\xi_2^{x+j} + S_{21}pT_{31}Q_{32}\xi_3^{x-i}\xi_2^{x}\xi_1^{x+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3.$$

The third series is:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_Y(x,x+i,x+i+j;122) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(pT_{32}pT_{31}\xi_3^{x}\xi_1^{x+i}\xi_2^{x+i+j}$$
$$+S_{21}pT_{31}pT_{32}\xi_3^{x}\xi_2^{x+i}\xi_1^{x+i+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3.$$

As before, we need to split each integral into a sum of integrals, and deal with each integral separately. There are 6 summands in the first integral, 4 summands in the second integral, and 2 summands in the third integral, and so there are 12 integrals to consider overall:

- **The first integral:**

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\xi_1^{x-i-j}\xi_2^{x-i}\xi_3^{x}\ \ W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3.$$

  This integral is the most basic, as there are no singularities to consider. First of all, observe that $\xi_1^{x-i-j}\xi_2^{x-i}\xi_3^{x} = \xi_1^{-j}(\xi_1\xi_2)^{-i}(\xi_1^x\xi_2^x\xi_3^x)$. And so, for the series to converge, we must have both $|\xi_1| > 1$ and $|\xi_1\xi_2| > 1$. For this, it suffices to deform only one contour c_1 into the contour

C_1 so that $\xi_1 \in C_1 \implies |\xi_1| > \max\{1; \max_{\xi_2 \in c_2} \left\{\frac{1}{|\xi_2|}\right\}\}$. Then the series becomes:

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1} \xi_1^{x-i-j}\xi_2^{x-i}\xi_3^{x}\ W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3\\
&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{C_1} \xi_1^{x-i-j}\xi_2^{x-i}\xi_3^{x}\ W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3\\
&=\int_{c_3}\int_{c_2}\int_{C_1} \frac{1}{\xi_1-1}\cdot\frac{1}{\xi_1\xi_2-1}\xi_1^{x}\xi_2^{x}\xi_3^{x}\ W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3\\
&=\int_{c_3}\int_{c_2}\int_{C_1} \frac{1}{\xi_1-1}\cdot\frac{1}{\xi_1}\cdot\frac{1}{\xi_2-\frac{1}{\xi_1}}\xi_1^{x}\xi_2^{x}\xi_3^{x}\ W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3\\
&=\int_{c_3}\int_{c_2}\int_{c_1} \frac{1}{\xi_1-1}\cdot\frac{1}{\xi_1\xi_2-1}\ W_{123}(\xi_1,\xi_2,\xi_3,x,t)\ d\xi_1 d\xi_2 d\xi_3\\
&\qquad+\left(\text{Residue at } \xi_2=\frac{1}{\xi_1}\right)+(\text{Residue at } \xi_1=1).
\end{aligned}$$

Next, we need to compute the residues.

The residue at $\xi_2 = \frac{1}{\xi_1}$ is:

$$\int_{c_3}\int_{S_1} \frac{1}{\xi_1-1}\cdot\frac{1}{\xi_1}\ W_{123}\left(\xi_1,\frac{1}{\xi_1},\xi_3,x,t\right)\ d\xi_1 d\xi_3.$$

where the contour S_1 is the contour such that $1 < |\xi_1| < \frac{1}{\xi_2}$. We deform the contour further, to get the next residue:

$$\begin{aligned}
&\int_{c_3}\int_{S_1} \frac{1}{\xi_1-1}\cdot\frac{1}{\xi_1}\ W_{123}\left(\xi_1,\frac{1}{\xi_1},\xi_3,x,t\right)\ d\xi_1 d\xi_3\\
&=\int_{c_3}\int_{c_1} \frac{1}{\xi_1-1}\cdot\frac{1}{\xi_1}\ W_{123}\left(\xi_1,\frac{1}{\xi_1},\xi_3,x,t\right)\ d\xi_1 d\xi_3+(\text{Residue at } \xi_1=1).
\end{aligned}$$

The residues at $\xi_1 = 1$ are:

$$\begin{aligned}
&\int_{c_3}\int_{c_2} \frac{1}{\xi_2}\cdot\frac{1}{1-\frac{1}{\xi_2}}\ W_{123}(1,\xi_2,\xi_3,x,t)\ d\xi_2 d\xi_3+\int_{c_3}\frac{1}{1}\ W_{123}\left(1,\frac{1}{1},\xi_3,x,t\right)\ d\xi_3\\
&=\int_{c_3}\int_{c_2}\frac{1}{\xi_2-1}\ W_{23}(\xi_2,\xi_3,x,t)\ d\xi_2 d\xi_3+\int_{c_3} W_3\left(\xi_3,x,t\right)\ d\xi_3.
\end{aligned}$$

- **The second integral:**

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\left(Q_{32}\xi_1^{x-i-j}\xi_3^{x-i}\xi_2^{x}\right)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3.$$

This time we need to take care of avoiding potential singularities of the function Q_{32}. Simplify a bit:

$$\Big(Q_{32}\xi_1^{x-i-j}\xi_3^{x-i}\xi_2^{x}\Big) = \Big(\frac{(p-q\xi_3)(\xi_2-1)}{p+q\xi_2\xi_3-\xi_2}\xi_1^{-j}\,(\xi_1\xi_3)^{-i}\,(\xi_1\xi_2\xi_3)^{x}\Big).$$

And so, we need to deform the contours so that we also have: $|q\xi_2\xi_3-\xi_2| \le |\xi_2|(q|\xi_3|+1) < |p| = p$. And so, deform the contour c_1 into C_1 so that $\xi_1 \in C_1 \implies |\xi_1| > \max\{1; \max_{\xi_3\in c_3}\left\{\frac{1}{|\xi_3|}\right\}\}$; also, make the contour c_2 small enough, so that $\xi_2 \in c_2 \implies |\xi_2| < \frac{p}{q|\xi_3|+1}$. Then the series becomes:

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(Q_{32}\xi_1^{x-i-j}\xi_3^{x-i}\xi_2^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3\\
&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{C_1}\Big(Q_{32}\xi_1^{x-i-j}\xi_3^{x-i}\xi_2^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3\\
&=\int_{c_3}\int_{c_2}\int_{C_1}\Big(Q_{32}\frac{1}{\xi_1-1}\cdot\frac{1}{\xi_1\xi_3-1}\xi_1^{x}\xi_3^{x}\xi_2^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3\\
&=\int_{c_3}\int_{c_2}\int_{c_1}\Big(Q_{32}\frac{1}{\xi_1-1}\cdot\frac{1}{\xi_1\xi_3-1}\Big)W_{123}(\xi_1,\xi_2,\xi_3,x,t)\;d\xi_1 d\xi_2 d\xi_3\\
&\quad+\Big(\text{Residue at }\xi_3=\frac{1}{\xi_1}\Big)+(\text{Residue at }\xi_1=1).
\end{aligned}$$

The residue at $\xi_3 = \frac{1}{\xi_1}$ is:

$$\begin{aligned}
&\int_{c_2}\int_{S_1}\Big(Q_{32}\frac{1}{\xi_1-1}\cdot\frac{1}{\xi_1}\Big)W_{123}\Big(\xi_1,\xi_2,\frac{1}{\xi_1},x,t\Big)\;d\xi_1 d\xi_2\\
&=\int_{c_2}\int_{S_1}\Big(\frac{(p\xi_1-q)(\xi_2-1)}{p\xi_1+q\xi_2-\xi_1\xi_2}\cdot\frac{1}{\xi_1-1}\cdot\frac{1}{\xi_1}\Big)W_{123}\Big(\xi_1,\xi_2,\frac{1}{\xi_1},x,t\Big)\;d\xi_1 d\xi_2\\
&=\int_{c_2}\int_{c_1}\Big(\frac{(p\xi_1-q)(\xi_2-1)}{p\xi_1+q\xi_2-\xi_1\xi_2}\cdot\frac{1}{\xi_1-1}\cdot\frac{1}{\xi_1}\Big)W_{123}\Big(\xi_1,\xi_2,\frac{1}{\xi_1},x,t\Big)\;d\xi_1 d\xi_2+(\text{Residue at }\xi_1=1).
\end{aligned}$$

The residues at $\xi_1 = 1$ are:

$$\begin{aligned}
&\int_{c_3}\int_{c_2}\Big(Q_{32}\frac{1}{1\cdot\xi_3-1}\Big)W_{123}(1,\xi_2,\xi_3,x,t)\;d\xi_2 d\xi_3+\int_{c_2}\Big(\frac{(p\cdot1-q)(\xi_2-1)}{p\cdot1+q\xi_2-1\cdot\xi_2}\cdot\frac{1}{1}\Big)W_{123}\Big(1,\xi_2,\frac{1}{1},x,t\Big)\;d\xi_2\\
&=\int_{c_3}\int_{c_2}\Big(Q_{32}\frac{1}{\xi_3-1}\Big)W_{23}(\xi_2,\xi_3,x,t)\;d\xi_2 d\xi_3+\int_{c_2}\Big(\frac{(p-q)(\xi_2-1)}{p+q\xi_2-\xi_2}\Big)W_2\,(\xi_2,x,t)\;d\xi_2\\
&=\int_{c_3}\int_{c_2}\Big(Q_{32}\frac{1}{\xi_3-1}\Big)W_{23}(\xi_2,\xi_3,x,t)\;d\xi_2 d\xi_3+\int_{c_2}\Big(\frac{(p-q)(\xi_2-1)}{p-p\xi_2}\Big)W_2\,(\xi_2,x,t)\;d\xi_2\\
&=\int_{c_3}\int_{c_2}\Big(Q_{32}\frac{1}{\xi_3-1}\Big)W_{23}(\xi_2,\xi_3,x,t)\;d\xi_2 d\xi_3+\int_{c_2}\Big(\frac{(q-p)}{p}\Big)W_2\,(\xi_2,x,t)\;d\xi_2.
\end{aligned}$$

- **The third integral:**

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}\xi_2^{x-i-j}\xi_1^{x-i}\xi_3^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3.$$

This integral is one of the toughest ones. As before, we want to enlarge the contours so that $|\xi_2| > 1$ and $|\xi_2\xi_1| > 1$, while also satisfying: $|q\xi_1\xi_2 - \xi_1| \le |\xi_1|(q|\xi_2| + 1) < |p| = p$. This time, it it also necessary to assume that $q < p$.

Now let us enlarge the contour c_2 into C_2 so that $\xi_2 \in C_2 \implies |\xi_2| > \frac{1}{|\xi_1|}$. Note that for small $|\xi_1|$, the requirement $|\xi_2| > 1$ is automatic. In particular, let c_1 and C_2 be circles, so that $|\xi_2| \equiv Const$, $|\xi_1| \equiv Const$. Then, it follows that $|\xi_2| = \frac{1}{|\xi_1|} + \epsilon$, where $\epsilon > 0$. Then, to avoid singularities, it suffices to have:

$$|\xi_1|(q|\xi_2| + 1) = |\xi_1|\left(q\left(\frac{1}{|\xi_1|} + \epsilon\right) + 1\right) = q + |\xi_1|(q\epsilon + 1) < p.$$

which is clearly true for sufficiently small circle c_1, provided that $q < p$. And so, finally the series becomes:

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}\xi_2^{x-i-j}\xi_1^{x-i}\xi_3^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3\\
&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{C_2}\int_{c_1}\Big(S_{21}\xi_2^{x-i-j}\xi_1^{x-i}\xi_3^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3\\
&=\int_{c_3}\int_{C_2}\int_{c_1}\Big(S_{21}\frac{1}{\xi_2-1}\cdot\frac{1}{\xi_1\xi_2-1}\xi_2^{x}\xi_1^{x}\xi_3^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3\\
&=\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}\frac{1}{\xi_2-1}\cdot\frac{1}{\xi_1\xi_2-1}\Big)W_{123}(\xi_1,\xi_2,\xi_3,x,t)\;d\xi_1 d\xi_2 d\xi_3\\
&\qquad+\left(\text{Residue at }\xi_2=\frac{1}{\xi_1}\right)+(\text{Residue at }\xi_2=1).
\end{aligned}$$

The residue at $\xi_2 = \frac{1}{\xi_1}$ is:

$$\begin{aligned}
&\int_{c_3}\int_{c_1}\Big(S_{21}\frac{1}{\frac{1}{\xi_1}-1}\cdot\frac{1}{\xi_1}\cdot\Big)W_{123}\left(\xi_1,\frac{1}{\xi_1},\xi_3,x,t\right)\;d\xi_1 d\xi_3\\
&=\int_{c_3}\int_{c_1}\Big(\frac{1}{\xi_1}\cdot\frac{\xi_1}{1-\xi_1}\cdot\frac{1}{\xi_1}\Big)W_{123}\left(\xi_1,\frac{1}{\xi_1},\xi_3,x,t\right)\;d\xi_1 d\xi_3\\
&=\int_{c_3}\int_{c_1}\Big(\frac{1}{1-\xi_1}\cdot\frac{1}{\xi_1}\Big)W_{123}\left(\xi_1,\frac{1}{\xi_1},\xi_3,x,t\right)\;d\xi_1 d\xi_3.
\end{aligned}$$

The residue at $\xi_2 = 1$ is:

$$\begin{aligned}
&\int_{c_3}\int_{c_1}\Big(S_{21}\frac{1}{\xi_1\cdot 1-1}\Big)W_{123}(\xi_1,1,\xi_3,x,t)\;d\xi_1 d\xi_3\\
&=\int_{c_3}\int_{c_1}\Big(\frac{q}{p}\cdot\frac{1}{\xi_1-1}\Big)W_{13}(\xi_1,\xi_3,x,t)\;d\xi_1 d\xi_3.
\end{aligned}$$

- **The fourth integral:**

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}Q_{31}\xi_2^{x-i-j}\xi_3^{x-i}\xi_1^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3.$$

As before, we want to enlarge the contour c_2 so that $|\xi_2|>1$, $|\xi_2\xi_3|>1$, $|q\xi_1\xi_2-\xi_1|\le|\xi_1|(q|\xi_2|+1)<|p|=p$, and $|q\xi_1\xi_3-\xi_1|\le|\xi_1|(q|\xi_3|+1)<|p|=p$. Deform the contour c_2 into C_2 so that $\xi_2\in C_2 \implies |\xi_2|>\max\{1;\max_{\xi_3\in c_3}\left\{\frac{1}{|\xi_3|}\right\}\}$; also, make the contour c_1 small enough, so that $\xi_1\in c_1 \implies |\xi_1|<\min\left\{\min_{\xi_2\in c_2}\left\{\frac{p}{q|\xi_2|+1}\right\};\min_{\xi_3\in c_3}\left\{\frac{p}{q|\xi_3|+1}\right\}\right\}=\min_{\xi_2\in c_2}\left\{\frac{p}{q|\xi_2|+1}\right\}$. Then, the series becomes:

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}Q_{31}\xi_2^{x-i-j}\xi_3^{x-i}\xi_1^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3\\
&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{C_2}\int_{c_1}\Big(S_{21}Q_{31}\xi_2^{x-i-j}\xi_3^{x-i}\xi_1^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3\\
&=\int_{c_3}\int_{C_2}\int_{c_1}\Big(S_{21}Q_{31}\frac{1}{\xi_2-1}\cdot\frac{1}{\xi_2\xi_3-1}\xi_2^{x}\xi_3^{x}\xi_1^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3\\
&=\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}Q_{31}\frac{1}{\xi_2-1}\cdot\frac{1}{\xi_2\xi_3-1}\Big)W_{123}(\xi_1,\xi_2,\xi_3,x,t)\ d\xi_1 d\xi_2 d\xi_3\\
&\qquad+\left(\text{Residue at }\xi_3=\frac{1}{\xi_2}\right)+(\text{Residue at }\xi_2=1).
\end{aligned}$$

The residue at $\xi_3=\frac{1}{\xi_2}$ is:

$$\begin{aligned}
&\int_{S_2}\int_{c_1}\Big(S_{21}Q_{31}\frac{1}{\xi_2-1}\cdot\frac{1}{\xi_2}\Big)W_{123}\left(\xi_1,\xi_2,\frac{1}{\xi_2},x,t\right)\ d\xi_1 d\xi_2\\
&=\int_{S_2}\int_{c_1}\Big(S_{21}\cdot\frac{(p\xi_2-q)(\xi_1-1)}{p\xi_2+q\xi_1-\xi_2\xi_1}\cdot\frac{1}{\xi_2-1}\cdot\frac{1}{\xi_2}\Big)W_{123}\left(\xi_1,\xi_2,\frac{1}{\xi_2},x,t\right)\ d\xi_1 d\xi_2\\
&=\int_{c_2}\int_{c_1}\Big(S_{21}\cdot\frac{(p\xi_2-q)(\xi_1-1)}{p\xi_2+q\xi_1-\xi_2\xi_1}\cdot\frac{1}{\xi_2-1}\cdot\frac{1}{\xi_2}\Big)W_{123}\left(\xi_1,\xi_2,\frac{1}{\xi_2},x,t\right)\ d\xi_1 d\xi_2+(\text{Residue at }\xi_2=1).
\end{aligned}$$

The residues at $\xi_2=1$ are:

$$\begin{aligned}
&\int_{c_3}\int_{c_1}\Big(S_{21}Q_{31}\frac{1}{1\cdot\xi_3-1}\Big)W_{123}(\xi_1,1,\xi_3,x,t)\ d\xi_1 d\xi_3\\
&+\int_{c_1}\Big(S_{21}\cdot\frac{(p\cdot 1-q)(\xi_1-1)}{p\cdot 1+q\xi_1-1\cdot\xi_1}\cdot\frac{1}{1}\Big)W_{123}\left(\xi_1,1,1,x,t\right)\ d\xi_1\\
&=\int_{c_3}\int_{c_1}\Big(\frac{q}{p}\cdot Q_{31}\frac{1}{\xi_3-1}\Big)W_{13}(\xi_1,\xi_3,x,t)\ d\xi_1 d\xi_3+\int_{c_1}\Big(\frac{q}{p}\cdot\frac{(p-q)(\xi_1-1)}{p+q\xi_1-\xi_1}\Big)W_1\left(\xi_1,x,t\right)\ d\xi_1\\
&=\int_{c_3}\int_{c_1}\Big(\frac{q}{p}\cdot Q_{31}\frac{1}{\xi_3-1}\Big)W_{13}(\xi_1,\xi_3,x,t)\ d\xi_1 d\xi_3+\int_{c_1}\Big(\frac{q}{p}\cdot\frac{(q-p)}{p}\Big)W_1\left(\xi_1,x,t\right)\ d\xi_1.
\end{aligned}$$

- **The fifth integral:**

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(Q_{32}S_{31}\xi_3^{x-i-j}\xi_1^{x-i}\xi_2^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3.$$

Shrink the contour c_1 and enlarge the contour c_3 into C_3 as in the third integral; next, shrink the contour c_2 as in the second integral. Then, provided that $q < p$, the series becomes:

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(Q_{32}S_{31}\xi_3^{x-i-j}\xi_1^{x-i}\xi_2^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3\\
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{C_3}\int_{c_2}\int_{c_1}\Big(Q_{32}S_{31}\xi_3^{x-i-j}\xi_1^{x-i}\xi_2^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3\\
&\int_{C_3}\int_{c_2}\int_{c_1}\Big(Q_{32}S_{31}\frac{1}{\xi_3-1}\cdot\frac{1}{\xi_1\xi_3-1}\xi_3^{x}\xi_1^{x}\xi_2^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3\\
&\int_{c_3}\int_{c_2}\int_{c_1}\Big(Q_{32}S_{31}\frac{1}{\xi_3-1}\cdot\frac{1}{\xi_1\xi_3-1}\Big)W_{123}(\xi_1,\xi_2,\xi_3,x,t)\;d\xi_1 d\xi_2 d\xi_3\\
&\qquad+\left(\text{Residue at }\xi_3=\frac{1}{\xi_1}\right)+(\text{Residue at }\xi_3=1).
\end{aligned}$$

The residue at $\xi_3 = \frac{1}{\xi_1}$ is:

$$\begin{aligned}
&\int_{c_2}\int_{c_1}\Big(Q_{32}S_{31}\frac{1}{\frac{1}{\xi_1}-1}\cdot\frac{1}{\xi_1}\Big)W_{123}\left(\xi_1,\xi_2,\frac{1}{\xi_1},x,t\right)\;d\xi_1 d\xi_2\\
&=\int_{c_2}\int_{c_1}\Big(\frac{(p\xi_1-q)(\xi_2-1)}{p\xi_1+q\xi_2-\xi_1\xi_2}\cdot\frac{1}{\xi_1}\cdot\frac{1}{1-\xi_1}\Big)W_{123}\left(\xi_1,\xi_2,\frac{1}{\xi_1},x,t\right)\;d\xi_1 d\xi_2.
\end{aligned}$$

The residue at $\xi_3 = 1$ is:

$$\begin{aligned}
&\int_{c_2}\int_{c_1}\Big(Q_{32}S_{31}\frac{1}{\xi_1\cdot 1-1}\Big)W_{123}(\xi_1,\xi_2,1,x,t)\;d\xi_1 d\xi_2\\
&=\int_{c_2}\int_{c_1}\Big(\frac{q-p}{p}\cdot\frac{q}{p}\cdot\frac{1}{\xi_1-1}\Big)W_{12}(\xi_1,\xi_2,x,t)\;d\xi_1 d\xi_2.
\end{aligned}$$

- **The sixth integral:**

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}Q_{31}S_{32}\xi_3^{x-i-j}\xi_2^{x-i}\xi_1^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,t)\;d\xi_1 d\xi_2 d\xi_3.$$

Shrink the contour c_2 and enlarge the contour c_3 into C_3 as in the third integral, and shrink the contour c_1 as in the fourth integral. Then, provided that $q < p$, the series becomes:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}Q_{31}S_{32}\xi_3^{x-i-j}\xi_2^{x-i}\xi_1^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3$$
$$=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{C_3}\int_{c_2}\int_{c_1}\Big(S_{21}Q_{31}S_{32}\xi_3^{x-i-j}\xi_2^{x-i}\xi_1^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3$$
$$\int_{C_3}\int_{c_2}\int_{c_1}\Big(S_{21}Q_{31}S_{32}\frac{1}{\xi_3-1}\cdot\frac{1}{\xi_2\xi_3-1}\xi_3^{x}\xi_2^{x}\xi_1^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3$$
$$=\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}Q_{31}S_{32}\frac{1}{\xi_3-1}\cdot\frac{1}{\xi_2\xi_3-1}\Big)W_{123}(\xi_1,\xi_2,\xi_3,x,t)\ d\xi_1 d\xi_2 d\xi_3$$
$$+\left(\text{Residue at } \xi_3=\frac{1}{\xi_2}\right)+(\text{Residue at } \xi_3=1).$$

The residue at $\xi_3 = \frac{1}{\xi_2}$ is:

$$\int_{c_2}\int_{c_1}\Big(S_{21}Q_{31}S_{32}\frac{1}{\frac{1}{\xi_2}-1}\cdot\frac{1}{\xi_2}\Big)W_{123}\left(\xi_1,\xi_2,\frac{1}{\xi_2},x,t\right)\ d\xi_1 d\xi_2$$
$$=\int_{c_2}\int_{c_1}\Big(S_{21}\cdot\frac{(p\xi_2-q)(\xi_1-1)}{p\xi_2+q\xi_1-\xi_2\xi_1}\cdot\frac{1}{\xi_2}\cdot\frac{1}{1-\xi_2}\Big)W_{123}\left(\xi_1,\xi_2,\frac{1}{\xi_2},x,t\right)\ d\xi_1 d\xi_2.$$

The residue at $\xi_3 = 1$ is:

$$\int_{c_2}\int_{c_1}\Big(S_{21}Q_{31}S_{32}\frac{1}{\xi_2\cdot 1-1}\Big)W_{123}(\xi_1,\xi_2,1,x,t)\ d\xi_1 d\xi_2$$
$$=\int_{c_2}\int_{c_1}\Big(S_{21}\cdot\frac{q-p}{p}\cdot\frac{q}{p}\cdot\frac{1}{\xi_2-1}\Big)W_{12}(\xi_1,\xi_2,x,t)\ d\xi_1 d\xi_2.$$

- **The sevenths integral:**

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(pT_{32}\xi_1^{x-i}\xi_3^{x}\xi_2^{x+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3.$$

Enlarge the contour c_1 into C_1 large enough, and shrink the contour c_2 small enough, so that the singularities are avoided. Then, the series becomes:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(pT_{32}\xi_1^{x-i}\xi_3^{x}\xi_2^{x+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3$$
$$=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{C_1}\Big(pT_{32}\xi_1^{x-i}\xi_3^{x}\xi_2^{x+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3$$
$$=\int_{c_3}\int_{c_2}\int_{C_1}\Big(pT_{32}\frac{1}{\xi_1-1}\cdot\frac{\xi_2}{1-\xi_2}\xi_1^{x}\xi_3^{x}\xi_2^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3$$
$$=\int_{c_3}\int_{c_2}\int_{c_1}\Big(pT_{32}\frac{1}{\xi_1-1}\cdot\frac{\xi_2}{1-\xi_2}\Big)W_{123}(\xi_1,\xi_2,\xi_3,x,t)\ d\xi_1 d\xi_2 d\xi_3$$
$$+(\text{Residue at } \xi_1=1).$$

The residue at $\xi_1 = 1$ is:

$$\int_{c_3}\int_{c_2}\Big(pT_{32}\frac{\xi_2}{1-\xi_2}\Big)W_{123}(1,\xi_2,\xi_3,x,t)\ d\xi_2 d\xi_3 = \int_{c_3}\int_{c_2}\Big(pT_{32}\frac{\xi_2}{1-\xi_2}\Big)W_{23}(\xi_2,\xi_3,x,t)\ d\xi_2 d\xi_3.$$

- **The eighths integral:**

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}pT_{31}\xi_2^{x-i}\xi_3^{x}\xi_1^{x+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3.$$

Enlarge the contour c_2 into C_2, and shrink the contour c_1. Then the series becomes:

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}pT_{31}\xi_2^{x-i}\xi_3^{x}\xi_1^{x+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3\\
&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{C_2}\int_{c_1}\Big(S_{21}pT_{31}\xi_2^{x-i}\xi_3^{x}\xi_1^{x+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3\\
&=\int_{c_3}\int_{C_2}\int_{c_1}\Big(S_{21}pT_{31}\frac{1}{\xi_2-1}\cdot\frac{\xi_1}{1-\xi_1}\xi_2^{x}\xi_3^{x}\xi_1^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3\\
&=\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}pT_{31}\frac{1}{\xi_2-1}\cdot\frac{\xi_1}{1-\xi_1}\Big)W_{123}(\xi_1,\xi_2,\xi_3,x,t)\ d\xi_1 d\xi_2 d\xi_3\\
&\qquad +(\text{Residue at } \xi_2 = 1).
\end{aligned}$$

The residue at $\xi_2 = 1$ is:

$$\begin{aligned}
&\int_{c_3}\int_{c_1}\Big(S_{21}pT_{31}\frac{\xi_1}{1-\xi_1}\Big)W_{123}(\xi_1,1,\xi_3,x,t)\ d\xi_1 d\xi_3\\
&=\int_{c_3}\int_{c_1}\Big(\frac{q}{p}\cdot pT_{31}\frac{\xi_1}{1-\xi_1}\Big)W_{13}(\xi_1,\xi_3,x,t)\ d\xi_1 d\xi_3\\
&=\int_{c_3}\int_{c_1}\Big(qT_{31}\frac{\xi_1}{1-\xi_1}\Big)W_{13}(\xi_1,\xi_3,x,t)\ d\xi_1 d\xi_3.
\end{aligned}$$

- **The ninth integral:**

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(pT_{32}Q_{31}\xi_3^{x-i}\xi_1^{x}\xi_2^{x+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3.$$

Enlarge the contour c_3 into C_3 and keep the contours c_1 and c_2 small enough. Then, the series becomes:

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(pT_{32}Q_{31}\xi_3^{x-i}\xi_1^{x}\xi_2^{x+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3\\
&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{C_3}\int_{c_2}\int_{c_1}\Big(pT_{32}Q_{31}\xi_3^{x-i}\xi_1^{x}\xi_2^{x+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3\\
&=\int_{C_3}\int_{c_2}\int_{c_1}\Big(pT_{32}Q_{31}\frac{1}{\xi_3-1}\cdot\frac{\xi_2}{1-\xi_2}\xi_3^{x}\xi_1^{x}\xi_2^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3\\
&=\int_{c_3}\int_{c_2}\int_{c_1}\Big(pT_{32}Q_{31}\frac{1}{\xi_3-1}\cdot\frac{\xi_2}{1-\xi_2}\Big)W_{123}(\xi_1,\xi_2,\xi_3,x,t)\;d\xi_1 d\xi_2 d\xi_3\\
&\qquad+(\text{Residue at }\xi_3=1).
\end{aligned}$$

The residue at $\xi_3 = 1$ is:

$$\begin{aligned}
&\int_{c_2}\int_{c_1}\Big(pT_{32}Q_{31}\frac{\xi_2}{1-\xi_2}\Big)W_{123}(\xi_1,\xi_2,1,x,t)\;d\xi_1 d\xi_2\\
&=\int_{c_2}\int_{c_1}\Big(p\cdot\frac{1}{p}\cdot\frac{q-p}{p}\frac{\xi_2}{1-\xi_2}\Big)W_{12}(\xi_1,\xi_2,x,t)\;d\xi_1 d\xi_2\\
&=\int_{c_2}\int_{c_1}\Big(\frac{q-p}{p}\frac{\xi_2}{1-\xi_2}\Big)W_{12}(\xi_1,\xi_2,x,t)\;d\xi_1 d\xi_2.
\end{aligned}$$

- **The tenth integral:**

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}pT_{31}Q_{32}\xi_3^{x-i}\xi_2^{x}\xi_1^{x+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3.$$

Enlarge the contour c_3 into C_3 and keep the contours c_1 and c_2 small enough. Then, we get:

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}pT_{31}Q_{32}\xi_3^{x-i}\xi_2^{x}\xi_1^{x+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3\\
&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{C_3}\int_{c_2}\int_{c_1}\Big(S_{21}pT_{31}Q_{32}\xi_3^{x-i}\xi_2^{x}\xi_1^{x+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3\\
&=\int_{C_3}\int_{c_2}\int_{c_1}\Big(S_{21}pT_{31}Q_{32}\frac{1}{\xi_3-1}\cdot\frac{\xi_1}{1-\xi_1}\xi_3^{x}\xi_2^{x}\xi_1^{x}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\;d\xi_1 d\xi_2 d\xi_3\\
&=\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}pT_{31}Q_{32}\frac{1}{\xi_3-1}\cdot\frac{\xi_1}{1-\xi_1}\Big)W_{123}(\xi_1,\xi_2,\xi_3,x,t)\;d\xi_1 d\xi_2 d\xi_3\\
&\qquad+(\text{Residue at }\xi_3=1).
\end{aligned}$$

The residue at $\xi_3 = 1$ is:

$$\begin{aligned}
&\int_{c_2}\int_{c_1}\Big(S_{21}pT_{31}Q_{32}\frac{\xi_1}{1-\xi_1}\Big)W_{123}(\xi_1,\xi_2,1,x,t)\;d\xi_1 d\xi_2\\
&=\int_{c_2}\int_{c_1}\Big(S_{21}p\cdot\frac{1}{p}\cdot\frac{q-p}{p}\cdot\frac{\xi_1}{1-\xi_1}\Big)W_{12}(\xi_1,\xi_2,x,t)\;d\xi_1 d\xi_2\\
&=\int_{c_2}\int_{c_1}\Big(S_{21}\cdot\frac{q-p}{p}\cdot\frac{\xi_1}{1-\xi_1}\Big)W_{12}(\xi_1,\xi_2,x,t)\;d\xi_1 d\xi_2.
\end{aligned}$$

- **The eleventh integral:**

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(pT_{32}pT_{31}\xi_3^x\xi_1^{x+i}\xi_2^{x+i+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3.$$

This time there is no need to deform anything; just choose contours small enough to avoid singularities. Then we get:

$$\begin{aligned}&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(pT_{32}pT_{31}\xi_3^x\xi_1^{x+i}\xi_2^{x+i+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3\\ &=\int_{c_3}\int_{c_2}\int_{c_1}\Big(pT_{32}pT_{31}\frac{\xi_2}{1-\xi_2}\cdot\frac{\xi_1\xi_2}{1-\xi_1\xi_2}\Big)W_{123}(\xi_1,\xi_2,\xi_3,x,t)\ d\xi_1 d\xi_2 d\xi_3.\end{aligned}$$

- **The twelveth integral:**

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}pT_{31}pT_{32}\xi_3^x\xi_2^{x+i}\xi_1^{x+i+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3.$$

Keep contours small enough. Then we get:

$$\begin{aligned}&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}pT_{31}pT_{32}\xi_3^x\xi_2^{x+i}\xi_1^{x+i+j}\Big)W_{123}(\xi_1,\xi_2,\xi_3,0,t)\ d\xi_1 d\xi_2 d\xi_3\\ &=\int_{c_3}\int_{c_2}\int_{c_1}\Big(S_{21}pT_{31}pT_{32}\frac{\xi_1}{1-\xi_1}\cdot\frac{\xi_1\xi_2}{1-\xi_1\xi_2}\Big)W_{123}(\xi_1,\xi_2,\xi_3,x,t)\ d\xi_1 d\xi_2 d\xi_3.\end{aligned}$$

Now, we collect the common terms together using the Mathematica software:

- **The integral over ξ_1,ξ_2,ξ_3:** This integral has 12 summands; after calculation, we get:

$$\oint_{c_3}\oint_{c_2}\oint_{c_1}\frac{(2p-1)^2(\xi_1-\xi_2)(\xi_1-\xi_3)(\xi_2-\xi_3)(\xi_1\xi_2\xi_3-1)}{(\xi_1-1)(\xi_2-1)(\xi_3-1)(\xi_1-p-q\xi_1\xi_2)(\xi_1-p-q\xi_1\xi_3)(\xi_2-p-q\xi_2\xi_3)}W_{123}\ d\xi_1 d\xi_2 d\xi_3.$$

- **The integral over ξ_1,ξ_2:**

$$\begin{aligned}&\oint_{c_2}\oint_{c_1}\Big(\frac{q-p}{p}\frac{q}{p}\frac{1}{\xi_1-1}\Big)W_{12}(\xi_1,\xi_2,x,t)\ d\xi_1 d\xi_2+\oint_{c_2}\oint_{c_1}\Big(S_{21}\frac{q-p}{p}\frac{q}{p}\frac{1}{\xi_2-1}\Big)W_{12}(\xi_1,\xi_2,x,t)\ d\xi_1 d\xi_2\\ &+\oint_{c_2}\oint_{c_1}\Big(\frac{q-p}{p}\frac{\xi_2}{1-\xi_2}\Big)W_{12}(\xi_1,\xi_2,x,t)\ d\xi_1 d\xi_2+\oint_{c_2}\oint_{c_1}\Big(S_{21}\frac{q-p}{p}\frac{\xi_1}{1-\xi_1}\Big)W_{12}(\xi_1,\xi_2,x,t)\ d\xi_1 d\xi_2\\ &=\frac{q-p}{p^2}\oint_{c_2}\oint_{c_1}-\frac{(2p-1)(\xi_1-\xi_2)(\xi_1\xi_2-1)}{(\xi_1-1)(\xi_2-1)(\xi_1-p-q\xi_1\xi_2)}W_{12}(\xi_1,\xi_2,x,t)\ d\xi_1 d\xi_2.\end{aligned}$$

- **The integral over ξ_1, ξ_3:**

$$\oint_{c_3}\oint_{c_1}\Big(\frac{q}{p}\frac{1}{\xi_1-1}\Big)W_{13}(\xi_1,\xi_3,x,t)\ d\xi_1 d\xi_3 + \oint_{c_3}\oint_{c_1}\Big(\frac{q}{p}Q_{31}\frac{1}{\xi_3-1}\Big)W_{13}(\xi_1,\xi_3,x,t)\ d\xi_1 d\xi_3$$
$$+\oint_{c_3}\oint_{c_1}\Big(qT_{31}\frac{\xi_1}{1-\xi_1}\Big)W_{13}(\xi_1,\xi_3,x,t)\ d\xi_1 d\xi_3$$
$$=\frac{q}{p}\oint_{c_3}\oint_{c_1}-\frac{(2p-1)(\xi_1-\xi_3)(\xi_1\xi_3-1)}{(\xi_1-1)(\xi_3-1)(\xi_1-p-q\xi_1\xi_3)}W_{13}(\xi_1,\xi_3,x,t)\ d\xi_1 d\xi_3.$$

- **The integral over ξ_2, ξ_3:**

$$\oint_{c_3}\oint_{c_2}\frac{1}{\xi_2-1}\xi_2^x\xi_3^x\ \ W_{23}(\xi_2,\xi_3,t)\ d\xi_2 d\xi_3 + \oint_{c_3}\oint_{c_2}\Big(Q_{32}\frac{1}{\xi_3-1}\Big)W_{23}(\xi_2,\xi_3,x,t)\ d\xi_2 d\xi_3$$
$$+\oint_{c_3}\oint_{c_2}\Big(pT_{32}\frac{\xi_2}{1-\xi_2}\Big)W_{23}(\xi_2,\xi_3,x,t)\ d\xi_2 d\xi_3$$
$$=\oint_{c_3}\oint_{c_2}-\frac{(2p-1)(\xi_2-\xi_3)(\xi_2\xi_3-1)}{(\xi_2-1)(\xi_3-1)(\xi_2-p-q\xi_2\xi_3)}W_{23}(\xi_2,\xi_3,x,t)\ d\xi_2 d\xi_3.$$

- **The integral over ξ_1:**

$$\oint_{c_1}\Big(\frac{q}{p}\cdot\frac{(q-p)}{p}\Big)W_1(\xi_1,x,t)\ \ d\xi_1.$$

- **The integral over ξ_2:**

$$\oint_{c_2}\Big(\frac{(q-p)}{p}\Big)W_2(\xi_2,x,t)\ \ d\xi_2.$$

- **The integral over ξ_3:**

$$\oint_{c_3}W_3(\xi_3,x,t)\ \ d\xi_3.$$

Notice that we didn't include the $\xi_\beta = \frac{1}{\xi_\alpha}$ residues in the computations of the double integrals above. This is because they cancel out with each other:

- ξ_1, ξ_2**:**

$$\oint_{c_2}\oint_{c_1}\Big(\frac{(p\xi_1-q)(\xi_2-1)}{p\xi_1+q\xi_2-\xi_1\xi_2}\cdot\frac{1}{\xi_1-1}\cdot\frac{1}{\xi_1}\Big)W_{123}\Big(\xi_1,\xi_2,\frac{1}{\xi_1},x,t\Big)\ d\xi_1 d\xi_2$$
$$+\oint_{c_2}\oint_{c_1}\Big(S_{21}\cdot\frac{(p\xi_2-q)(\xi_1-1)}{p\xi_2+q\xi_1-\xi_2\xi_1}\cdot\frac{1}{\xi_2-1}\cdot\frac{1}{\xi_2}\Big)W_{123}\Big(\xi_1,\xi_2,\frac{1}{\xi_2},x,t\Big)\ d\xi_1 d\xi_2$$
$$+\oint_{c_2}\oint_{c_1}\Big(\frac{(p\xi_1-q)(\xi_2-1)}{p\xi_1+q\xi_2-\xi_1\xi_2}\cdot\frac{1}{\xi_1}\cdot\frac{1}{1-\xi_1}\Big)W_{123}\Big(\xi_1,\xi_2,\frac{1}{\xi_1},x,t\Big)\ d\xi_1 d\xi_2$$
$$+\oint_{c_2}\oint_{c_1}\Big(S_{21}\cdot\frac{(p\xi_2-q)(\xi_1-1)}{p\xi_2+q\xi_1-\xi_2\xi_1}\cdot\frac{1}{\xi_2}\cdot\frac{1}{1-\xi_2}\Big)W_{123}\Big(\xi_1,\xi_2,\frac{1}{\xi_2},x,t\Big)\ d\xi_1 d\xi_2 = 0.$$

- ξ_1, ξ_3:

$$\oint_{c_3}\oint_{c_1} \frac{1}{\xi_1 - 1}\cdot\frac{1}{\xi_1}\, W_{123}\left(\xi_1, \frac{1}{\xi_1}, \xi_3, x, t\right)\, d\xi_1 d\xi_3 + \oint_{c_3}\oint_{c_1} \Big(\frac{1}{1-\xi_1}\cdot\frac{1}{\xi_1}\Big) W_{123}\left(\xi_1, \frac{1}{\xi_1}, \xi_3, x, t\right)\, d\xi_1 d\xi_3 = 0.$$

Remark 3.2.3. These cancellations seem somewhat miraculous; however, it becomes much clearer if we didn't make any evaluations in the first place: for example, consider the last cancellation in the variables ξ_1, ξ_3. They came from the permutations $\sigma = 123$ and $\sigma = 213$. Then, the corresponding sum would be:

$$\begin{aligned} &\int_{c_3}\int_{c_1} \left(\frac{1}{\xi_1 - 1}\cdot\frac{1}{\xi_1}\right)\; W_{123}\left(\xi_1, \frac{1}{\xi_1}, \xi_3, x, t\right)\, d\xi_1 d\xi_3 \\ &+ \int_{c_3}\int_{c_1} \left(S_{21}|_{\xi_2 = \frac{1}{\xi_1}} \frac{1}{\xi_2 - 1}\cdot\frac{1}{\xi_1}\right)\; W_{123}\left(\xi_1, \frac{1}{\xi_1}, \xi_3, x, t\right)\, d\xi_1 d\xi_3 \\ &= \frac{1}{\xi_1}\int_{c_3}\int_{c_1} \left.\frac{q(\xi_1-\xi_2)(\xi_1\xi_2 - 1)}{(\xi_1 - 1)(\xi_2 - 1)(\xi_1 - p - q\xi_1\xi_2)}\right|_{\xi_2 = \frac{1}{\xi_1}} W_{123}\left(\xi_1, \frac{1}{\xi_1}, \xi_3, x, t\right)\, d\xi_1 d\xi_3 = 0. \end{aligned}$$

Remark 3.2.4. When we deformed the contours, we assumed that $q < p$ to avoid some singularities. As it turns out, there is no need to be afraid of these singularities: Tracy and Widom in [5] gave the argument that the residue that arises because of the denominator $p + q\xi_\alpha\xi_\beta - \xi_\alpha$ (which is given by $\xi_\alpha = \frac{p}{1-q\xi_\beta}$) is $O(1)$ at infinity in the variable ξ_β, and so it vanishes when integrated over the variable ξ_β.

3.3 Some simplifications for further work

After some calculations, we can verify the following main identities:

$$Q_{\beta\alpha} = S_{\beta\alpha} - pT_{\beta\alpha};$$

Similarly:

$$T_{\beta\alpha} = S_{\beta\alpha} + 1;$$

Moreover:

$$\begin{aligned} &\oint_{c_2}\oint_{c_1} -\frac{(2p-1)(\xi_1-\xi_2)(\xi_1\xi_2-1)}{(\xi_1-1)(\xi_2-1)(\xi_1-p-q\xi_1\xi_2)} W_{12}(\xi_1,\xi_2,x,t)\, d\xi_1 d\xi_2 \\ &= \oint_{c_2}\oint_{c_1} \frac{(p-q)(1-\xi_1\xi_2)}{(1-\xi_1)(1-\xi_2)} T_{21} W_{12}(\xi_1,\xi_2,x,t)\, d\xi_1 d\xi_2; \end{aligned}$$

$$\oint_{c_3}\oint_{c_2}\oint_{c_1} \frac{(2p-1)^2(\xi_1-\xi_2)(\xi_1-\xi_3)(\xi_2-\xi_3)(\xi_1\xi_2\xi_3-1)}{(\xi_1-1)(\xi_2-1)(\xi_3-1)(\xi_1-p-q\xi_1\xi_2)(\xi_1-p-q\xi_1\xi_3)(\xi_2-p-q\xi_2\xi_3)} W_{123}\, d\xi_1 d\xi_2 d\xi_3$$
$$= \oint_{c_3}\oint_{c_2}\oint_{c_1} \frac{(p-q)^2(1-\xi_1\xi_2\xi_3)}{(1-\xi_1)(1-\xi_2)(1-\xi_3)} \prod_{\alpha<\beta} T_{\beta\alpha} W_{123}(\xi_1,\xi_2,\xi_3,x,t)\, d\xi_1 d\xi_2 d\xi_3$$

Let us consider the sum inside the integral:

$$\begin{aligned}
&\frac{1}{\xi_1-1}\frac{1}{\xi_1\xi_2-1} + Q_{32}\frac{1}{\xi_1-1}\cdot\frac{1}{\xi_1\xi_3-1} + S_{21}\frac{1}{\xi_2-1}\cdot\frac{1}{\xi_1\xi_2-1} + S_{21}Q_{31}\frac{1}{\xi_2-1}\cdot\frac{1}{\xi_2\xi_3-1}\\
&+Q_{32}S_{31}\frac{1}{\xi_3-1}\cdot\frac{1}{\xi_1\xi_3-1} + S_{21}Q_{31}S_{32}\frac{1}{\xi_3-1}\cdot\frac{1}{\xi_2\xi_3-1} + pT_{32}\frac{1}{\xi_1-1}\cdot\frac{\xi_2}{1-\xi_2}\\
&+S_{21}pT_{31}\frac{1}{\xi_2-1}\cdot\frac{\xi_1}{1-\xi_1} + pT_{32}Q_{31}\frac{1}{\xi_3-1}\cdot\frac{\xi_2}{1-\xi_2} + S_{21}pT_{31}Q_{32}\frac{1}{\xi_3-1}\cdot\frac{\xi_1}{1-\xi_1}\\
&+pT_{32}pT_{31}\frac{\xi_2}{1-\xi_2}\cdot\frac{\xi_1\xi_2}{1-\xi_1\xi_2} + S_{21}pT_{31}pT_{32}\frac{\xi_1}{1-\xi_1}\cdot\frac{\xi_1\xi_2}{1-\xi_1\xi_2}
\end{aligned}$$

The first thing that we need to do is to remove the $Q_{\beta\alpha}$ terms by writing: $Q_{\beta\alpha} = S_{\beta\alpha} - pT_{\beta\alpha}$. Then, this will yield:

$$\begin{aligned}
&\sum_{\sigma\in\mathbb{S}} A_\sigma \frac{1}{\xi_{\sigma(1)}-1}\cdot\frac{1}{\xi_{\sigma(1)}\xi_{\sigma(2)}-1} - pT_{32}\frac{1}{\xi_1-1}\cdot\frac{1}{\xi_1\xi_3-1} - S_{21}pT_{31}\frac{1}{\xi_2-1}\cdot\frac{1}{\xi_2\xi_3-1}\\
&-pT_{32}S_{31}\frac{1}{\xi_3-1}\cdot\frac{1}{\xi_1\xi_3-1} - S_{21}pT_{31}S_{32}\frac{1}{\xi_3-1}\cdot\frac{1}{\xi_2\xi_3-1} + pT_{32}\frac{1}{\xi_1-1}\cdot\frac{\xi_2}{1-\xi_2}\\
&+S_{21}pT_{31}\frac{1}{\xi_2-1}\cdot\frac{\xi_1}{1-\xi_1} + pT_{32}S_{31}\frac{1}{\xi_3-1}\cdot\frac{\xi_2}{1-\xi_2} - pT_{32}pT_{31}\frac{1}{\xi_3-1}\cdot\frac{\xi_2}{1-\xi_2}\\
&+S_{21}pT_{31}S_{32}\frac{1}{\xi_3-1}\cdot\frac{\xi_1}{1-\xi_1} - S_{21}pT_{31}pT_{32}\frac{1}{\xi_3-1}\cdot\frac{\xi_1}{1-\xi_1}\\
&+pT_{32}pT_{31}\frac{\xi_2}{1-\xi_2}\cdot\frac{\xi_1\xi_2}{1-\xi_1\xi_2} + S_{21}pT_{31}pT_{32}\frac{\xi_1}{1-\xi_1}\cdot\frac{\xi_1\xi_2}{1-\xi_1\xi_2}
\end{aligned}$$

The first term we recognize as the integrand from the paper by Tracy and Widom, which equals:

$$\sum_{\sigma\in\mathbb{S}} A_\sigma \frac{1}{\xi_{\sigma(i)}-1}\cdot\frac{1}{\xi_{\sigma(i)}\xi_{\sigma(j)}-1} = qT_{21}qT_{31}qT_{32}\frac{(1-\xi_1\xi_2\xi_3)}{(1-\xi_1)(1-\xi_2)(1-\xi_3)}$$

We now gather the remaining terms by the $pT_{\beta\alpha}$ factors:

- **The pT_{31} factor:**

$$-S_{21}pT_{31}\frac{1}{\xi_2-1}\cdot\frac{1}{\xi_2\xi_3-1}-S_{21}pT_{31}S_{32}\frac{1}{\xi_3-1}\cdot\frac{1}{\xi_2\xi_3-1}+S_{21}pT_{31}\frac{1}{\xi_2-1}\cdot\frac{\xi_1}{1-\xi_1}$$
$$+S_{21}pT_{31}S_{32}\frac{1}{\xi_3-1}\cdot\frac{\xi_1}{1-\xi_1}=-S_{21}pT_{31}qT_{32}\frac{(1-\xi_1\xi_2\xi_3)}{(1-\xi_1)(1-\xi_2)(1-\xi_3)};$$

- **The pT_{32} factor:**

$$-pT_{32}\frac{1}{\xi_1-1}\cdot\frac{1}{\xi_1\xi_3-1}-pT_{32}S_{31}\frac{1}{\xi_3-1}\cdot\frac{1}{\xi_1\xi_3-1}+pT_{32}\frac{1}{\xi_1-1}\cdot\frac{\xi_2}{1-\xi_2}$$
$$+pT_{32}S_{31}\frac{1}{\xi_3-1}\cdot\frac{\xi_2}{1-\xi_2}=-pT_{32}qT_{31}\frac{(1-\xi_1\xi_2\xi_3)}{(1-\xi_1)(1-\xi_2)(1-\xi_3)};$$

- **The $pT_{31}pT_{32}$ factor:**

$$-pT_{32}pT_{31}\frac{1}{\xi_3-1}\cdot\frac{\xi_2}{1-\xi_2}-S_{21}pT_{31}pT_{32}\frac{1}{\xi_3-1}\cdot\frac{\xi_1}{1-\xi_1}+pT_{32}pT_{31}\frac{\xi_2}{1-\xi_2}\cdot\frac{\xi_1\xi_2}{1-\xi_1\xi_2}$$
$$+S_{21}pT_{31}pT_{32}\frac{\xi_1}{1-\xi_1}\cdot\frac{\xi_1\xi_2}{1-\xi_1\xi_2}=pT_{21}pT_{31}pT_{32}\frac{(1-\xi_1\xi_2\xi_3)}{(1-\xi_1)(1-\xi_2)(1-\xi_3)};$$

And so, the coefficient before the $\frac{(1-\xi_1\xi_2\xi_3)}{(1-\xi_1)(1-\xi_2)(1-\xi_3)}$-factor becomes:

$$qT_{21}qT_{31}qT_{32}-S_{21}pT_{31}qT_{32}-pT_{32}qT_{31}+pT_{21}pT_{31}pT_{32};$$

We now observe the following identities:

$$S_{21}+1=T_{21};$$
$$q^3+p^3=(p+q)(p^2-pq+q^2)=1\cdot(p^2-pq+q^2)=p^2-pq+q^2$$

and so we finally get:

$$qT_{21}qT_{31}qT_{32}-S_{21}pT_{31}qT_{32}-pT_{32}qT_{31}+pT_{21}pT_{31}pT_{32}$$
$$=(p^2-pq+q^2)T_{21}T_{31}T_{32}-pqT_{21}T_{31}T_{32}=(p-q)^2T_{21}T_{31}T_{32};$$

This simplification suggests that we should always remove $Q_{\beta\alpha}$ terms, then we need to take the summations over the columns in the $[A_\sigma]$ table with $T_{\beta\alpha}$ rearranged into the next column.

σ	$[\mathbf{A}_\sigma]_{2221,2221}$	$[\mathbf{A}_\sigma]_{2212,2221}$	$[\mathbf{A}_\sigma]_{2122,2221}$	$[\mathbf{A}_\sigma]_{1222,2221}$
1234	1	0	0	0
1243	Q_{43}	pT_{43}	0	0
1324	S_{32}	0	0	0
1342	$S_{32}Q_{42}$	$S_{32}pT_{42}$	0	0
1423	$Q_{43}S_{42}$	$pT_{43}Q_{42}$	$pT_{43}pT_{42}$	0
1432	$S_{32}Q_{42}S_{43}$	$S_{32}pT_{42}Q_{43}$	$pT_{43}pT_{42}S_{32}$	0
2134	S_{21}	0	0	0
2143	$S_{21}Q_{43}$	$S_{21}pT_{43}$	0	0
2314	$S_{21}S_{31}$	0	0	0
2341	$S_{21}S_{31}Q_{41}$	$S_{21}S_{31}pT_{41}$	0	0
2413	$S_{21}Q_{43}S_{41}$	$S_{21}pT_{43}Q_{41}$	$S_{21}pT_{43}pT_{41}$	0
2431	$S_{21}S_{31}Q_{41}S_{43}$	$S_{21}S_{31}pT_{41}Q_{43}$	$S_{21}pT_{43}pT_{41}S_{31}$	0
3124	$S_{32}S_{31}$	0	0	0
3142	$S_{32}S_{31}Q_{42}$	$S_{32}S_{31}pT_{42}$	0	0
3214	$S_{32}S_{31}S_{21}$	0	0	0
3241	$S_{32}S_{31}S_{21}Q_{41}$	$S_{32}S_{31}S_{21}pT_{41}$	0	0
3412	$S_{32}S_{31}Q_{42}S_{41}$	$S_{32}S_{31}pT_{42}Q_{41}$	$S_{32}S_{31}pT_{42}pT_{41}$	0
3421	$S_{32}S_{31}S_{21}Q_{41}S_{42}$	$S_{32}S_{31}S_{21}pT_{41}Q_{42}$	$S_{32}S_{31}pT_{42}pT_{41}S_{21}$	0
4123	$Q_{43}S_{42}S_{41}$	$pT_{43}Q_{42}S_{41}$	$pT_{43}pT_{42}Q_{41}$	$pT_{43}pT_{42}pT_{41}$
4132	$S_{32}Q_{42}S_{43}S_{41}$	$S_{32}pT_{42}Q_{43}S_{41}$	$S_{32}pT_{42}pT_{43}Q_{41}$	$S_{32}pT_{42}pT_{43}pT_{41}$
4213	$S_{21}Q_{43}S_{41}S_{42}$	$S_{21}pT_{43}Q_{41}S_{42}$	$S_{21}pT_{43}pT_{41}Q_{42}$	$S_{21}pT_{43}pT_{41}pT_{42}$
4231	$S_{21}S_{31}Q_{41}S_{43}S_{42}$	$S_{21}S_{31}pT_{41}Q_{43}S_{42}$	$S_{21}S_{31}pT_{41}pT_{43}Q_{42}$	$S_{21}S_{31}pT_{41}pT_{43}pT_{42}$
4312	$S_{32}S_{31}Q_{42}S_{41}S_{43}$	$S_{32}S_{31}pT_{42}Q_{41}S_{43}$	$S_{32}S_{31}pT_{42}pT_{41}Q_{43}$	$S_{32}S_{31}pT_{42}pT_{41}pT_{43}$
4321	$S_{32}S_{31}S_{21}Q_{41}S_{42}S_{43}$	$S_{32}S_{31}S_{21}pT_{41}Q_{42}S_{43}$	$S_{32}S_{31}S_{21}pT_{41}pT_{42}Q_{43}$	$S_{32}S_{31}S_{21}pT_{41}pT_{42}pT_{43}$

3.4 General formula conjectured

We have the following table for the case $N = 4$ (see [2]):

After computing manually the formula for $N = 4$, we get the following expression for the 4-fold integral:

$$(p-q)^3(q+p^2)\oint_c\oint_c\oint_c\oint_c\prod_{j>i}\frac{\xi_j-\xi_i}{p+q\xi_i\xi_j-\xi_i}\cdot\frac{1-\xi_1\xi_2\xi_3\xi_4}{\prod_{i=1}^{4}(1-\xi_i)}W_{1234}(\xi_1,\xi_2,\xi_3,\xi_4,x,t)\ d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

which equals:

$$(p-q)\cdot(p^2-q^2)\cdot(p^3-q^3)\oint_c\oint_c\oint_c\oint_c\prod_{j>i}\frac{\xi_j-\xi_i}{p+q\xi_i\xi_j-\xi_i}\cdot\frac{1-\xi_1\xi_2\xi_3\xi_4}{\prod_{i=1}^{4}(1-\xi_i)}W_{1234}(\xi_1,\xi_2,\xi_3,\xi_4,x,t)\ d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

This suggests the following conjecture for the N-fold integral for general N:

$$\prod_{i=1}^{N-1}(p^i-q^i)\oint_c\dots\oint_c\oint_c\prod_{\beta>\alpha}T_{\beta\alpha}\cdot\frac{1-\prod_{i=1}^{N}\xi_i}{\prod_{i=1}^{N}(1-\xi_i)}W_{1\dots N}(\xi_1,\xi_2,\dots,\xi_N,x,t)\ d\xi_1 d\xi_2\dots d\xi_N.$$

Chapter 4

Main results

Theorem 4.0.1. Let $S = \{s_1, s_2, ..., s_k\}$ be a subset of $\{1, 2, ..., N\}$ of cardinality $|S| = k$, and define:

$$I(x, Y_S) := \oint_{c_{s_k}} \dots \oint_{c_{s_1}} \prod_{\substack{\alpha<\beta \\ \alpha,\beta\in S}} T_{\beta\alpha} \frac{\left(1 - \prod_{i=1}^{|S|} \xi_{s_i}\right)}{\prod_{i=1}^{|S|} (1 - \xi_{s_i})} \cdot W_{s_1 \dots s_k}(\xi_{s_1}, ..., \xi_{s_k}, x, t)\, d\xi_{s_1} \dots d\xi_{s_k};$$

Then, the distribution of the unique species-1 particle which is right-most at time $t = 0$ is given by:

$$\mathbb{P}_Y(\eta(t) = x) = \sum_{\substack{S\subseteq\{1,2,...,N\} \\ \xi_N \in S}} \prod_{i=1}^{|S|-1} (p^i - q^i) \left(\frac{q}{p}\right)^{\Sigma(S^c) - \frac{|S^c|\cdot(|S^c|+1)}{2}} I(x, Y_S)$$

$$- \sum_{\substack{S\subsetneq\{1,2,...,N\} \\ \xi_N \notin S}} \frac{\prod_{i=1}^{|S|} (p^i - q^i)}{p^{|S|}} \left(\frac{q}{p}\right)^{\Sigma(S^c\setminus\{\xi_N\}) - \frac{|S^c\setminus\{\xi_N\}|\cdot(|S^c\setminus\{\xi_N\}|+1)}{2}} I(x, Y_S).$$

where the sum is over all subsets S of $\{1, 2, ..., N\}$.

Remark 4.0.2. Note that the number of terms grows exponentially as N increases: there are $2^N - 1$ different integrals.

Remark 4.0.3. The formula suggests that the integrals that contain the ξ_N variable behave differently from those that don't. Indeed, to compute the lower-dimensional integrals, we will need to evaluate $[A_\sigma]$ at the variables that are not present in the integral. The coefficients of $[A_\sigma]$ in the variable ξ_m, $m < N$ are not different from the single-species coefficients and are given by $\prod S_{m\alpha}$, where the product is over all inversions in σ. However, the coefficients of $[A_\sigma]$ in the variable ξ_N contain the terms $T_{N\alpha}$ and the unique $Q_{N\alpha}$, and so this case needs special attention.

Before we give a proof of this theorem, we need to establish some preliminary results.

4.1 Brackets and q-Brackets.

Our goal in this section is to introduce and develop the machinery that is necessary to study the kind of problems that we are studying.

Definition 4.1.1. Let n be a positive integer. A q-bracket is defined by:

$$[n]_\tau := \frac{1-\tau^n}{1-\tau} \equiv 1+\tau+\ldots+\tau^{n-2}+\tau^{n-1}.$$

Similarly, the q-factorial and the q-binomial coefficients are defined by:

$$[n]_\tau! := [1]_\tau \cdot [2]_\tau \cdot \ldots \cdot [n]_\tau = \prod_{k=1}^{n} [k]_\tau; \qquad \begin{bmatrix} n \\ k \end{bmatrix}_\tau := \frac{[n]_\tau!}{[n-k]_\tau! \cdot [k]_\tau!}.$$

Warning 4.1.2. This is a departure from the standard notation: it is customary to write $[n]_q$ for the q-bracket. However, it creates potential confusion with our probabilistic q, so we prefer to write $[n]_\tau$ instead.

Remark 4.1.3. The q-bracket $[n]_\tau$ can be seen as a continuous deformation of the usual integer n: we recover the usual integer by taking the limit $\lim\limits_{\tau\to 1} [n]_\tau = n$. The q-brackets are not made-up, but rather arise naturally in mathematics, as in the following theorem (which we will need later):

Theorem 4.1.4. (Cauchy binomial theorem): We have:

$$\prod_{k=1}^{n} (1+y\tau^k) = \sum_{k=0}^{n} y^k \tau^{\frac{k(k+1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_\tau.$$

Definition 4.1.5. Let n be a positive integer. We define a bracket by:

$$[n] := \frac{p^n - q^n}{p-q} \equiv p^{n-1} + p^{n-2}q + \ldots + pq^{n-2} + q^{n-1}.$$

Similarly, we define the factorial and the binomial coefficient:

$$[n]! := [1] \cdot [2] \cdot \ldots \cdot [n] = \prod_{k=1}^{n} [k]; \qquad \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[n-k]! \cdot [k]!}.$$

Remark 4.1.6. Let $\tau = \frac{q}{p}$. Then, the following relations between brackets and q-brackets hold:

$$[n] = p^{n-1}[n]_\tau; \qquad \begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_\tau p^{m(n-m)};$$

Proposition 4.1.7. The coefficient $\prod\limits_{i=1}^{N-1} (p^i - q^i)$ can be expanded as follows:

$$\prod_{i=1}^{N-1} (p^i - q^i) = \sum_{k=0}^{N-1} (-1)^k \begin{bmatrix} N-1 \\ k \end{bmatrix} q^{\frac{k(k+1)}{2}} p^{\frac{(N-k)(N-k-1)}{2}}.$$

Proof:

First of all, we have:

$$\prod_{i=1}^{N-1}(p^i - q^i) = \prod_{i=1}^{N-1} p^i \cdot \prod_{i=1}^{N-1}\left(1 - \left(\frac{q}{p}\right)^k\right) = \left(\prod_{j=1}^{N-1} p^j\right) \cdot \prod_{i=1}^{N-1}\left(1 + (-1)\cdot\left(\frac{q}{p}\right)^k\right).$$

The first factor we can evaluate immediately:

$$\left(\prod_{j=1}^{N-1} p^j\right) = p \cdot p^2 \cdot \ldots \cdot p^{N-1} = p^{\sum_{k=1}^{N-1} k} = p^{\frac{(N-1)N}{2}}.$$

The second factor is handled by the Cauchy binomial theorem: set $y = -1$, $\tau = \frac{q}{p}$; then:

$$\prod_{i=1}^{N-1}\left(1 + (-1)\cdot\left(\frac{q}{p}\right)^k\right) = \sum_{k=0}^{N-1}(-1)^k\left(\frac{q}{p}\right)^{\frac{k(k+1)}{2}}\begin{bmatrix} N-1 \\ k\end{bmatrix}_\tau = \sum_{k=0}^{N-1}(-1)^k\left(\frac{q}{p}\right)^{\frac{k(k+1)}{2}}\begin{bmatrix} N-1 \\ k\end{bmatrix}\cdot p^{-k(N-k-1)}.$$

Putting everything together yields:

$$\prod_{i=1}^{N-1}(p^i - q^i) = \left(\prod_{j=1}^{N-1} p^j\right) \cdot \prod_{i=1}^{N-1}\left(1 + (-1)\cdot\left(\frac{q}{p}\right)^k\right) = p^{\frac{(N-1)N}{2}} \cdot \sum_{k=0}^{N-1}(-1)^k\left(\frac{q}{p}\right)^{\frac{k(k+1)}{2}}\begin{bmatrix} N-1 \\ k\end{bmatrix}\cdot p^{-Nk}\cdot p^{k(k+1)}$$

$$= p^{\frac{(N-1)N}{2}} \cdot \sum_{k=0}^{N-1}(-1)^k q^{\frac{k(k+1)}{2}}\begin{bmatrix} N-1 \\ k\end{bmatrix}\cdot p^{-Nk}\cdot p^{\frac{k(k+1)}{2}} = \sum_{k=0}^{N-1}(-1)^k q^{\frac{k(k+1)}{2}}\begin{bmatrix} N-1 \\ k\end{bmatrix}\cdot p^{\frac{N^2}{2}-\frac{N}{2}-Nk+\frac{k^2}{2}+\frac{k}{2}}$$

$$= \sum_{k=0}^{N-1}(-1)^k q^{\frac{k(k+1)}{2}}\begin{bmatrix} N-1 \\ k\end{bmatrix}\cdot p^{\frac{(N-k)(N-k-1)}{2}}. \quad \blacksquare$$

4.2 The $[A_\sigma]$ and $[A^\circ_\sigma]$ coefficients.

Since we will work with the $[A_\sigma]$ coefficients extensively, we will need to study them thoroughly. The material of this section will be primarily used in the section about the lower-dimensional integrals, but the construction $[A^\circ_\sigma]$ will be needed soon.

It was shown in [2] that the coefficients $[A_\sigma]_{2\ldots212\ldots2,2\ldots21}$ are zero whenever $\sigma(j) = N$, $\forall j > N-m+1$, where the particle of species 1 is mth from the right, and otherwise is given by:

$$[A_\sigma]_{2\ldots212\ldots2,2\ldots21} = \prod_{\alpha<\beta<N} S_{\beta,\alpha} \cdot \prod_{k=1}^{N-m} S_{N,\sigma(k)} \cdot \prod_{k=1}^{m-1} pT_{N,\sigma(N+1-k)} \cdot Q_{N,\sigma(N+1-m)}.$$

where the product is taken over all inversions in σ.

As was noted in the previous chapter, we don't want the $Q_{\beta\alpha}$ to appear when we add the different summands together. With this in mind, we introduce the coefficient $[A^\circ_\sigma]$ with $Q_{N\alpha}$ replaced by $S_{N\alpha}$:

Definition 4.2.1. The coefficients $[A^{\circ}_{\sigma}]_{2...212...2,2...21}$ are defined by:

$$[A^{\circ}_{\sigma}]_{2...212...2,2...21} = \prod_{\alpha<\beta<N} S_{\beta,\alpha} \cdot \prod_{k=1}^{N-m+1} S_{N,\sigma(k)} \cdot \prod_{k=1}^{m-1} pT_{N,\sigma(N+1-k)}.$$

where the product is taken over all inversions in σ, where we also have that $\sigma(j) = N$, $\exists j \leq N-m+1$, and are zero otherwise.

Now, let us pay attention to the coefficients $[A_{\sigma}]_{2...212...2,2...21}$ again. The very first thing that we observe is that, if it wasn't for the ξ_N variable, the coefficient would look exactly as in the single-species case: $\prod_{\alpha<\beta} S_{\beta\alpha}$. As we shall soon see, this is exactly what makes the integrals with ξ_N variable so different from the integrals without it.

Proposition 4.2.2. Let $S = \{s_1, s_2, ..., s_k\} \subsetneq \{1, 2, ..., N\}$ be a proper subset, let $\sigma \in \mathbb{S}_N$ be a permutation that is order-preserving in the first $|S^c| < N$ variables, suppose that $\sigma(1) = n_1$, $\sigma(2) = n_2, ..., \sigma(|S^c|) = n_{|S^c|}$, and suppose that $n_{|S^c|} \neq N$. Let $[A_{\sigma}]_{2...212...2,\,2...21}$ be the coefficient where the 1 appears in the mth position from the right, $m \leq N - |S^c| = |S|$. Then:

$$[A_{\sigma}]_{2...212...2,\,2...21}\big|_{\xi_{n_1},...,\xi_{n_{|S^c|}}=1} = \left(\frac{q}{p}\right)^{\Sigma(S^c) - \frac{|S^c|\cdot(|S^c|+1)}{2}} \cdot [A_{\gamma}]_{2...212...2,\,2...21}$$

where $[A_{\gamma}]_{2...212...2,\,2...21}$ is now the coefficient on the $N - |S^c| = |S|$ particles, γ is the permutation on the set S such that $\gamma = \sigma|_S$, and the position of 1 from the **right** is m (i.e. the position from the right doesn't change).

Proof:

First of all, since the map σ is order-preserving in the first $|S^c|$ variables, it follows that there are no inversions among the first $|S^c|$ variables. We now need to count the number of inversions (β, α) such that $\beta \in S^c$ and $\alpha \in S$.

- We have that $\sigma(1) = n_1$. Therefore, there are exactly $n_1 - 1$ inversions (n_1, α) in σ: $(n_1, 1), ..., (n_1, n_1 - 1)$.
- We have that $\sigma(2) = n_2$. Then, there are exactly $n_2 - 2$ inversions (n_2, α) in σ: it's $(n_2, 1), ..., (n_2, n_2 - 1)$, but without (n_2, n_1).
- More generally, we have that $\sigma(i) = n_i$, and there are exactly $n_i - i$ inversions (n_i, α) in σ: it's $(n_i, 1), ..., (n_i, n_i - 1)$, but without $(n_i, n_1), (n_i, n_2), ..., (n_i, n_{i-1})$.

We now need to count all there inversions:

$$(n_1 - 1) + (n_2 - 2) + ... + (n_{S^c} - |S^c|) = \sum_{i=1}^{|S^c|} n_i - \sum_{i=1}^{|S^c|} i = \sum(|S^c|) - \frac{|S^c| \cdot (|S^c| + 1)}{2}.$$

We now observe that, since we have that $n_i \neq N$, $\forall i$, then it follows that every inversion (n_i, i) in σ gives rise to the $S_{n_i i}$ factor, and so when evaluated at $(\xi_{n_1}, \xi_{n_2}, ..., \xi_{n_{|S^c|}}) = (1, 1, ..., 1)$ gives the factor $\left(\frac{q}{p}\right)^{\Sigma(S^c) - \frac{|S^c|\cdot(|S^c|+1)}{2}}$.

We finally observe that the position m from the right in the factor $[A_\gamma]_{2...212...2,\, 2...21}$ doesn't change: indeed, since $\gamma = \sigma|_S$, the position from the right remains unchanged. ∎

Corollary 4.2.3. Under the hypotheses above, suppose that $\sigma(1) = n_1$, $\sigma(2) = n_2, ...,$ $\sigma(|S^c|) = n_{|S^c|}$, and suppose now also that $n_{|S^c|} = N$. Then:

$$[A_\sigma]_{2...212...2,\, 2...21}\big|_{\xi_{n_1},...,\xi_{n_{|S^c|}}=1} = \left(\frac{q-p}{p}\right) \cdot \left(\frac{q}{p}\right)^{|S|-m} \left(\frac{q}{p}\right)^{\Sigma(S^c\setminus\{\xi_N\}) - \frac{|S^c\setminus\{\xi_N\}|\cdot(|S^c\setminus\{\xi_N\}|+1)}{2}} \cdot \prod_{\substack{\alpha<\beta \\ \alpha,\beta\in S}} S_{\beta\alpha}$$

Proof:

Apply the previous proposition to the first $|S^c| - 1$ variables to obtain:

$$[A_\sigma]_{2...212...2,\, 2...21}\big|_{\xi_{n_1},...,\xi_{n_{|S^c|-1}}=1} = \left(\frac{q}{p}\right)^{\Sigma(S^c\setminus\{\xi_N\}) - \frac{|S^c\setminus\{\xi_N\}|\cdot(|S^c\setminus\{\xi_N\}|+1)}{2}} \cdot [A_\gamma]_{2...212...2,\, 2...21}.$$

Now evaluate at $\xi_N = 1$. Since we have that $\sigma(|S^c|) = N$, then it follows that there are exactly $N - |S^c| = |S|$ inversions in σ of the form (N, α): $(N, s_1), ..., (N, s_k)$. Next, since the position of the 1 is mth from the right, then it follows that we have $(m-1)$ $pT_{N\alpha}$-factors, a single $Q_{N\alpha}$-factor, and $(|S| - m)$ $S_{N\alpha}$-factors. So that, when we evaluate those at $\xi_N = 1$ we get $p \cdot \frac{1}{p} = 1$ to the power of $m - 1$, $\frac{q-p}{p}$ to the power of 1, and $\frac{q}{p}$ to the power of $|S| - m$, thereby obtaining:

$$\left(\frac{q}{p}\right)^{\Sigma(S^c\setminus\{\xi_N\}) - \frac{|S^c\setminus\{\xi_N\}|\cdot(|S^c\setminus\{\xi_N\}|+1)}{2}} \cdot [A_\gamma]_{2...212...2,\, 2...21}\big|_{\xi_N=1}$$
$$= \left(\frac{q-p}{p}\right) \cdot \left(\frac{q}{p}\right)^{|S|-m} \left(\frac{q}{p}\right)^{\Sigma(S^c\setminus\{\xi_N\}) - \frac{|S^c\setminus\{\xi_N\}|\cdot(|S^c\setminus\{\xi_N\}|+1)}{2}} \cdot \prod_{\substack{\alpha<\beta \\ \alpha,\beta\in S}} S_{\beta\alpha}. \blacksquare$$

4.3 The N-fold integral.

Our main goal in this section is to prove the following proposition:

Proposition 4.3.1. The N-fold integral in the distribution of the right-most species-1 particle is given by:

$$\prod_{i=1}^{N-1}(p^i - q^i)\oint_c ... \oint_c \oint_c \prod_{\beta>\alpha} T_{\beta\alpha} \cdot \frac{1-\prod_{i=1}^{N}\xi_i}{\prod_{i=1}^{N}(1-\xi_i)} W_{1...N}(\xi_1,\xi_2,...,\xi_N,x,t)\, d\xi_1 d\xi_2 ... d\xi_N.$$

As always, before we need to establish some results.

Proposition 4.3.2. Let m be some fixed positive integer, $m \le N$. Then, we have the following identity:

$$\sum_{|S|=m} \prod_{\substack{\beta>\alpha \\ \alpha,\beta\in S^c}} T_{\beta\alpha} \cdot \prod_{\substack{\beta>\alpha \\ \alpha,\beta\in S}} T_{\beta\alpha} \cdot \prod_{\substack{\beta>\alpha \\ \alpha\in S \\ \beta\in S^c}} S_{\beta\alpha} = \begin{bmatrix} N \\ m \end{bmatrix} \cdot \prod_{\beta>\alpha} T_{\beta\alpha}.$$

where the sum is over all subsets $S \subseteq \{1,2,...,N\}$ with $|S| = m$.

Proof:

To prove this, it suffices to show that

$$\frac{\sum_{|S|=m} \prod_{\substack{\beta>\alpha \\ \alpha,\beta\in S^c}} T_{\beta\alpha} \cdot \prod_{\substack{\beta>\alpha \\ \alpha,\beta\in S}} T_{\beta\alpha} \cdot \prod_{\substack{\beta>\alpha \\ \alpha\in S \\ \beta\in S^c}} S_{\beta\alpha}}{\prod_{\beta>\alpha} T_{\beta\alpha}} = \begin{bmatrix} N \\ m \end{bmatrix}.$$

Now observe that the left-hand side becomes:

$$\sum_{|S|=m} \prod_{\substack{\beta>\alpha \\ \alpha\in S \\ \beta\in S^c}} \frac{S_{\beta\alpha}}{T_{\beta\alpha}} \cdot \prod_{\substack{\beta>\alpha \\ \alpha\in S^c \\ \beta\in S}} \frac{1}{T_{\beta\alpha}} = \sum_{|S|=m} \prod_{\substack{\beta>\alpha \\ \alpha\in S \\ \beta\in S^c}} \left(-\frac{p+q\xi_\alpha\xi_\beta-\xi_\beta}{\xi_\beta-\xi_\alpha}\right) \cdot \prod_{\substack{\beta>\alpha \\ \alpha\in S^c \\ \beta\in S}} \left(\frac{p+q\xi_\alpha\xi_\beta-\xi_\alpha}{\xi_\beta-\xi_\alpha}\right).$$

Now, observe that, if we multiply this expression by the Vandermonde $\prod_{\alpha<\beta}(\xi_\alpha - \xi_\beta)$, then we get an antisymmetric polynomial which is $O(\xi_i^{N-1})$ at infinity, and so has the degree at most $N-1$. And so, it is a constant times the Vandermonde, because any antisymmetric polynomial in $\xi_i, ..., \xi_N$ is divisible by the Vandermonde. Therefore, the original sum is a constant.

We now prove the result by induction. Assume that the identity holds for $N-1$. Then, to evaluate the constant C, let us evaluate the expression at $\xi_N = 1$:

$$C_{N,m} = \sum_{|S|=m} \prod_{\substack{\beta>\alpha \\ \alpha\in S \\ \beta\in S^c}} \left(-\frac{p+q\xi_\alpha\xi_\beta-\xi_\beta}{\xi_\beta-\xi_\alpha}\right)\Bigg|_{\xi_N=1} \cdot \prod_{\substack{\beta>\alpha \\ \alpha\in S^c \\ \beta\in S}} \left(\frac{p+q\xi_\alpha\xi_\beta-\xi_\alpha}{\xi_\beta-\xi_\alpha}\right)\Bigg|_{\xi_N=1}$$

Now, we will need the following expressions:

$$-\left.\frac{p+q\xi_\alpha\xi_N-\xi_N}{\xi_N-\xi_\alpha}\right|_{\xi_N=1}=-\frac{q\xi_\alpha-q}{1-\xi_\alpha}=q;\qquad \left.\frac{p+q\xi_\alpha\xi_N-\xi_\alpha}{\xi_N-\xi_\alpha}\right|_{\xi_N=1}=\frac{p-p\xi_\alpha}{1-\xi_\alpha}=p.$$

so that we get:

$$C_{N,m}=\sum_{\substack{|S|=m\\ N\in S}}\prod_{\substack{\beta>\alpha\\ \alpha\in S\\ \beta\in S^c}}\left(-\frac{p+q\xi_\alpha\xi_\beta-\xi_\beta}{\xi_\beta-\xi_\alpha}\right)\cdot\prod_{\substack{\beta>\alpha\\ \alpha\in S^c\\ \beta\in S-\{N\}}}\left(\frac{p+q\xi_\alpha\xi_\beta-\xi_\alpha}{\xi_\beta-\xi_\alpha}\right)\cdot p^{N-m}+$$
$$+\sum_{\substack{|S|=m\\ N\notin S}}\prod_{\substack{\beta>\alpha\\ \alpha\in S\\ \beta\in S^c-\{N\}}}\left(-\frac{p+q\xi_\alpha\xi_\beta-\xi_\beta}{\xi_\beta-\xi_\alpha}\right)\cdot\prod_{\substack{\beta>\alpha\\ \alpha\in S^c\\ \beta\in S}}\left(\frac{p+q\xi_\alpha\xi_\beta-\xi_\alpha}{\xi_\beta-\xi_\alpha}\right)\cdot q^m.$$

By inductive hypothesis, this equals:

$$C_{N,m}=\begin{bmatrix}N-1\\ m-1\end{bmatrix}\cdot p^{N-m}+\begin{bmatrix}N-1\\ m\end{bmatrix}\cdot q^m.$$

We now observe that this is identically equal to what we need:

$$C_{N,m}=\begin{bmatrix}N-1\\ m-1\end{bmatrix}\cdot p^{N-m}+\begin{bmatrix}N-1\\ m\end{bmatrix}\cdot q^m=\frac{[N-1]!}{[N-m]!\cdot[m-1]!}\cdot p^{N-m}+\frac{[N-1]!}{[N-m-1]!\cdot[m]!}\cdot q^m$$
$$=\frac{[N-1]!\cdot[m]\cdot p^{N-m}+[N-1]!\cdot[N-m]\cdot q^m}{[N-m]!\cdot[m]!}=\left(\frac{[N-1]!}{[N-m]!\cdot[m]!}\right)\cdot\left(\frac{p^N-q^mp^{N-m}+q^mp^{N-m}-q^N}{p-q}\right)$$
$$=\left(\frac{[N-1]!}{[N-m]!\cdot[m]!}\right)\cdot\left(\frac{p^N-q^N}{p-q}\right)=\left(\frac{[N-1]!}{[N-m]!\cdot[m]!}\right)\cdot[N]=\frac{[N]!}{[N-m]!\cdot[m]!}=\begin{bmatrix}N\\ m\end{bmatrix}.\quad\blacksquare$$

To prove the theorem of this section, we will need to sum over each column separately, as was remarked in the previous chapter:

Proposition 4.3.3. Split the factor $Q_{\beta\alpha}$ as $S_{\beta\alpha}-pT_{\beta\alpha}$ so that we are working in the $[A^\circ_\sigma]$ table. Then the coefficient before the sum of N-fold integrands over the k-th column of this table is given by:

$$C_{N,k}=(-1)^{N-k}\begin{bmatrix}N-1\\ N-k\end{bmatrix}q^{\frac{(N-k)(N-k+1)}{2}}p^{\frac{k(k-1)}{2}}.$$

Proof:

First of all, the summation over the geometric series yields:

$$\sum_{\sigma\in\mathbb{S}_N}\left[\frac{1}{(\xi_{\sigma(1)}\cdots\xi_{\sigma(N-k+1)}-1)\ldots(\xi_{\sigma(1)}-1)}\right]\cdot\left[\frac{\xi_{\sigma(N)}\cdots\xi_{\sigma(N-k+3)}}{1-\xi_{\sigma(N)}\cdots\xi_{\sigma(N-k+3)}}\cdot\ldots\cdot\frac{\xi_{\sigma(N)}}{1-\xi_{\sigma(N)}}\right]\cdot(-1)\cdot[A^\circ_\sigma]_{2\ldots212\ldots2,2\ldots21}$$
$$+\left[\frac{1}{(\xi_{\sigma(1)}\cdots\xi_{\sigma(N-k)}-1)\ldots(\xi_{\sigma(1)}-1)}\right]\cdot\left[\frac{\xi_{\sigma(N)}\cdots\xi_{\sigma(N-k+2)}}{1-\xi_{\sigma(N)}\cdots\xi_{\sigma(N-k+2)}}\cdot\ldots\cdot\frac{\xi_{\sigma(N)}}{1-\xi_{\sigma(N)}}\right]\cdot[A^\circ_\sigma]_{2\ldots212\ldots2,2\ldots21}\cdot$$

We now observe that the numerator of this expression becomes:

$$-(1-\xi_{\sigma(N)}...\xi_{\sigma(N-k+2)})\cdot(\xi_{\sigma(N)}...\xi_{\sigma(N-k+3)})\cdot...\cdot\xi_{\sigma(N)}+(\xi_{\sigma(1)}...\xi_{\sigma(N-k+1)}-1)\cdot(\xi_{\sigma(N)}...\xi_{\sigma(N-k+2)})\cdot...\cdot\xi_{\sigma(N)}$$
$$=\{\xi_{\sigma(N)}...\xi_{\sigma(N-k+2)}-1+(\xi_{\sigma(1)}...\xi_{\sigma(N-k+1)}-1)\cdot(\xi_{\sigma(N)}...\xi_{\sigma(N-k+2)})\}\cdot\xi_{\sigma(N-k+3)}...\xi_{\sigma(N)}^{k-2}$$
$$=\{\xi_{\sigma(1)}...\xi_{\sigma(N)}-1\}\xi_{\sigma(N-k+3)}...\xi_{\sigma(N)}^{k-2}=-\{1-\xi_{\sigma(1)}...\xi_{\sigma(N)}\}\xi_{\sigma(N-k+3)}...\xi_{\sigma(N)}^{k-2}=-\{1-\xi_1...\xi_N\}\xi_{\sigma(N-k+3)}...\xi_{\sigma(N)}^{k-2}.$$

So that the whole expression becomes:

$$\sum_{\sigma\in\mathbb{S}_N}\left[\frac{1}{(\xi_{\sigma(1)}...\xi_{\sigma(N-k)}-1)...(\xi_{\sigma(1)}-1)}\right]\cdot\left[\frac{1-\xi_1...\xi_N}{(1-\xi_{\sigma(1)}...\xi_{\sigma(N-k+1)})\cdot(1-\xi_{\sigma(N)}...\xi_{\sigma(N-k+2)})}\right]$$
$$\cdot\left[\frac{\xi_{\sigma(N)}...\xi_{\sigma(N-k+3)}}{1-\xi_{\sigma(N)}...\xi_{\sigma(N-k+3)}}\cdot...\cdot\frac{\xi_{\sigma(N)}}{1-\xi_{\sigma(N)}}\right]\cdot[A_\sigma^\circ]_{2...212...2,2...21}.$$

To go further, we need to write out the coefficient $[A_\sigma^\circ]_{2...212...2,2...21}$ explicitly:

$$\sum_{\sigma\in\mathbb{S}_N}\left[\frac{1}{(\xi_{\sigma(1)}...\xi_{\sigma(N-k)}-1)...(\xi_{\sigma(1)}-1)}\right]\cdot\left[\frac{1-\xi_1...\xi_N}{(1-\xi_{\sigma(1)}...\xi_{\sigma(N-k+1)})\cdot(1-\xi_{\sigma(N)}...\xi_{\sigma(N-k+2)})}\right]$$
$$\cdot\left[\frac{\xi_{\sigma(N)}...\xi_{\sigma(N-k+3)}}{1-\xi_{\sigma(N)}...\xi_{\sigma(N-k+3)}}\cdot...\cdot\frac{\xi_{\sigma(N)}}{1-\xi_{\sigma(N)}}\right]\prod_{\alpha<\beta<N}S_{\beta\alpha}\cdot\prod_{n=1}^{N-k+1}S_{N,\sigma(n)}\cdot\prod_{n=1}^{k-1}pT_{N,\sigma(N+1-n)}.$$

Let $S^c=\{\sigma(1),...,\sigma(N-k+1)\}$, and $S=\{\sigma(N-k+2),...,\sigma(N)\}$. Now we split the coefficient $\prod_{\alpha<\beta<N}S_{\beta\alpha}$ as $\prod_{\substack{\alpha<\beta<N\\ \alpha\in S\\ \beta\in S^c}}S_{\beta\alpha}\cdot\prod_{\substack{\alpha<\beta<N\\ \beta,\alpha\in S}}S_{\beta\alpha}\cdot\prod_{\substack{\alpha<\beta<N\\ \beta,\alpha\in S^c}}S_{\beta\alpha}$. We now split the sum over all permutations $\sigma\in\mathbb{S}_N$ into a triple sum: the sum over all permutations in the set $\{1,...,N-k+1\}$, the sum over all permutations in the set $\{N-k+2,...,N\}$, and finally the sum over all subsets $S^c\subset\{1,2,...,N\}$ of cardinality $N-k+1$.

Next, we observe that $\prod_{n=1}^{N-k+1}S_{N,\sigma(n)}\cdot\prod_{n=1}^{k-1}T_{N,\sigma(N+1-n)}=\prod_{\alpha\in S^c}S_{N\alpha}\prod_{\alpha\in S}T_{N\alpha}$ .We now make the following important observation: the coefficient $[A_\sigma^\circ]$ is nonzero at first place if and only if $N\in S^c$. Therefore, it follows that the whole product becomes:

$$\prod_{\substack{\alpha<\beta<N\\ \alpha\in S\\ \beta\in S^c}}S_{\beta\alpha}\prod_{\substack{\alpha<\beta<N\\ \beta,\alpha\in S}}S_{\beta\alpha}\prod_{\substack{\alpha<\beta<N\\ \beta,\alpha\in S^c}}S_{\beta\alpha}\prod_{\substack{\alpha\in S^c\\ N\in S^c}}S_{N\alpha}\prod_{\substack{\alpha\in S\\ N\in S^c}}T_{N\alpha}=\prod_{\substack{\alpha<\beta<N\\ \alpha\in S\\ \beta\in S^c}}S_{\beta\alpha}\prod_{\beta,\alpha\in S}S_{\beta\alpha}\prod_{\beta,\alpha\in S^c}S_{\beta\alpha}\prod_{\substack{\alpha\in S\\ N\in S^c}}T_{N\alpha}.$$

Moreover, the sum over all subsets S^c of $\{1,2,...,N\}$ with cardinality $N-k+1$ reduces to the sum over all subsets $S^c\backslash\{N\}$ of $\{1,2,...,N-1\}$ of cardinality $N-k$.

Therefore, since we have the factors $\prod_{\beta,\alpha\in S}S_{\beta\alpha}$ and $\prod_{\beta,\alpha\in S^c}S_{\beta\alpha}$, we can apply the Tracy-Widom integral formula for the left-most particle in the variables $\xi_{\sigma(N-k+2)},...,\xi_{\sigma(N)}$ and right-most particle in the variables $\xi_{\sigma(1)},...,\xi_{\sigma(N-k+1)}$:

$$\sum_{\substack{|S^c|=N-k+1\\ N\in S^c}}\left[q^{\frac{(N-k)(N-k+1)}{2}}(-1)^{N-k}\frac{1-\xi_1...\xi_{N-k+1}}{(1-\xi_1)...(1-\xi_{N-k+1})}\right]\prod_{\beta,\alpha\in S^c}T_{\beta\alpha}$$
$$\cdot\left[\frac{1-\xi_1...\xi_N}{(1-\xi_1...\xi_{N-k+1})\cdot(1-\xi_N...\xi_{N-k+2})}\right]\cdot p^{k-1}\prod_{\substack{\alpha<\beta<N\\ \alpha\in S\\ \beta\in S^c}}S_{\beta\alpha}\prod_{\substack{\alpha\in S\\ N\in S^c}}T_{N\alpha}$$
$$\cdot\left[p^{(k-2)(k-1)/2}(-1)^{1-1}\frac{1-\xi_{N-k+2}...\xi_N}{(1-\xi_{N-k+2})...(1-\xi_N)}\right]\cdot\prod_{\beta,\alpha\in S}T_{\beta\alpha}$$
$$=\sum_{\substack{|S^c|=N-k+1\\ N\in S^c}}\left[q^{\frac{(N-k)(N-k+1)}{2}}p^{(k-1)k/2}(-1)^{N-k}\frac{1-\xi_1...\xi_N}{(1-\xi_1)...(1-\xi_N)}\right]$$
$$\cdot\prod_{\beta,\alpha\in S}T_{\beta\alpha}\prod_{\beta,\alpha\in S^c}T_{\beta\alpha}\prod_{\substack{\alpha<\beta<N\\ \alpha\in S\\ \beta\in S^c}}S_{\beta\alpha}\prod_{\substack{\alpha\in S\\ N\in S^c}}T_{N\alpha}.$$

We now write: $\prod_{\beta,\alpha\in S^c}T_{\beta\alpha}\prod_{\substack{\alpha\in S\\ N\in S^c}}T_{N\alpha}=\prod_{\substack{\alpha<\beta<N\\ \beta,\alpha\in S^c}}T_{\beta\alpha}\prod_{\alpha<N}T_{N\alpha}$, and apply the previous proposition to the variables $\xi_1,...,\xi_{N-1}$ to obtain:

$$\sum_{\substack{|S^c|=N-k+1\\ N\in S^c}}\left[q^{\frac{(N-k)(N-k+1)}{2}}p^{(k-1)k/2}(-1)^{N-k}\frac{1-\xi_1...\xi_N}{(1-\xi_1)...(1-\xi_N)}\right]\prod_{\beta,\alpha\in S}T_{\beta\alpha}\prod_{\substack{\alpha<\beta<N\\ \beta,\alpha\in S^c}}T_{\beta\alpha}\prod_{\substack{\alpha<\beta<N\\ \alpha\in S\\ \beta\in S^c}}S_{\beta\alpha}\prod_{\alpha<N}T_{N\alpha}$$
$$=\begin{bmatrix}N-1\\ N-k\end{bmatrix}\cdot\left(q^{\frac{(N-k)(N-k+1)}{2}}p^{(k-1)k/2}(-1)^{N-k}\frac{1-\xi_1...\xi_N}{(1-\xi_1)...(1-\xi_N)}\right)\prod_{\alpha<\beta<N}T_{\beta\alpha}\prod_{\alpha<N}T_{N\alpha}$$
$$=\begin{bmatrix}N-1\\ N-k\end{bmatrix}\cdot\left(q^{\frac{(N-k)(N-k+1)}{2}}p^{(k-1)k/2}(-1)^{N-k}\frac{1-\xi_1...\xi_N}{(1-\xi_1)...(1-\xi_N)}\right)\prod_{\alpha<\beta}T_{\beta\alpha}.\ \blacksquare$$

Corollary 4.3.4. The N-fold integral in the distribution of the right-most species-1 particle is given by:

$$\prod_{i=1}^{N-1}(p^i-q^i)\oint_c...\oint_c\oint_c\prod_{\beta>\alpha}T_{\beta\alpha}\cdot\frac{1-\prod_{i=1}^{N}\xi_i}{\prod_{i=1}^{N}(1-\xi_i)}W_{1...N}(\xi_1,\xi_2,...,\xi_N,x,t)\,d\xi_1d\xi_2...d\xi_N.$$

Proof:

We have that the coefficient before the sum of the integrands over the kth column in the $[A_\sigma^\circ]$ table is given by $C_{N,k}=(-1)^{N-k}\begin{bmatrix}N-1\\ N-k\end{bmatrix}q^{\frac{(N-k)(N-k+1)}{2}}p^{\frac{k(k-1)}{2}}$. What remains is to sum all these coefficients over the whole table (that is, over $k=1,...,N$):

$$\sum_{k=1}^{N} C_{N,k} = \sum_{k=1}^{N} (-1)^{N-k} \begin{bmatrix} N-1 \\ N-k \end{bmatrix} q^{\frac{(N-k)(N-k+1)}{2}} p^{\frac{k(k-1)}{2}} = \sum_{k=0}^{N-1} (-1)^{k} \begin{bmatrix} N-1 \\ k \end{bmatrix} q^{\frac{k(k+1)}{2}} p^{\frac{(N-k)(N-k-1)}{2}} = \prod_{i=1}^{N-1} (p^i - q^i). \blacksquare$$

4.4 Lower-dimensional integrals

We conclude the chapter with the study of lower-dimensional integrals, leading us to the final form of the integral formula.

Proposition 4.4.1. Consider an arbitrary integrand of the form:

$$\left[\frac{1}{(\xi_{\sigma(1)} \dots \xi_{\sigma(N-k)} - 1) \dots (\xi_{\sigma(1)} - 1)}\right] \cdot \left[\frac{\xi_{\sigma(N)} \cdots \xi_{\sigma(N-k+2)}}{1 - \xi_{\sigma(N)} \cdots \xi_{\sigma(N-k+2)}} \cdot \ldots \cdot \frac{\xi_{\sigma(N)}}{1 - \xi_{\sigma(N)}}\right] \cdot [A_\sigma]_{2\dots212\dots2,2\dots21} \cdot$$

Let $c_1 < c_2 < \dots < c_n$. Then, the residues at $\xi_{c_n} = \frac{1}{\xi_{c_1}\xi_{c_2}\cdots\xi_{c_{n-1}}}$ vanish.

Remark 4.4.2. There is nothing special about taking the residue at the value with the largest index only; it is just the convention that we stick to.

Proof:

Let σ be an arbitrary permutation, and let $[A_\sigma]_{2\dots212\dots2,2\dots21}$ be the coefficient with the 1 being kth from the right, and let $\xi_{c_n} = \frac{1}{\xi_{c_1}\xi_{c_2}\cdots\xi_{c_{n-1}}}$ be an arbitrary residue. We will group up this residue with other residues of this type, evaluate at $\xi_{c_n} = \frac{1}{\xi_{c_1}\xi_{c_2}\cdots\xi_{c_{n-1}}}$ and obtain zero.

First of all, we remark that $n \le N - k$, because the maximum number of variables in the geometric series when k is the position of 1 from the right is $N - k$.

Next, from the shape of the denominator we see that if there is a residue at $\xi_{c_n} = \frac{1}{\xi_{c_1}\xi_{c_2}\cdots\xi_{c_{n-1}}}$, then it means that the map σ takes the set $\{1, \dots, n\}$ to the set $\{c_1, \dots, c_n\}$. Taking the residue at $\xi_{c_n} = \frac{1}{\xi_{c_1}\xi_{c_2}\cdots\xi_{c_{n-1}}}$ (without evaluating) gives:

$$\frac{1}{\xi_{c_1}\xi_{c_2}\cdots\xi_{c_{n-1}}} \cdot \frac{1}{(\xi_{\sigma(1)} \dots \xi_{\sigma(n-1)} - 1) \dots (\xi_{\sigma(1)} - 1)} \cdot \left[\frac{1}{(\xi_{\sigma(1)} \dots \xi_{\sigma(n+1)} - 1) \dots (\xi_{\sigma(1)} - 1)}\right]$$
$$\dots \cdot \left[\frac{\xi_{\sigma(N)} \cdots \xi_{\sigma(N-k+2)}}{1 - \xi_{\sigma(N)} \cdots \xi_{\sigma(N-k+2)}} \cdot \ldots \cdot \frac{\xi_{\sigma(N)}}{1 - \xi_{\sigma(N)}}\right] \cdot [A_\sigma]_{2\dots212\dots2,2\dots21}$$

Now let us sum this integrand over all permutations σ such that $\sigma(n+1), \sigma(n+2), \dots, \sigma(N)$ are all fixed (so that it is possible to group them up by the $W_{1\dots N}$ term). In other words, we sum over all permutations in $\{c_1, c_2, \dots, c_n\}$.

We now recall that $[A_\sigma]_{2\dots212\dots2,2\dots21} = \prod_{\alpha<\beta<N} S_{\beta,\alpha} \cdot \prod_{i=1}^{N-k} S_{N,\sigma(i)} \cdot \prod_{i=1}^{k-1} pT_{N,\sigma(N+1-i)} \cdot Q_{N,\sigma(N+1-k)}$ and $\sigma(j) = N$, $\exists j \le N - k + 1$. We also observe the following: since $\sigma(n+1), \sigma(n+2), \dots, \sigma(N)$ are

all fixed and $n \le N - k$, then the coefficient $\prod_{i=1}^{k-1} pT_{N,\sigma(N+1-i)} \cdot Q_{N,\sigma(N+1-k)}$ doesn't change as σ changes; similarly, the part of the coefficient $\prod_{\alpha<\beta<N} S_{\beta,\alpha} \cdot \prod_{i=1}^{N-k} S_{N,\sigma(i)}$ that doesn't involve ξ_{c_i} variables also remains unchanged. Therefore, when we sum over the permutations in $\{c_1, c_2, ..., c_n\}$, we get the Tracy-Widom's top-dimensional integrand for the right-most particle in n particles, thereby obtaining:

$$\frac{1}{\xi_{c_1}\xi_{c_2}\cdots\xi_{c_{n-1}}} \cdot \left[\frac{1}{(\xi_{\sigma(1)}\cdots\xi_{\sigma(n+1)} - 1)\ldots(\xi_{\sigma(1)} - 1)}\right] \cdot \ldots \cdot \left[\frac{\xi_{\sigma(N)}\cdots\xi_{\sigma(N-k+2)}}{1 - \xi_{\sigma(N)}\cdots\xi_{\sigma(N-k+2)}} \cdot \ldots \cdot \frac{\xi_{\sigma(N)}}{1 - \xi_{\sigma(N)}}\right] \cdot$$

$$\cdot q^{n(n-1)/2}(-1)^{n-1} \prod_{\substack{\beta>\alpha \\ \alpha,\beta\in S^c}} T_{\beta\alpha} \cdot \frac{1 - \prod_{i=1}^{n} \xi_{c_i}}{\prod_{i=1}^{n} (1 - \xi_{c_i})} \cdot \prod_{(\beta,\alpha)} S_{\beta\alpha} \cdot \prod_{i=1}^{k-1} pT_{N,\sigma(N+1-i)} \cdot Q_{N,\sigma(N+1-k)} \cdot$$

where the set S^c is given by $\{c_1, c_2, ..., c_n\}$, and the product $\prod_{(\beta,\alpha)} S_{\beta\alpha}$ is the product over all pairs of inversions (β, α) that remain after the factors $\prod_{\substack{\beta>\alpha \\ \alpha,\beta\in S^c}} S_{\beta\alpha}$ are gone.

It now becomes clear that when we evaluate this expression at $\xi_{c_n} = \frac{1}{\xi_{c_1}\xi_{c_2}\cdots\xi_{c_{n-1}}}$, everything becomes zero (because of the $1 - \prod_{i=1}^{n} \xi_{c_i}$ factor). ■

Corollary 4.4.3. Let $\sigma \in \mathbb{S}_N$, and consider an arbitrary integrand of the form:

$$\left[\frac{1}{(\xi_{\sigma(1)}\cdots\xi_{\sigma(N-k)} - 1)\ldots(\xi_{\sigma(1)} - 1)}\right] \cdot \left[\frac{\xi_{\sigma(N)}\cdots\xi_{\sigma(N-k+2)}}{1 - \xi_{\sigma(N)}\cdots\xi_{\sigma(N-k+2)}} \cdot \ldots \cdot \frac{\xi_{\sigma(N)}}{1 - \xi_{\sigma(N)}}\right] \cdot [A_\sigma]_{2\ldots212\ldots2,2\ldots21} \cdot$$

Suppose that the integrand gives rise to a nonvanishing residue in $|S|$ variables. Then, the map σ is necessarily order-preserving in the first $N - |S|$ variables.

Proof:

To get a residue in $|S|$ variables, we need to have the $N - |S|$-fold residue. Moreover, since the residue is nonvanishing, all $\xi_{c_n} = \frac{1}{\xi_{c_1}\xi_{c_2}\cdots\xi_{c_{n-1}}}$ factors must disappear. This is possible only if all the residues are given by: $\xi_{c_n} = \frac{1}{\xi_{c_1}\xi_{c_2}\cdots\xi_{c_{n-1}}}, ..., \xi_{c_2} = \frac{1}{\xi_{c_1}}$, and $\xi_{c_1} = 1$. And this is possible only if the map σ is order-preserving in $1, 2, ..., n$, $n = N - |S|$. ■

Remark 4.4.4. Notice that every permutation σ is always order-preserving in the first variable, trivially. Therefore, almost every integrand gives rise to an $N - 1$-dimensional residue; the only exception are the integrands with the coefficient $[A_\sigma]_{12\ldots2,2\ldots21}$, which don't require any contour deformations, and hence don't give any residues at all.

Proposition 4.4.5. Let $S = \{s_1, ..., s_k\} \subseteq \{1, 2, ..., N\}$, $\xi_N \in S$. Next, let $S^c = \{c_1, ..., c_{N-k}\}$ with $c_1 < c_2 < ... < c_{N-k}$. Then, the k-fold integral over S is given by:

$$\prod_{i=1}^{|S|-1} (p^i - q^i) \left(\frac{q}{p}\right)^{\Sigma(S^c) - \frac{|S^c|\cdot(|S^c|+1)}{2}} I(x, Y_S).$$

Proof:

By virtue of the previous corollary, the integrals over $S \subset \{1, ..., N\}$, $|S| = k$ are exactly the sums over σ such that $\sigma(1) = c_1, ..., \sigma(N-k) = c_{N-k}$, with the residues taken at $\xi_{c_{N-k}} = \frac{1}{\xi_{c_1} \cdots \xi_{c_{N-k-1}}}, ..., \xi_{c_2} = \frac{1}{\xi_{c_1}}, \xi_{c_1} = 1$, which is equivalent to the evaluation at $\xi_{c_{N-k}} = 1, ..., \xi_{c_2} = 1, \xi_{c_1} = 1$. Therefore, the $|S|$-fold integrand over $\xi_{s_1}, ..., \xi_{s_k}$ (without the W-term) becomes:

$$\sum_{\substack{\sigma \in \mathbb{S}_N: \\ \sigma(1)=c_1 \\ \cdots \\ \sigma(N-k)=c_{N-k}}} \frac{1}{\xi_{\sigma(N-k+1)} - 1} \cdot ... \cdot \frac{1}{\xi_{\sigma(N-k+1)} \cdots \xi_{\sigma(N-1)} - 1} [A_\sigma]_{2...21,\, 2...21} \Big|_{\substack{\xi_{\sigma(i)}=1 \\ 1 \le i \le N-k}}$$

$$+ ... + \sum_{\substack{\sigma \in \mathbb{S}_N: \\ \sigma(1)=c_1 \\ \cdots \\ \sigma(N-k)=c_{N-k}}} \frac{\xi_{\sigma(N-k+2)} \cdots \xi_{\sigma(N)}}{1 - \xi_{\sigma(N-k+2)} \cdots \xi_{\sigma(N)}} \cdot ... \cdot \frac{\xi_{\sigma(N)}}{1 - \xi_{\sigma(N)}} [A_\sigma]_{2...212...2,\, 2...21} \Big|_{\substack{\xi_{\sigma(i)}=1 \\ 1 \le i \le N-k}} .$$

We now observe that by the proposition 4.2.2, $[A_\sigma]_{2...212...2,\, 2...21} \Big|_{\substack{\xi_{\sigma(i)}=1 \\ 1 \le i \le N-k}} = \left(\frac{q}{p}\right)^{\Sigma(S^c) - \frac{|S^c|\cdot(|S^c|+1)}{2}} [A_\gamma]_{2...212...2,\, 2...21}$ with the position of 1 from the right unchanged. Next, since the coefficient $\left(\frac{q}{p}\right)^{\Sigma(S^c) - \frac{|S^c|\cdot(|S^c|+1)}{2}}$ doesn't depend on the permutation σ, we can put it outside of the sum. Therefore, we obtain:

$$\left(\frac{q}{p}\right)^{\Sigma(S^c) - \frac{|S^c|\cdot(|S^c|+1)}{2}} \cdot \Bigg[\sum_{\gamma \in \mathbb{S}_S} \frac{1}{\xi_{\sigma(N-k+1)} - 1} \cdot ... \cdot \frac{1}{\xi_{\sigma(N-k+1)} \cdots \xi_{\sigma(N-1)} - 1} [A_\gamma]_{2...21,\, 2...21}$$
$$+ ... + \sum_{\gamma \in \mathbb{S}_S} \frac{\xi_{\sigma(N-k+2)} \cdots \xi_{\sigma(N)}}{1 - \xi_{\sigma(N-k+2)} \cdots \xi_{\sigma(N)}} \cdot ... \cdot \frac{\xi_{\sigma(N)}}{1 - \xi_{\sigma(N)}} [A_\gamma]_{12...2,\, 2...21} \Bigg].$$

We finally observe that the sum inside the brackets is nothing but the sum that we had for the N-fold integral, but over the variables $\xi_{s_1}, ..., \xi_{s_k}$. Applying proposition 4.3.1 to the expression in the brackets gives:

$$\left(\frac{q}{p}\right)^{\Sigma(S^c) - \frac{|S^c|\cdot(|S^c|+1)}{2}} \cdot \left[\prod_{i=1}^{k-1} (p^i - q^i) \prod_{\substack{\beta > \alpha: \\ \alpha, \beta \in S}} T_{\beta\alpha} \cdot \frac{1 - \prod_{i=1}^{k} \xi_{s_i}}{\prod_{i=1}^{k} (1 - \xi_{s_i})} \right]. \blacksquare$$

Proposition 4.4.6. We have the following identity:

$$\sum_{m=1}^{|S|} p^{\frac{(|S|-m)(|S|-m+1)}{2}} q^{\frac{m(m-1)}{2}} (-1)^{m-1} \begin{bmatrix} |S|-1 \\ m-1 \end{bmatrix} \cdot (p-q) \cdot p^{|S|-m} q^{m-1} = \prod_{i=1}^{|S|} (p^i - q^i).$$

Proof:

By one of the propositions above, this is equivalent to showing that:

$$\sum_{m=1}^{|S|} p^{\frac{(|S|-m)(|S|-m+1)}{2}} q^{\frac{m(m-1)}{2}} (-1)^{m-1} \begin{bmatrix} |S|-1 \\ m-1 \end{bmatrix} \cdot (p-q) \cdot p^{|S|-m} q^{m-1} = \sum_{k=0}^{|S|} (-1)^k \begin{bmatrix} |S| \\ k \end{bmatrix} q^{\frac{k(k+1)}{2}} p^{\frac{(|S|-k+1)(|S|-k)}{2}}.$$

For this, it suffices to demonstrate the equality termwise. Now the first thing that we do is that we change the index set: $k = m-1$, so that $m = k+1$; then, our expression becomes:

$$\begin{aligned}
&\sum_{m=1}^{|S|} p^{\frac{(|S|-m)(|S|-m+1)}{2}} q^{\frac{m(m-1)}{2}} (-1)^{m-1} \begin{bmatrix} |S|-1 \\ m-1 \end{bmatrix} \cdot (p-q) \cdot p^{|S|-m} q^{m-1} \\
&= \sum_{k=0}^{|S|-1} p^{\frac{(|S|-k-1)(|S|-k)}{2}} q^{\frac{(k+1)k}{2}} (-1)^{k} \begin{bmatrix} |S|-1 \\ k \end{bmatrix} \cdot (p-q) \cdot p^{|S|-k-1} q^{k} \\
&= \sum_{k=0}^{|S|-1} p^{\frac{(|S|-k-1)(|S|-k)}{2}} q^{\frac{(k+1)k}{2}} (-1)^{k} \begin{bmatrix} |S|-1 \\ k \end{bmatrix} \cdot \left(p^{|S|-k} q^{k} - p^{|S|-k-1} q^{k+1}\right);
\end{aligned}$$

We now observe that $p^{\frac{(|S|-k-1)(|S|-k)}{2}} \cdot p^{|S|-k} = p^{\frac{(|S|-k)(|S|-k+1)}{2}}$, and $q^{\frac{(k+1)k}{2}} \cdot q^{k+1} = q^{\frac{(k+2)(k+1)}{2}}$, so that our expression becomes:

$$\sum_{k=0}^{|S|-1} \left(p^{\frac{(|S|-k)(|S|-k+1)}{2}} q^{\frac{(k+1)k}{2}} (-1)^k \begin{bmatrix} |S|-1 \\ k \end{bmatrix} \cdot q^k - p^{\frac{(|S|-k-1)(|S|-k)}{2}} q^{\frac{(k+2)(k+1)}{2}} (-1)^k \begin{bmatrix} |S|-1 \\ k \end{bmatrix} \cdot p^{|S|-k-1} \right);$$

We now rearrange the terms so that our sum runs over $k = 0, ..., |S|$ as follows:

- $k = 0:$

$$p^{\frac{(|S|-k)(|S|-k+1)}{2}} q^{\frac{(k+1)k}{2}} (-1)^k \begin{bmatrix} |S|-1 \\ k \end{bmatrix} \cdot q^k = p^{\frac{(|S|)(|S|+1)}{2}};$$

- $k = 1, ..., |S|-1:$

$$\begin{aligned}
&-p^{\frac{(|S|-k)(|S|-k+1)}{2}} q^{\frac{(k+1)k}{2}} (-1)^{k-1} \begin{bmatrix} |S|-1 \\ k-1 \end{bmatrix} \cdot p^{|S|-k} + p^{\frac{(|S|-k)(|S|-k+1)}{2}} q^{\frac{(k+1)k}{2}} (-1)^{k} \begin{bmatrix} |S|-1 \\ k \end{bmatrix} \cdot q^k \\
&= p^{\frac{(|S|-k)(|S|-k+1)}{2}} q^{\frac{(k+1)k}{2}} (-1)^{k} \begin{bmatrix} |S|-1 \\ k-1 \end{bmatrix} \cdot p^{|S|-k} + p^{\frac{(|S|-k)(|S|-k+1)}{2}} q^{\frac{(k+1)k}{2}} (-1)^{k} \begin{bmatrix} |S|-1 \\ k \end{bmatrix} \cdot q^k \\
&= p^{\frac{(|S|-k)(|S|-k+1)}{2}} q^{\frac{(k+1)k}{2}} (-1)^{k} \begin{bmatrix} |S| \\ k \end{bmatrix}.
\end{aligned}$$

- $k = |S|$:

$$-q^{\frac{(|S|+1)(|S|)}{2}}(-1)^{|S|-1}\begin{bmatrix}|S|-1\\|S|-1\end{bmatrix} = q^{\frac{(|S|+1)(|S|)}{2}}(-1)^{|S|}.$$

We finally observe that our new rearranged sum is exactly equal to $\sum_{k=0}^{|S|}(-1)^k\begin{bmatrix}|S|\\k\end{bmatrix}q^{\frac{k(k+1)}{2}}p^{\frac{(|S|-k+1)(|S|-k)}{2}}$ termwise. ■

Corollary 4.4.7. Let $S = \{s_1, ..., s_k\} \subseteq \{1, 2, ..., N\}$, $\xi_N \notin S$. Next, let $S^c = \{c_1, ..., c_{N-k}\}$ with $c_1 < c_2 < ... < c_{N-k}$. Then, the k-fold integral over S is given by:

$$-\frac{\prod_{i=1}^{|S|}(p^i - q^i)}{p^{|S|}}\left(\frac{q}{p}\right)^{\Sigma(S^c\setminus\{\xi_N\})-\frac{|S^c\setminus\{\xi_N\}|\cdot(|S^c\setminus\{\xi_N\}|+1)}{2}} I(x, Y_S).$$

Proof:

As before, we have that the integrand over $\xi_{s_1}, \xi_{s_2}, ..., \xi_{s_k}$ (without the W term) is given by:

$$\sum_{\substack{\sigma\in\mathbb{S}_N:\\ \sigma(1)=c_1\\ \vdots\\ \sigma(N-k)=c_{N-k}}} \frac{1}{\xi_{\sigma(N-k+1)}-1}\cdot\ldots\cdot\frac{1}{\xi_{\sigma(N-k+1)}\cdots\xi_{\sigma(N-1)}-1}[A_\sigma]_{2\ldots21,\,2\ldots21}\Big|_{\substack{\xi_{\sigma(i)}=1\\ 1\le i\le N-k}}$$
$$+\ldots+\sum_{\substack{\sigma\in\mathbb{S}_N:\\ \sigma(1)=c_1\\ \vdots\\ \sigma(N-k)=c_{N-k}}} \frac{\xi_{\sigma(N-k+2)}\cdots\xi_{\sigma(N)}}{1-\xi_{\sigma(N-k+2)}\cdots\xi_{\sigma(N)}}\cdot\ldots\cdot\frac{\xi_{\sigma(N)}}{1-\xi_{\sigma(N)}}[A_\sigma]_{2\ldots212\ldots2,\,2\ldots21}\Big|_{\substack{\xi_{\sigma(i)}=1\\ 1\le i\le N-k}}.$$

By the Corollary 4.2.3, this sum becomes:

$$\left(\frac{q}{p}\right)^{\Sigma(S^c\setminus\{\xi_N\})-\frac{|S^c\setminus\{\xi_N\}|\cdot(|S^c\setminus\{\xi_N\}|+1)}{2}}\cdot\Bigg[\sum_{\gamma\in\mathbb{S}_S}\frac{1}{\xi_{\sigma(N-k+1)}-1}\cdot\ldots\cdot\frac{1}{\xi_{\sigma(N-k+1)}\cdots\xi_{\sigma(N-1)}-1}$$
$$\cdot\left(\frac{q}{p}\right)^{|S|-1}\frac{q-p}{p}\prod_{\substack{\alpha<\beta\\ \alpha,\beta\in S}}S_{\beta\alpha}+\ldots+\sum_{\gamma\in\mathbb{S}_S}\frac{\xi_{\sigma(N-k+2)}\cdots\xi_{\sigma(N)}}{1-\xi_{\sigma(N-k+2)}\cdots\xi_{\sigma(N)}}\cdot\ldots\cdot\frac{\xi_{\sigma(N)}}{1-\xi_{\sigma(N)}}\frac{q-p}{p}\prod_{\substack{\alpha<\beta\\ \alpha,\beta\in S}}S_{\beta\alpha}\Bigg].$$

We now observe that each sum individually is nothing but the coefficient for the N-fold integrand in the Tracy-Widom integral formula for the kth rightmost particle (equivalently, the $|S| - k + 1$-th leftmost particle), $1 \le k \le |S|$. Applying the Tracy-Widom integral formula gives:

$$\frac{q-p}{p}\cdot\left(\frac{q}{p}\right)^{\Sigma(S^c\setminus\{\xi_N\})-\frac{|S^c\setminus\{\xi_N\}|\cdot(|S^c\setminus\{\xi_N\}|+1)}{2}}\sum_{k=1}^{|S|}\left(\frac{q}{p}\right)^{|S|-k}p^{(k-1)k/2}q^{(|S|-k+1)(|S|-k)/2}(-1)^{|S|-k}\begin{bmatrix}|S|-1\\|S|-k\end{bmatrix}.$$

We now factor out $\frac{1}{p^{|S|-1}}$ out of the sum, and put $p-q$ inside the sum to obtain:

$$-\frac{1}{p}\cdot\frac{1}{p^{|S|-1}}\cdot\left(\frac{q}{p}\right)^{\Sigma(S^c\setminus\{\xi_N\})-\frac{|S^c\setminus\{\xi_N\}|\cdot(|S^c\setminus\{\xi_N\}|+1)}{2}}\sum_{k=1}^{|S|}q^{|S|-k}\cdot p^{k-1}\cdot(p-q)p^{(k-1)k/2}q^{(|S|-k+1)(|S|-k)/2}(-1)^{|S|-k}\begin{bmatrix}|S|-1\\|S|-k\end{bmatrix}.$$

Applying the previous proposition gives:

$$-\frac{1}{p}\cdot\frac{1}{p^{|S|-1}}\cdot\left(\frac{q}{p}\right)^{\Sigma(S^c\setminus\{\xi_N\})-\frac{|S^c\setminus\{\xi_N\}|\cdot(|S^c\setminus\{\xi_N\}|+1)}{2}}\prod_{i=1}^{|S|}(p^i-q^i)=-\left(\frac{q}{p}\right)^{\Sigma(S^c\setminus\{\xi_N\})-\frac{|S^c\setminus\{\xi_N\}|\cdot(|S^c\setminus\{\xi_N\}|+1)}{2}}\frac{\prod_{i=1}^{|S|}(p^i-q^i)}{p^{|S|}}.\blacksquare$$

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