Variety of Bicommutative Algebras defined by identity $\gamma[(\mathbf{ab})\mathbf{c} - \mathbf{2}(\mathbf{ba})\mathbf{c} + (\mathbf{ca})\mathbf{b}] + \delta[\mathbf{c}(\mathbf{ba}) - \mathbf{2c}(\mathbf{ab}) + \mathbf{b}(\mathbf{ac})] = \mathbf{0}$

by

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Abstract

One of the important problem of the theory of polynomial identitites in algebra is describe all varieties of algebras with given system of identities. Our aim is to classify all subvarieties of the variety of bicommutative algebras. Classifying is usually done in the language of lattices. Of course this problem is equivalent to describing of T-ideals. In order to construct a lattice of subvarieties of given variety of algebras, we need to define the following 1) determine the module structure of $P_n(\mathfrak{M})$ over the symmetric group; 2) find for each irreducible S_n-module in $P_n(\mathfrak{M})$ a consequence in $P_{n+1}(\mathfrak{M})$.

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Contents

1 Introduction

One of the important problems of modern algebra is to study algebras satisfying some identities. In the theory of polynomial identities in algebras there are two main questions: 1) describe algebras with identities; 2) describe identities in algebras. Studying above the questions leads us to the study free algebras, construction bases of free algebras, finding Hilbert series, finding codimension sequences, finding codimension growth, finding Gelfand-Kirillov dimension, finding cocharacter sequences, finding colengths, investigating Specht problem and etc.

In the theory of polynomial identities, identity and algebra are mutually defining concepts. Their interconnection is determined by varieties of algebras. The language of varieties of algebras allows one to freely pass from identity to algebra and conversely. Therefore, studying varieties of algebras is one of important problems of the theory of polynomial identities.

In 1950, A.I. Malcev [1] and W.Specht [2] first time and independently used the representation theory of symmetric group to classify polynomial identities of algebraic structures. It is known that if K is a field of characteristic 0, then every polynomial is equivalent to a finite set of multilinear polynomials. Therefore, in this work we focus on module structures of multilinear components (parts) of free algebras. Since the multilinear parts of a free algebras contain a lot of useful and important informations about varieties of algebras. Here the module structure means modules over the symmetric group and the general linear group.

The methods of representation theory of the symmetric group and general linear group in many cases are considered to be the best methods in studying multilinear components of free algebras. There are known S_n and GL_n -module structures of several free algebras. For example, associative algebra, Leibniz algebra, Zinbiel algebra, Lie algebra, right-symmetric algebra, Novikov algebra and etc.

In many cases it is more convenient to apply the methods of the theory of representations of general linear group than representation theory of the symmetric group to varieties of algebras. We may see in the following works. For instance, in [3] there are fullly described varieties of associative algebras with identity of degree three by methods of the theory of representations of the general linear group. Criterion for the distributivity of the lattice of subvarieties of varieties of associative algebras using the methods of the theory of representations of general linear group [4]. Criterion to the distributivity of the lattice of subvarieties of varieties of alternative algebras is given using the methods of the theory of representations of general linear group as well [5].

Bicommutative algebras first appear in the work of Dzhumadil'daev and Tulenbaev [6]. In [6] authors proved that if A is a bicommutative algebra then A^2 is commutative and associative. In 2011 Dzhumadil'daev, Ismailov and Tulenbaev [7] calculated multiplicities of irreducible S_n-representations in decomposition of multilinear component via combinatorial method. Moreover, there are given construction of a basis of free bicommutative algebras, description of multiplication of basis elements, given cocharacter and codimension sequences, and calculated Hilbert series. It is also proved that bicommutative operad is not Koszul and the growth of codimension sequence of bicommutative algebra is equal to 2. Later there was given an alternative proof of the formula for the cocharacter sequence of bicommutative algebra [8]. In 2018 Drensky and Zhakhayev [9] proved that the Specht problem for varieties of bicommutative algebras is solved positively. The connection of bicommutative algebra with the filtration and grading of free Novikov algebra gives us great motivation to study bicommutative algebra in more depth.

In this work we study variety of bicommutative algebras defined by identity

$$\gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + b(ac)] = 0,$$

where $\gamma, \delta \in K$.

One of the important problem of the theory of polynomial identities in algebra is describe all varieties of algebras with given system of identities. Our aim is to classify all subvarieties of the variety of bicommutative algebras. Classifying is usually done in the language of lattices. Of course this problem is equivalent to describing of T-ideals. In order to construct a lattice of subvarieties of given variety of algebras, we need to define the following 1) determine the module structure of $P_n(\mathfrak{M})$ over the symmetric group; 2) find for each irreducible S_n-module in $P_n(\mathfrak{M})$ a consequence in $P_{n+1}(\mathfrak{M})$.

2 Preliminaries

In this section we will consider some definitions that we will use to formulate a theorem. Sources are indicated next to them.

Definition 2.1. ([10]) A vector space A is called an *algebra* if A is equipped with a binary operation $\cdot$, called multiplication, such that for any $a, b, c \in A$ and any $\alpha \in K$

$$(a+b)\cdot c = a\cdot c + a\cdot b,$$

$$a\cdot(b+c) = a\cdot b + a\cdot c,$$

$$\alpha(a\cdot b) = (\alpha a)\cdot b = a\cdot(\alpha b).$$

Definition 2.2. ([6]) An algebra $(A, \cdot)$ is called *bicommutative* if any $a, b, c \in A$ are satisfied the following identities

$$a\cdot(b\cdot c) = b\cdot(a\cdot c) \tag{1}$$

$$(a\cdot b)\cdot c = (a\cdot c)\cdot b \tag{2}$$

Let X be a set of generators $x_1, \ldots, x_n$.

Definition 2.3. ([10]) Let $\mathfrak{B}$ be a class of algebras and let $F(X) \in \mathfrak{B}$ be an algebra generated by a set X. The algebra $F(X)$ is called a *free algebra* in the class $\mathfrak{B}$, if for any algebra $A \in \mathfrak{B}$, every mapping $X \to A$ can be extended to a homomorphism $F(X) \to A$.

Let $K\{X\}$ be a free non-associative algebra generated by set $X = \{x_1, \ldots, x_n\}$.

Definition 2.4. ([10]) Let $f = f(x_1, ..., x_n) \in K\{X\}$ and let A be a non-associative algebra. We say that $f = 0$ is a *polynomial identity* for A if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$

Definition 2.5. ([10]) Let $\{f_i(x_1, ..., x_n) \in K\{X\} \mid i \in I\}$ be a system of polynomials. The class $\mathfrak{V}$ of all non-associative algebras satisfying the polynomial identities f_i, $i \in I$, is called the *variety* defined by the system of polynomial identities $\{f_i \mid i \in I\}$. The variety $\mathfrak{M}$ is called a *subvariety* of $\mathfrak{V}$ if $\mathfrak{M} \subset \mathfrak{V}$

Definition 2.6. ([10]) The set $T(\mathfrak{V})$ of all polynomial identities satisfied by the variety $\mathfrak{V}$ is called the *T-ideal* of $\mathfrak{V}$. We say, that the T-ideal $T(\mathfrak{V})$ is generated as a T-ideal by the defining set of identities $\{f_i | i \in I\}$ of the variety $\mathfrak{V}$.

Definition 2.7. Let V be a vector space over $\mathbb{R}$ and let G be a group. Then V is a *G-module* if a multiplication vg $(v \in V,\ g \in G)$ is defined, satisfying the following conditions for all $u, v \in V,\ \lambda \in \mathbb{R}$ and $g, h \in G$.

(1) $ug \in V$;
(2) $v(gh) = (vg)h$;
(3) $v1 = v$;
(4) $(\lambda v)g = \lambda(vg)$;
(5) $(u + v)g = ug + vg$.

Definition 2.8. ([11]) A polynomial $f = f(x_1, \ldots, x_n)$ is called *linear* in the variable x_i if x_i occurs in every monomial of f with degree 1. A polynomial f is called *multilinear* if it is linear in each variable.

3 Variety of Bicommutative Algebras defined by identity $\gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + b(ac)] = 0$

3.1 Main statements

Let $\mathfrak{M}$ be a variety of bicommutative algebras over a field K of characteristic 0 defined by identity

$$\gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + b(ac)] = 0, \qquad (3)$$

where $(\gamma, \delta) \neq (0,0)$, $\gamma, \delta \in K$.

Let $F(\mathfrak{M})$ be a free algebra in variety $\mathfrak{M}$, and let $F_n(\mathfrak{M})$ be a free algebra in $\mathfrak{M}$ generated by $X = \{x_1, \ldots, x_n\}$.

Let P_n be a space of multilinear polynomials of $F_n(\mathfrak{M})$ of degree n. Further we call the space of multilinear polynomials as multilinear component.

We have four cases depending on coefficients γ and δ
(I) $\gamma\delta(\gamma - \delta)(\gamma + \delta) \neq 0$
(II) $\gamma = 0,\ \delta \neq 0$ ($\gamma \neq 0,\ \delta = 0$ analogous)
(III) $\gamma + \delta = 0$
(IV) $\gamma - \delta = 0$

3.2 Case I: $\gamma\delta(\gamma + \delta)(\gamma - \delta) \neq 0$

Let $\mathfrak{M}$ be a variety of bicommutative algebras defined by identity

$$\gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + b(ac)] = 0 \qquad (4)$$

where $\gamma\delta(\gamma + \delta)(\gamma - \delta) \neq 0$.

Then we have the following theorem.

Theorem 3.1. As S_n-module

$$P_1(\mathfrak{M}) \cong S^{(1)}, P_2(\mathfrak{M}) \cong S^{(2)} \oplus S^{(1,1)},$$

$$P_3(\mathfrak{M}) \cong 2S^{(3)} \oplus S^{(2,1)}, P_4(\mathfrak{M}) \cong 2S^{(4)}$$

$$P_n(\mathfrak{M}) \cong S^{(n)} \text{ for } n \geq 5.$$

Proof. Let $n = 3$. The number of base elements of the multilinear component of degree 3 of free bicommutative algebra $F(\mathfrak{B})$ is equal to 6. There are 2 types of the base elements.

First type

$$V_3((**)*) = \{(ab)c, (ba)c, (ca)b\}.$$

Second type

$$V_3(*(**)) = \{c(ba), c(ab), b(ac)\}.$$

In [7] proved that $V_3((**)*)$ and $V_3(*(**))$ are invariant under action of symmetric group S_3 and as S_3-module

$$V_3((**)*) \cong S^{(3)} \oplus S^{(2,1)},$$
$$V_3(*(**)) \cong S^{(3)} \oplus S^{(2,1)},$$
$$P_3(\mathfrak{B}) \cong V_3((**)*) \oplus V_3(*(**)) \cong$$
$$2S^{(3)} \oplus 2S^{(2,1)}.$$

Since

$$(ab)c - 2(ba)c + (ca)b = -\frac{\delta}{\gamma}[c(ba) - 2c(ab) + b(ac)]$$

then we have

$$P_3(\mathfrak{M}) \cong 2S^{(3)} \oplus S^{(2,1)}.$$

Based on result the number of base elements of $P_3(\mathfrak{M})$ is equal to 4.

Let $n = 4$. The number of base elements of the multilinear component of degree 4 of free bicommutative algebra $F(\mathfrak{B})$ is equal to 14. There are 3 types of the base elements.

First type

$$V_4(((**)*)*) = \{((ab)c)d, ((ba)c)d, ((ca)b)d, ((da)b)c\}.$$

Second type

$$V_4(*((**)*)) = \{c((ab)d), b((ac)d), b((ad)c), a((bc)d), a((bd)c), a((cd)b)\}.$$

Third type

$$V_4(*(*(**))) = \{d(c(ba)), d(c(ab)), d(b(ac)), c(b(ad))\}.$$

In [7] proved that $V_4(((**)*)*)$, $V_4(*((**)*))$ and $V_4(*(*(**)))$ are invariant under action of symmetric group S_4 and as S_4-module

$$V_4(((**)*)*) \cong S^{(4)} \oplus S^{(3,1)},$$

$$V_4(*((**)*)) \cong S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)},$$
$$V_4(*(*(**))) \cong S^{(4)} \oplus S^{(3,1)},$$

and

$$P_4(\mathfrak{B}) \cong V_4(((**)*)*) \oplus V_4(*((**)*)) \oplus V_4(*(*(**))) \cong$$
$$3S^{(4)} \oplus 3S^{(3,1)} \oplus S^{(2,2)}.$$

Let $W_2^{(4)}(*(*(**)))$ be a space. The space $V_4(*(*(**)))$ as GL_4-module is isomorphic to

$$V_4(*(*(**))) \cong W^{(4)} \oplus W^{(3,1)}.$$

The irreducible GL_4-module $W^{(4)}$ is generated by the following polynomial (highest weight vector)

$$\begin{array}{|c|c|c|c|}\hline \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} \\ \hline \end{array} = a(a(aa)),$$

and the irreducible GL_4-module $W^{(3,1)}$ is generated by the following polynomial

$$\begin{array}{|c|c|c|}\hline a & a & a \\ \hline b \\ \cline{1-1} \end{array} = a(a(ab)) - a(a(ba)).$$

By substituting $a := aa,\ b := a,\ c := a$ into (4) we get a new identity

$$\gamma[((aa)a)a - 2(a(aa))a + (a(aa))a] + \delta[a(a(aa)) - 2a((aa)a) + a((aa)a)] = 0$$

$$\gamma[((aa)a)a - (a(aa))a] + \delta[a(a(aa)) - a((aa)a)] = 0$$
$$\gamma((aa)a)a - \gamma a((aa)a) + \delta a(a(aa)) - \delta a((aa)a)] = 0$$
$$\gamma((aa)a)a - (\gamma + \delta)a((aa)a) = -\delta a(a(aa)) \qquad (5)$$

Hence, the element $a(a(aa))$ can be expressed through elements $((aa)a)a$ and $a((aa)a)$.

By substituting $a := a,\ b := b,\ c := a$ into (4) we get a new identity

$$\gamma[(ab)a - 2(ba)a + (aa)b] + \delta[a(ba) - 2a(ab) + b(aa)] = 0$$

$$\gamma[2(aa)b - 2(ba)a] + \delta[2b(aa) - 2a(ab)] = 0$$
$$\gamma[(aa)b - (ba)a] + \delta[b(aa) - a(ab)] = 0 \qquad (6)$$

We multiply identity (6) by the generator a from the left side

$$\gamma[a((aa)b) - a((ba)a)] + \delta[a(b(aa)) - a(a(ab))] = 0$$

$$\gamma[a((aa)b) - b((aa)a)] + \delta[a((ba)) - a(a(ab))] = 0 \tag{7}$$

Hence, the element $a(a(ab)) - a(a(ba))$ can be expressed as linear combinations of elements of the type $*((**)*)$.

Now we consider elements of type $*((**)*)$.

The irreducible GL_4-submodule $W^{(4)}$ is generated by the following polynomial

$$\begin{array}{|c|c|c|c|}\hline a & a & a & a \\ \hline \end{array} = a((aa)a).$$

The irreducible GL_4-submodule $W^{(3,1)}$ is generated by the following polynomial

$$\begin{array}{|c|c|c|}\hline a & a & a \\ \hline b \\ \cline{1-1} \end{array} = a((ab)a) - a((ba)a).$$

The irreducible GL_4-submodule $W^{(2,2)}$ is generated by the following polynomial

$$\begin{array}{|c|c|}\hline a & a \\ \hline b & b \\ \hline \end{array} = (ab - ba)(ab - ba) =$$

$$(ab)(ab) - (ab)(ba) - (ba)(ab) + (ba)(ba) =$$

$$a((ab)b) - b((ab)a) - a((ba)b) + b((ba)a) =$$

$$a((ab)b) - b((aa)b) - b((aa)b) + b((ba)a)) =$$

$$a((ab)b) - 2b((aa)b) + b((ba)a).$$

We multiply identity (6) by the generator b from the right side

$$\gamma[((aa)b)b - ((ba)a)b] + \delta[(b(aa))b - (a(ab))b] = 0$$

Since there is no elements of type $((**)*)*$ which contains two a and two b. Therefore we have

$$((aa)b)b - ((ba)a)b = 0.$$

Hence, we obtain a new identity

$$b((aa)b) = a((ab)b). \tag{8}$$

We multiply identity (6) by the generator b from the left side

$$\gamma[b((aa)b) - b((ba)a)] + \delta[b(b(aa)) - b(a(ab))] = 0.$$

Since there is no elements of type $*(*(**))$ which contains two a and two b. Therefore we have

$$b(b(aa)) - b(a(ab)) = 0.$$

Hence we obtain a new identity

$$b((aa)b) = b((ba)a). \tag{9}$$

Using identitties (8) and (9) we get the following identity

$$\begin{array}{|c|c|}\hline a & a \\ \hline b & b \\ \hline \end{array} = a((ab)b) - 2b((aa)b) + b((ba)a) = 0.$$

We multiply identity (6) by the generator a from the right side

$$\gamma[((aa)b)a - ((ba)a)a] + \delta[(b(aa))a - (a(ab))a] = 0$$

$$\gamma[((ab)a)a - ((ba)a)a] + \delta[(a(ba))a - a((ab)a)] = 0$$

$$\gamma[((ab)a)a - ((ba)a)a] + \delta[a((ba)a) - a((ab)a)] = 0 \tag{10}$$

Thus the elements $a((ab)a) - a((ba)a)$ can be expressed as linear combination of elements of the type $((**)*)*$.

Now we consider elements of type $*(*(**))$.

$$\begin{array}{|c|c|c|c|}\hline a & a & a & a \\ \hline \end{array} = a(a(aa))$$

$$\begin{array}{|c|c|c|}\hline a & a & a \\ \hline b & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} = a(a(ab)) - a(a(ba))$$

By substituting $a := aa,\ b := b,\ c := a$ into (4) we get a new identity

$$\gamma[((aa)b)a - 2(b(aa))a + (a(aa))b] + \delta[a(b(aa)) - 2a((aa)b) + b((aa)a)] = 0$$

$$\gamma((aa)a)b + \delta b(a(aa)) - 2\gamma b((aa))a) + (\gamma - \delta)a((aa)b) + \delta(-a((aa)b) + b((aa)a)) = 0$$

$$\gamma((aa)a)b + \delta b(a(aa)) - 2\gamma b((aa)a) + (\gamma - \delta)a((aa)b) - \gamma(((aa)a)b - ((ba)a)a) = 0$$

$$\gamma b((aa)a) + \delta a((aa)b) = \gamma((ba)a)a + \delta a(a(ab)) \tag{11}$$

By summing the identities (7) and (10) we obtain

$$(\gamma+\delta)a((aa)b) - (\gamma+\delta)b((aa)a) = \gamma((aa)a)b - \gamma((ba)a)a + \delta a(a(ab)) - \delta a(a(ba))$$

$$(\gamma+\delta)a((aa)b) - (\gamma+\delta)b((aa)a) = (\gamma+\delta)((aa)a)b - \gamma((ba)a)a - \delta b(a(aa))$$

$$(\gamma + \delta)b((aa)a) = \gamma((ba)a)a + \delta b(a(aa)) \tag{12}$$

By summing the identities (12) and (7) we obtain

$$\gamma a((aa)b) + \delta b((aa)a) = \delta a(a(ab)) + \gamma((ba)a)a \tag{13}$$

By subtracting the identities (11) and (13) we obtain

$$(\gamma-\delta)b((aa)a)+(\delta-\gamma)a((aa)b)=0$$

$$b((aa)a)-a((aa)b)=0 \tag{14}$$

By identities (14) and (10) we get that elements $((ab)a)a-((ba)a)a=0$

Based on result we claim that the number of base elements of $P_4(\mathfrak{M})$ is equal to 2. They are

$$\{((ab)c)d, a((bc)d)\}.$$

Let $n=5$. The number of base elements of the multilinear component of degree 5 of free bicommutative algebra $F(\mathfrak{B})$ is equal to 30. There are 4 types of the base elements.

First type

$$V_5((((**)*)*)*)=\{(((ab)c)d)e, (((ba)c)d)e, (((ca)b)d)e, (((da)b)c)e, (((ea)b)c)d\}.$$

Second type

$$V_5(*(((**)*)*))=\{a(((bc)d)e), a(((cb)d)e), a(((db)c)e), a(((eb)c)d), b(((ca)d)e),$$
$$b(((da)c)e), b(((ea)c)d), c(((da)b)e), c(((ea)b)d), d(((ea)b)c)\}.$$

Third type

$$V_5(*(*((**)*)))=\{a(b((cd)e)), a(b((dc)e)), a(b((ec)d)), a(c((db)e)), a(c((eb)d)),$$
$$a(d((eb)c)), b(c((da)e)), b(c((ea)d)), b(d((ea)c)), c(d((ea)b))\}.$$

Fourth type

$$V_5(*(*(*(**))))=\{a(b(c(de))), b(c(d(ea))), c(d(e(ab))), d(e(a(bc))), e(a(b(cd)))\}.$$

In [7] proved that $V_5((((**)*)*)*)$, $V_5(*(((**)*)*))$, $V_5(*(*((**)*)))$ and $V_5(*(*(*(**))))$ are invariant under action of symmetric group S_5 and as S_5-module

$$V_5((((**)*)*)*)\cong S^{(5)}\oplus S^{(4,1)},$$
$$V_5(*(((**)*)*))\cong S^{(5)}\oplus S^{(4,1)}\oplus S^{(3,2)},$$
$$V_5(*(*((**)*)))\cong S^{(5)}\oplus S^{(4,1)}\oplus S^{(3,2)},$$
$$V_5(*(*(*(**))))\cong S^{(5)}\oplus S^{(4,1)},$$

and

$$P_5(\mathfrak{B}) \cong V_5((((**)*)*)*) \oplus V_5(*(((**)*)*)) \oplus V_5(*(*((**)*))) \oplus V_5(*(*(*(**)))) \cong$$

$$4S^{(5)} \oplus 4S^{(4,1)} \oplus 2S^{(3,2)}.$$

Let's consider elements of the type $*(*(*(**)))$

$$\begin{array}{|c|c|c|c|c|}\hline a & a & a & a & a \\ \hline\end{array} = a(a(a(aa)))$$

By substituting $b := a$ into (11) we obtain

$$(\gamma+\delta)a((aa)a) = \gamma((aa)a)a + \delta a(a(aa)) \tag{15}$$

We multiply identity (15) by the generator a from the left side

$$(\gamma+\delta)a(a((aa)a)) = \gamma a(((aa)a)a) + \delta a(a(a(aa)))$$

$$-\gamma a(((aa)a)a) + (\gamma+\delta)a(a((aa)a)) = \delta a(a(a(aa)))$$

So the element $a(a(a(aa)))$ can be expressed through elements $a(((aa)a)a)$ and $a(a((aa)a))$.

$$\begin{array}{|c|c|c|c|}\hline a & a & a & a \\ \hline b & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} = a(a(a(ab))) - a(a(a(ba))) = 0$$

(analogous to case $n = 4$).

Now we consider elements of the type $*(*((**)*))$

$$\begin{array}{|c|c|c|c|c|}\hline a & a & a & a & a \\ \hline\end{array} = a(a((aa)a)).$$

We multiply identity (15) by the generator a from the right side

$$(\gamma+\delta)(a((aa)a))a = \gamma(((aa)a)a)a + \delta(a(a(aa)))a$$

$$-\gamma(((aa)a)a)a + (\gamma+\delta)a(((aa)a)a) = \delta a((a(aa))a)$$

$$-\gamma(((aa)a)a)a + (\gamma+\delta)a(((aa)a)a) = \delta a(a((aa)a))$$

Thus the element $a(a((aa)a))$ can be expressed through elements $(((aa)a)a)a$ and $a(((aa)a)a)$.

$$\begin{array}{|c|c|c|c|}\hline a & a & a & a \\ \hline b & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} = 0 \text{ and } \begin{array}{|c|c|c|}\hline a & a & a \\ \hline b & b & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} = 0$$

(analogous to case $n = 4$).

Let's consider elements of the type $*(((**)*)*)$

$$\begin{array}{|c|c|c|c|c|}\hline a & a & a & a & a \\ \hline \end{array} = a(((aa)a)a).$$

By substituting $b := aa$ into (11) and using (15) we get

$$\gamma(aa)((aa)a) + \delta a((aa)(aa)) = \gamma(((aa)a)a)a + \delta a(a(a(aa)))$$

$$\gamma a(((aa)a)a) + \delta a(a((aa)a)) = \gamma(((aa)a)a)a - \gamma a(((aa)a)a) + (\gamma+\delta)a(a((aa)a))$$

$$2\gamma a(((aa)a)a) - \gamma a(a((aa)a)a) = \gamma(((aa)a)a$$

$$-\frac{\gamma}{\delta}(((aa)a)a)a + \frac{\gamma+\delta}{\delta}a(((aa)a)a) = a(a((aa)a))$$

$$2\gamma a(((aa)a)a) + \frac{\gamma^2}{\delta}(((aa)a)a)a - \frac{\gamma(\gamma+\delta}{\delta}a(((aa)a)a) = \gamma(((aa)a)a)a$$

$$(2\gamma - \frac{\gamma(\gamma+\delta)}{\delta})a(((aa)a)a) = (\frac{\delta\gamma-\gamma^2}{\delta})(((aa)a)a)a$$

$$(2\gamma\delta - \gamma^2 - \gamma\delta)a(((aa)a)a) = \delta\gamma - \gamma^2(((aa)a)a)a$$

$$a(((aa)a)a) = (((aa)a)a)a$$

Hence, the element $a(a((aa)a))$ can be expressed through element $(((aa)a)a)a$.

$$\begin{array}{|c|c|c|c|}\hline a & a & a & a \\ \hline b & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} = 0 \text{ and } \begin{array}{|c|c|c|}\hline a & a & a \\ \hline b & b & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} = 0$$

(analogous to case $n = 4$).

Based on result we claim that the number of base elements of $P_5(\mathfrak{M})$ is equal to 1. This is

$$\{(((ab)c)d)e\}$$

$\square$

This implies the number of base elements of $P_n(\mathfrak{M})$ is the following

n	1	2	3	4	5	...	n	...
$dim(P_n(\mathfrak{M}))$	1	2	4	2	1	...	1	...

Set

$$f_n = (\cdots((\underbrace{xx)x)\cdots)x}_{n},$$

$$f_n' = \underbrace{x(\cdots(x(xx}_{n})) \cdots),$$

$$g_n = (\cdots([y, \underbrace{x]x) \cdots)x}_{n-1}.$$

Theorem 3.2. Let a linearization of f generates an irreducible S_n-submodule of $P_n(\mathfrak{M})$. Then the consequences of higher degrees from the f are equivalent to the following identities

(a) f_{n+1} if $f = f_n,\ n \geq 1$;

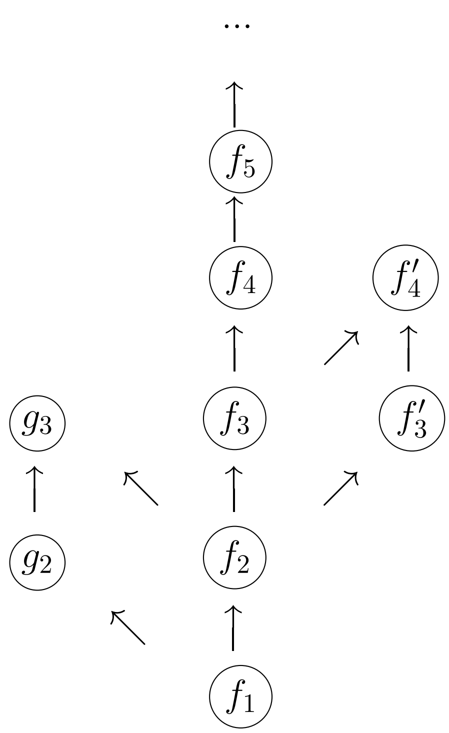
(b) f'_{n+1} if $f = f_n,\ n = 1, 2, 3$;

(c) f'_{n+1} if $f = f'_n,\ n = 1, 2, 3$;

(d) g_{n+1} if $f = f_n,\ n = 1, 2$;

(e) g_{n+1} if $f = g_n,\ n = 1, 2$.

This theorem can be illustrated in the following lattice:



Proof. Notice that $f_1 = f_1' = g_1$ and $f_2 = f_2'$.

(a) For $n = 1:\ f_1 = x = 0$ implies $f_2 = f_1 x = xx = 0 \cdot x = 0$.

For $n = 2:\ f_2 = xx = 0$ implies $f_3 = (xx)x = 0 \cdot x = 0$.

For $n = k$ we have $f_k = 0$ implies $f_{k+1} = f_k \cdot x = 0 \cdot x = 0$.

(b) For $n = 2:\ f_2 = xx = 0$ implies $f_3' = x(xx) = x \cdot 0 = 0$.

For $n = 3:\ f_3 = (xx)x = 0$ implies $f_4' = x((xx)x) = x \cdot f_3 = x \cdot 0 = 0$.

(c) For $n = 3:\ f_3' = x(xx) = 0$ implies $f_4' = x((xx)x) = (xx)(xx) = (x(xx))x = f_3' \cdot x = 0 \cdot x = 0$.

(d) For $n = 1$: $f_1 = x = 0$ implies $g_2 = xy - yx = 0 \cdot y - y \cdot 0 = 0$
For $n = 2$: $f_2 = xx = 0$ implies $g_3 = (xy)x - (yx)x$.
By substituting $x := x + y$ into f_2 we obtain

$$xx = (x + y)(x + y) = xx + xy + yx + yy = 0$$

xx and yy are equal to 0 since $f_2 = 0$. This means that $xy + yx = 0$ and we obtain:

$$xy = -yx$$

So $g_3 = (xy)x - (yx)x = (xy)x + (xy)x = 2(xx)y = 2f_2 \cdot y = 0$
(e) For $n = 2$: $g_2 = xy - yx = 0$ implies $g_3 = (xy)x - (yx)x = (xy - yx)x = g_2 \cdot x = 0 \cdot x = 0$. $\square$

3.3 Case II: $\gamma = 0,\ \delta \neq 0$

Let $\mathfrak{M}$ be a variety of bicommutative algebras defined by identity

$$c(ba) - 2c(ab) + b(ac) = 0 \tag{16}$$

Then we have the following theorem.

Theorem 3.3. As S_n-module

$$P_1(\mathfrak{M}) \cong S^{(1)}, P_2(\mathfrak{M}) \cong S^{(2)} \oplus S^{(1,1)},$$

$$P_n(\mathfrak{M}) \cong 2S^{(n)} \oplus S^{(n-1,1)},\ \text{for } n \geq 3.$$

Proof. By replacing b and c in the identity (16) we get $b(ca) - 2b(ac) + c(ab) = 0$. Subtracting (16) from this equation

$$3c(ab) = 3b(ac)$$

$$c(ab) = b(ac)$$

We get a new identity

$$a(bc) = c(ba) \tag{17}$$

Returning to the equation (16) and inserting $c(ab)$ instead of $b(ac)$ by identity (17)

$$c(ba) - 2c(ab) + c(ab) = 0$$

$$c(ba) - 2c(ab) = 0$$

$$c(ba) = c(ab)$$

Based on result we get one more new identity

$$a(bc) = a(cb) \tag{18}$$

Let $n = 3$.

We get $b(ac) = c(ab)$ and $c(ba) = c(ab)$ by identities (17) and (18) respectively.

We obtain the following base elements of $P_3(\mathfrak{M})$

$$\{(ab)c, (ba)c, (ca)b, c(ab)\}.$$

Based on result we claim that the number of base elements of $P_3(\mathfrak{M})$ is equal to 4.

Let $n = 4$.

Let's consider elements of the type $*(*(**))$. By using (2), (17) and (18) identities we get

$$b(c(da)) = c(b(da)) = c(b(ad)) = a(b(cd))$$
$$c(d(ab)) = a(d(cb)) = a(b(cd))$$
$$d(a(bc)) = a(d(bc)) = a(c(bd)) = a(b(cd))$$

It follows that we can consider only one element $a(b(cd))$ of the type $*(*(**))$.

Now we consider elements of the type $*((**)*)$. By using (18) identity we obtain following equalities.

$$a((bc)d) = a(d(bc)); a((cb)d) = a(d(cb)); a((db)c) = a(c(db));$$

$$b((ca)d) = b(d(ca)); b((da)c) = b(c(da)); c((da)b) = c(b(da))$$

All these elements are elements of the type $*(*(**)$. This means we can express elements of the type $*((**)*)$ through elements of the type $*(*(**)$.

First type $((**)*)*$ has all 4 elements and we can not eliminate them.

We obtain the following base elements of $P_4(\mathfrak{M})$

$$\{((ab)c)d, ((bc)d)a, ((cd)a)b, ((da)b)c, a(b(cd))\}.$$

Based on result we claim that the number of base elements of $P_4(\mathfrak{M})$ is equal to 5.

Let $n = 5$.

Let's consider elements of the type $*(*(*(**)))$. By using (2), (17) and (18) identities we get

$$b(c(d(ea))) = b(c(d(ae))) = a(b(c(de)))$$
$$c(d(e(ab))) = c(d(b(ae))) = a(b(c(de)))$$
$$d(e(a(bc))) = d(a(b(ec))) = d(a(b(ce))) = a(b(c(de)))$$
$$e(a(b(cd))) = a(b(c(ed))) = a(b(c(de)))$$

It follows that we can consider only one element $a(b(c(de)))$ of the type $*(*(*(**)))$.

Let's consider elements of the type $*(*((**)*))$. By using (18) identity we obtain following equalities

$$a(b((cd)e)) = a(b(e(cd))); a(b((dc)e)) = a(b(e(dc)));$$
$$a(b((ec)d)) = a(b(d(ec))); a(c((db)e)) = a(c(e(db)));$$
$$a(c((eb)d)) = a(c(d(eb))); a(d((eb)c)) = a(d(c(eb)));$$
$$b(c((da)e)) = b(c(e(da))); b(c((ea)d)) = b(c((dea)));$$
$$b(d((ea)c)) = b(d(c(ea))); c(d((ea)b)) = c(d(b(ea))).$$

All these elements are elements of the type $*(*(*(**)))$. This means we can express elements of the type $*(*((**)*))$ through elements of the type $*(*(*(**)))$.

Now consider elements of the type $*(((**)*)*)$. By using (18) identity we obtain following equalities

$$a(((bc)d)e) = a(e((bc)d)) = a(e(d(bc)))$$
$$a(((cb)d)e) = a(e((cb)d)) = a(e(d(cb)))$$
$$a(((db)c)e) = a(e((db)c)) = a(e(c(db)))$$
$$a(((eb)c)d) = a(d((eb)c)) = a(d(c(eb)))$$
$$b(((ca)d)e) = b(e((ca)d)) = b(d(e(ca)))$$
$$b(((da)c)e) = b(e((da)c)) = b(e(c(da)))$$
$$b(((ea)c)d) = b(d((ea)c)) = b(d(c(ea)))$$
$$c(((da)b)e) = c((e(da)b)) = c(e(b(da)))$$
$$c(((ea)b)d) = c(d((ea)b)) = c(d(b(ea)))$$
$$d(((ea)b)c) = d(c((ea)b)) = d(c(b(ea)))$$

All these elements are elements of the type $*(*(*(**)))$. This means we can express elements of the type $*(((**)*)*)$ through elements of the type $*(*(*(**)))$.

First type $(((**)*)*)*$ has all 5 elements and we can not eliminate them.

We obtain the following base elements of $P_5(\mathfrak{M})$

$$\{(((ab)c)d)e, (((bc)d)e)a, (((cd)e)a)b, (((de)a)b)c, (((ea)b)c)d, a(b(c(de)))\}.$$

Based on result we claim that the number of base elements of $P_5(\mathfrak{M})$ is equal to 6. □

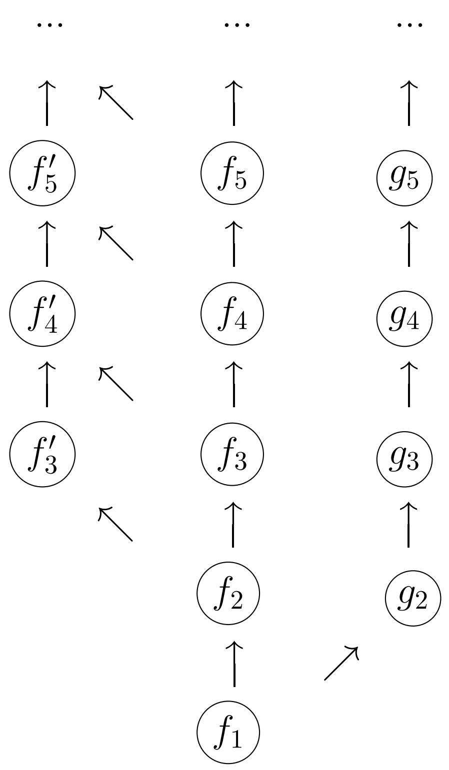
This implies the number of base elements of $P_n(\mathfrak{M})$

n	1	2	3	4	5	6	...	n	...
$dim(P_n(\mathfrak{M}))$	1	2	4	5	6	7	...	$n+1$	...

Theorem 3.4. Let a linearization of f generates an irreducible S_n-submodule of $P_n(\mathfrak{M})$. Then the consequences of higher degrees from the f are equivalent to the following identities

(a) f_{n+1} if $f = f_n,\ n \geq 1$;
(b) f'_{n+1} if $f = f_n,\ n \geq 1$;
(c) f'_{n+1} if $f = f'_n,\ n \geq 1$;
(d) g_{n+1} if $f = g_n,\ n \geq 1$.

This theorem can be illustrated in the following lattice



Proof. Notice that $f_1 = f'_1 = g_1$ and $f_2 = f'_2$.

(a) For $n = 1:\ \ f_1 = x = 0$ implies $f_2 = xx = 0 \cdot x = 0$.

For $n = 2:\ \ f_2 = xx = 0$ implies $f_3 = (xx)x = 0 \cdot x = 0$.

For $n = k$: $f_k = 0$ implies $f_{k+1} = f_k \cdot x = 0 \cdot x = 0$.

(b) For $n = 2:\ \ f_2 = xx = 0$ implies $f'_3 = x(xx) = x \cdot 0 = 0$.

For $n = 3:\ \ f_3 = (xx)x = 0$ implies $f'_4 = x(x(xx)) = x((xx)x) = x \cdot f_3 = x \cdot 0 = 0$.

For $n = k$ we have $f_k = 0$ implies $f'_{k+1} = x \cdot f_k = x \cdot 0 = 0$.

(c) For $n = 3$: $f'_3 = x(xx) = 0$ implies $f'_4 = x(x(xx)) = x \cdot f'_3 = x \cdot 0 = 0$.

For $n = 4$: $f'_4 = x(x(xx)) = 0$ implies $f'_5 = x(x(x(xx))) = x \cdot f'_4 = x \cdot 0 = 0$.

For $n = k$ we have $f'_k = 0$ implies $f'_{k+1} = x \cdot f'_k = x \cdot 0 = 0$.

(d) $f_1 = x = 0$ implies $g_2 = xy - yx = 0 \cdot y - y \cdot 0 = 0$

For $n = 2$: $g_2 = xy - yx = 0$ implies $g_3 = (xy)x - (yx)x = (xy - yx)x = g_2 \cdot x = 0 \cdot x = 0$.

For $n = 3$: $g_3 = (xy - yx)x = 0$ implies $g_4 = ((xy)x)x - ((yx)x)x = ((xy - yx)x)x = g_3 \cdot x = 0 \cdot x = 0$

For $n = k$ we have $g_k = 0$ implies $g_{k+1} = g_k \cdot x = 0 \cdot x = 0$. □

3.4 Case III: $\gamma + \delta = 0$

Let $\mathfrak{M}$ be a variety of bicommutative algebras defined by identity

$$(ab)c - 2(ba)c + (ca)b - c(ba) + 2c(ab) - b(ac) = 0 \tag{19}$$

Then we have the following theorem.

Theorem 3.5. As S_n-module

$$P_1(\mathfrak{M}) \cong S^{(1)}, P_2(\mathfrak{M}) \cong S^{(2)} \oplus S^{(1,1)},$$

$$P_3(\mathfrak{M}) \cong 2S^{(3)} \oplus S^{(2,1)}, P_4(\mathfrak{M}) \cong 2S^{(4)}$$

$$P_n(\mathfrak{M}) \cong S^{(n)} \text{ for } n \geq 5.$$

Proof. Let $n = 3$.

By replacing a and b in identity (19) and substracting (19) from this equation

$$3(ba)c - 3(ab)c = 3c(ab) - 3c(ba)$$

$$(ba)c - (ab)c = c(ab) - c(ba) \tag{20}$$

From identity (19) we have

$$(ba)c - (ab)c = (ca)b - (ba)c - c(ba) + 2c(ab) - b(ac)$$

Using identity (20)

$$c(ab) - c(ba) = (ca)b - (ba)c - c(ba) + 2c(ab) - b(ac)$$

$$(ca)b - (ba)c = b(ac) - c(ab) \tag{21}$$

From identities (20) and (21) respectively we have

$$\begin{aligned} c(ab) &= (ba)c + c(ba) - (ab)c \\ b(ac) &= (ca)b + c(ab) - (ba)c \\ &= (ca)b + (ba)c + c(ba) - (ab)c - (ba)c \\ &= (ca)b + c(ba) - (ab)c \end{aligned}$$

This means that we can represent elements $c(ab)$ and $b(ac)$ by others. So we obtain the following base elements of $P_3(\mathfrak{M})$

$$\{(ab)c, (ba)c, (ca)b, c(ba)\}$$

Based on result we claim that the number of base elements of $P_3(\mathfrak{M})$ is equal to 4.

By substituting $a := a,\ b := a,\ c := b$ into (19) we get new identity

$$(aa)b - 2(aa)b + (ba)a = b(aa) - 2b(aa) + a(ab)$$

$$(ba)a - (aa)b = a(ab) - b(aa) \tag{22}$$

Let $n = 4$.

We can get new identity using (1) and (2) that we'll use further

$$(a(bc))d = (b(ac))d = (bd)(ac) = a((bd)c) = a((bc)d)$$

Thus we get new identity

$$(a(bc))d = a((bc)d) \tag{23}$$

$$\begin{array}{|c|c|c|}\hline a & a & a \\ \hline b & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad = \quad ((ab)a)a - ((ba)a)a = ((aa)a)b - ((ba)a)a$$

$$\begin{array}{|c|c|c|}\hline a & a & a \\ \hline b & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad = \quad a((ab)a) - a((ba)a) = a((aa)b) - b((aa)a)$$

$$\begin{array}{|c|c|c|}\hline a & a & a \\ \hline b & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad = \quad a(a(ab)) - a(a(ba)) = a(a(ab)) - b(a(aa))$$

We multiply identity (22) by the generator a from the right side

$$((ba)a)a - ((aa)b)a = (a(ab))a - (b(aa))a$$

$$((ba)a)a - ((aa)a)b = a((aa)b) - b((aa)a) \tag{24}$$

We multiply identity (22) by the generator a from the left side

$$a((ba)a) - a((aa)b) = a(a(ab)) - a(b(aa))$$

$$b((aa)a) - a((aa)b) = a(a(ab)) - b(a(aa)) \tag{25}$$

We multiply identity (19) by the generator d from the right side

$$((ab)c)d - 2((ba)c)d + ((ca)b)d = (c(ba))d - 2(c(ab))d + (b(ac))d$$

$$((ab)c)d - 2((ba)c)d + ((ca)b)d = c((ba)d) - 2c((ab)d) + b((ac)d)$$

We multiply identity (19) by the generator d from the left side

$$d((ab)c) - 2d((ba)c) + d((ca)b) = d(c(ba)) - 2d(c(ab)) + d(b(ac))$$

$$d((ab)c) - 2d((ba)c) + d((ca)b) = d(c(ba)) - 2d(c(ab)) + d(b(ac))$$

By substituting $a := a,\ b := b,\ c := cd$ into (19) we get new identity

$$(ab)(cd) - 2(ba)(cd) + ((cd)a)b = (cd)(ba) - 2(cd)(ab) + b(a(cd))$$

$$c((ab)d) - 2c((ba)d) + ((cd)a)b = b((cd)a) - 2a((cd)b) + b(a(cd))$$

$$3c((ab)d) + ((cd)a)b = 3b((cd)a) + b(a(cd)) \tag{26}$$

By substituting $a := ad,\ b := b,\ c := c$ into (19) we get new identity

$$((ad)b)c - 2(b(ad))c + (c(ad))b = c(b(ad)) - 2c((ad)b) + b((ad)c)$$

$$((ad)b)c - 2b((ad)c) + c((ad)b) = c(b(ad)) - 2c((ad)c)$$

$$3c((ad)b) + ((ad)b)c = 3b((ad)c) + c(b(ad)) \tag{27}$$

By substituting $a := a,\ b := bd,\ c := c$ into (19) we get new identity

$$(a(bd))c - 2((bd)a)c + (ca)(bd) = c((bd)a) - 2c(a(bd)) + (bd)(ac)$$

$$a((bd)c) - 2((bd)a)c + b((ac)d) = c((bd)a) - 2c(a(bd)) + a((bd)c)$$

$$-2((bd)a)c = -2c(a(bd))$$

$$((bd)a)c = c(a(bd)) \tag{28}$$

Using the last identity we get

$$\begin{array}{|c|c|c|}\hline a & a & a \\ \hline b & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} = ((ab)a)a - ((ba)a)a = ((aa)a)b - ((ba)a)a =$$

$$= b(a(aa)) - b(a(aa)) = 0$$

$$((ab)a)a = ((ba)a)a \tag{29}$$

Using identities (28) and (29) we get that elements of the type $((**)*)*$ eliminate elements of the type $*(*(**))$.

From identities (26) and (28) we get

$$3c((ab)d) = 3b((cd)a)$$

$$c((ab)d) = c((ba)d) \tag{30}$$

$$\begin{array}{|c|c|}\hline a & a \\ \hline b & b \\ \hline\end{array} = (ab - ba)(ab - ba) =$$

$$(ab)(ab) - (ab)(ba) - (ba)(ab) + (ba)(ba) =$$
$$a((ab)b) - b((ab)a) - a((ba)b) + b((ba)a) =$$
$$a((ab)b) - 2b((aa)b) + b((ba)a) =$$
$$2b((aa)b - 2b((aa)b) = 0$$

This means that elements of the type $*((**)*)$ can be represented by only one of them.

We obtain the following base elements of $P_4(\mathfrak{M})$

$$\{((ab)c)d, c((ab)d)\}$$

Based on result we claim that the number of base elements of $P_4(\mathfrak{M})$ is equal to 2.

Let $n = 5$. Using identities we get we can claim that the number of base elements of $P_5(\mathfrak{M})$ is equal to 1

$$\{(((ab)c)d)e\}$$

□

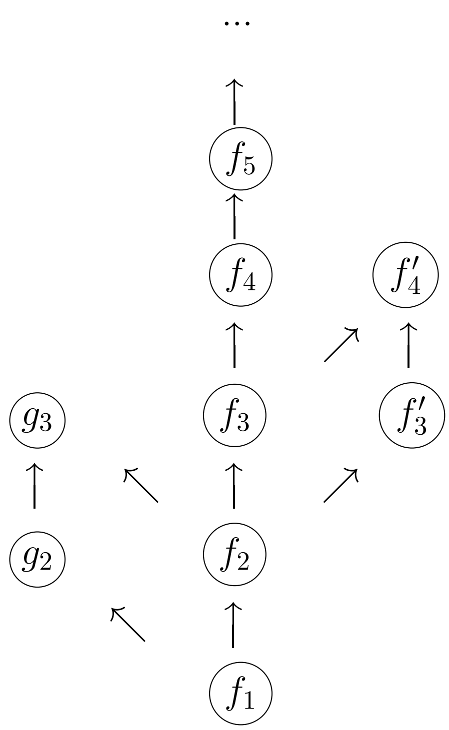
This implies the number of base elements of $P_n(\mathfrak{M})$

n	1	2	3	4	5	...	n	...
$dim(P_n(\mathfrak{M}))$	1	2	4	2	1	...	1	...

Theorem 3.6. Let a linearization of f generates an irreducible S_n-submodule of $P_n(\mathfrak{M})$. Then the consequences of higher degrees from the f are equivalent to the following identities

(a) f_{n+1} if $f = f_n,\ n \geq 1$;
(b) f'_{n+1} if $f = f_n,\ n = 1, 2, 3$;
(c) f'_{n+1} if $f = f'_n,\ n = 1, 2, 3$;
(d) g_{n+1} if $f = f_n,\ n = 1, 2$;
(e) g_{n+1} if $f = g_n,\ n = 1, 2$.

This theorem can be illustrated in the following lattice



Proof. Notice that $f_1 = f_1' = g_1$ and $f_2 = f_2'$.

(a) For $n = 1:\ f_1 = x = 0$ implies $f_2 = xx = 0 \cdot x = 0$.

For $n = 2:\ f_2 = xx = 0$ implies $f_3 = (xx)x = 0 \cdot x = 0$.

For $n = k$ we have $f_k = 0$ implies $f_{k+1} = f_k \cdot x = 0 \cdot x = 0$.

(b) For $n = 2:\ f_2 = xx = 0$ implies $f_3' = x(xx) = x \cdot 0 = 0$.

For $n = 3:\ f_3 = (xx)x = 0$ implies $f_4' = x((xx)x) = x \cdot f_3 = x \cdot 0 = 0$.

(c) For $n = 3:\ f_3' = x(xx) = 0$ implies $f_4' = x((xx)x) = (xx)(xx) = (x(xx))x = f_3' \cdot x = 0 \cdot x = 0$.

(d) For $n = 1$: $f_1 = x = 0$ implies $g_2 = xy - yx = 0 \cdot y - y \cdot 0 = 0$

For $n = 2$: $f_2 = xx = 0$ implies $g_3 = (xy)x - (yx)x$.

By substituting $x := x + y$ into f_2 we obtain

$$xx = (x + y)(x + y) = xx + xy + yx + yy = 0$$

xx and yy are equal to 0 since $f_2 = 0$. This means that $xy + yx = 0$ and we obtain

$$xy = -yx$$

So $g_3 = (xy)x - (yx)x = (xy)x + (xy)x = 2(xx)y = 2f_2 \cdot y = 0$

(e) For $n = 2$: $g_2 = xy - yx = 0$ implies $g_3 = (xy)x - (yx)x = (xy - yx)x = g_2 \cdot x = 0 \cdot x = 0$.

□

3.5 Case IV: $\gamma - \delta = 0$

Let $\mathfrak{M}$ be a variety of bicommutative algebras defined by identity

$$(ab)c - 2(ba)c + (ca)b + c(ba) - 2c(ab) + b(ac) = 0 \tag{31}$$

Then we have the following theorem.

Theorem 3.7. As S_n-module

$$P_1(\mathfrak{M}) \cong S^{(1)}, P_2(\mathfrak{M}) \cong S^{(2)} \oplus S^{(1,1)},$$

$$P_n(\mathfrak{M}) \cong 2S^{(n)} \oplus S^{(n-1,1)}, \text{ for } n \geq 3.$$

Proof. Let $n = 3$.

By replacing a and b in identity (31) and subtracting (31) from this equation

$$3(ba)c - 3(ab)c + 3c(ab) - 3c(ba) = 0$$

$$(ba)c + c(ab) = (ab)c + c(ba) \tag{32}$$

From identity (31) we have

$$(ab)c + c(ba) = 2(ba)c + 2c(ab) - (ca)b - b(ac)$$

Using identity (32)

$$(ba)c + c(ab) = 2(ba)c + 2c(ab) - (ca)b - b(ac)$$

$$(ca)b + b(ac) = (ba)c + c(ab) \tag{33}$$

$$(ca)b + b(ac) = (ab)c + c(ba) \tag{34}$$

From identities (34) and (33) respectively we have

$$\begin{aligned} b(ac) &= (ab)c + c(ba) - (ca)b \\ c(ab) &= (ca)b + b(ac) - (ba)c \\ &= (ca)b + (ab)c + c(ba) - (ca)b - (ba)c \\ &= (ab)c + c(ba) - (ba)c \end{aligned}$$

We obtain the following base elements of $P_3(\mathfrak{M})$

$$\{(ab)c, (ba)c, (ca)b, c(ba)\}$$

Based on result we claim that the number of base elements of $P_3(\mathfrak{M})$ is equal to 4.

Let $n = 4$.
We can get new identity using (1) and (2) that we'll use further

$$(a(bc))d = (b(ac))d = (bd)(ac) = a((bd)c) = a((bc)d)$$

So we get new identity

$$(a(bc))d = a((bc)d) \tag{35}$$

We multiply identity (31) by fourth element d by the right side

$$\big((ab)c - 2(ba)c + (ca)b + c(ba) - 2c(ab) + b(ac)\big)d = 0 \cdot d = 0$$

$$((ab)c)d - 2((ba)c)d + ((ca)b)d + (c(ba))d - 2(c(ab))d + (b(ac))d = 0$$

$$((ab)c)d - 2((ba)c)d + ((ca)b)d + c((ba)d) - 2c((ab)d) + b((ac)d) = 0$$

We multiply identity (31) by the geberator d from the left side

$$d\big((ab)c - 2(ba)c + (ca)b + c(ba) - 2c(ab) + b(ac)\big) = d \cdot 0 = 0$$

$$d((ab)c) - 2d((ba)c) + d((ca)b) + d(c(ba)) - 2d(c(ab)) + d(b(ac)) = 0$$

By substituting $a := ad,\ b := b,\ c := c$ into (31) we get new identity

$$((ad)b)c - 2(b(ad))c + (c(ad))b + c(b(ad)) - 2c((ad)b) + b((ad(c) = 0$$

$$((ad)b)c - 2b((ad)c) + c((ad)b) + c(b(ad)) - 2c((ad)b) + b((ad)c) = 0$$

$$((ad)b)c + c(b(ad)) - b((ad)c) - c((ad)b) = 0 \tag{36}$$

By substituting $a := a,\ b := bd,\ c := c$ into (31) we get new identity

$$(a(bd))c - 2((bd)a)c + (ca)(bd) + c((bd)a) - 2c(a(bd)) + (bd)(ac) = 0$$

$$a((bd)c) - 2((bd)a)c + b((ca)d) + c((bd)a) - 2c(a(bd)) + a((bd)c) = 0$$

$$2b((ca)d) + 2a((bd)c) - 2((bd)a)c - 2c(a(bd)) = 0 \tag{37}$$

By substituting $a := a,\ b := b,\ c := cd$ into (31) we get new identity

$$(ab)(cd) - 2(ba)(cd) + ((cd)a)b + (cd)(ba) - 2(cd)(ab) + b(a(cd)) = 0$$

$$c((ab)d) - 2c((ba)d) + ((cd)a)b + b((cd)a) - 2a((cd)b) + b(a(cd)) = 0$$

$$((cd)a)b + b(a(cd)) - c((ab)d) - c((ba)d) = 0 \tag{38}$$

$$\begin{array}{|c|c|}\hline a & a \\ \hline b \\ \cline{1-1}\end{array} \quad = \quad (ab)a - (ba)a = (aa)b - (ba)a$$

$$\begin{array}{|c|c|}\hline a & a \\ \hline b & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad = \quad a(ab) - a(ba) = a(ab) - b(aa)$$

$$\begin{array}{|c|c|}\hline a & a \\ \hline b & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline a & a \\ \hline b & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad = \quad (aa)b - (ba)aa(ab) - b(aa) = 0$$

$$= \quad (aa)b - (ba)a = b(aa) - a(ab) \tag{39}$$

$$\begin{array}{|c|c|c|}\hline a & a & a \\ \hline b & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad = \quad ((ab)a)a - ((ba)a)a = ((aa)a)b - ((ba)a)a$$

$$\begin{array}{|c|c|c|}\hline a & a & a \\ \hline b & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad = \quad a((ab)a) - a((ba)a) = a((aa)b) - b((aa)a)$$

$$\begin{array}{|c|c|c|}\hline a & a & a \\ \hline b & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad = \quad a(a(ab)) - a(a(ba)) = a(a(ab)) - b(a(aa))$$

By substituting $a = b = c = d$ into (25) we get

$$((aa)a)a + a(a(aa)) - a((aa)a) - a((aa)a) = 0$$

$$a(a(aa)) = 2a((aa)a) - ((aa)a)a$$

We multiply identity (26) by the generator b from the right side

$$((aa)b)b - ((ba)a)b = (b(aa))b - (a(ab))b$$

$$((aa)b)b - ((ba)a)b = b((aa)b) - a((ab)b)$$

We multiply identity (26) by b by the generator from the left side

$$b((aa)b) - b((ba)a) = b(b(aa)) - b(a(ab))$$

By substituting $a := aa,\ b := b,\ c := b$ into (31) we get new identity

$$((aa)b)b - 2(b(aa))b + (b(aa))b + b(b(aa)) - 2b((aa)b) + b((aa)b) = 0$$

$$((aa)b)b + b(b(aa)) - b((aa)b) - b((aa)b) = 0$$

$$((aa)b)b + b(b(aa)) - 2b((aa)b) = 0 \tag{40}$$

By substituting $a := a,\ b := bb,\ c := a$ into (31) we get new identity

$$(a(bb))a - 2((bb)a)a + (aa)(bb) + a((bb)a) - 2a(a(bb)) + (bb)(aa) = 0$$

$$a((bb)a) - 2((ba)a)b + b((aa)b) + a((bb)a) - 2b(a(ab)) + a((bb)a) = 0$$
$$4b((aa)b) - 2((ba)a)b - 2b(a(ab)) = 0$$
$$2b((aa)b) - ((ba)a)b - b(a(ab)) = 0 \tag{41}$$

Using identities (27) and (28) we get

$$((aa)b)b + b(b(aa)) = ((ba)a)b + b(a(ab))$$

By substituting $a := ab,\ b := b,\ c := a$ into (6) we get a new identity

$$((ab)b)a - 2(b(ab))a + (a(ab))b + a(b(ab)) - 2a((ab)b) + b((ab)a) = 0$$

$$((aa)b)b - 2b((aa)b) + a((ab)b) + b(a(ab)) - 2a((ab)b) + b((aa)b) = 0$$
$$((aa)b)b + b(a(ab)) - a((ab)b) = 0 \tag{42}$$

By substituting $a := a,\ b := ba,\ c := b$ into (31) we get new identity

$$(a(ba))b - 2((ba)a)b + (ba)(ba) + b((ba)a) - 2b(a(ba)) + (ba)(ab) = 0$$

$$b((aa)b) - 2((ba)a)b + b((ba)a) + b((ba)a) - 2b(b(aa)) = 0$$
$$2b((aa)b) + 2((ba)a)b - 2((ba)a)b - 2b(a(ba)) = 0$$
$$b((aa)b) + b((ba)a) = ((ba)a)b + b(a(ba)) \tag{43}$$

$$\begin{array}{|c|c|}\hline a & a \\ \hline b & b \\ \hline \end{array} \quad = \quad (ab - ba)(ab - ba)a =$$

$$(ab)(ab) - (ab)(ba) - (ba)(ab) + (ba)(ba) =$$

$$a((ab)b) - b((ab)a) - a((ba)b) + b((ba)a) = a((ab)b) - 2b((aa)b) + b((ba)a) =$$

$$((aa)b) + b(a(ab)) - ((ba)a)b - b(a(ab)) + ((ba)a)b + b(a(ba)) - b((aa)b) =$$

$$((aa)b) + b(b(aa)) - b((aa)b)$$

$$2b((aa)b) = ((aa)b)b + b(b(aa))$$

$$2b((aa)b) = ((ba)a)b + b(a(ab))$$

We obtain the following base elements of $P_4(\mathfrak{M})$

$$\{((ab)c)d, ((bc)d)a, ((cd)a)b, ((da)b)c, a(b(cd))\}.$$

Based on result we claim that the number of base elements of $P_4(\mathfrak{M})$ is equal to 5.

Let $n = 5$. By the same way we claim that the number of base elements of $P_5(\mathfrak{M})$ is equal to 6.

We obtain the following base elements of $P_5(\mathfrak{M})$

$$\{(((ab)c)d)e, (((bc)d)e)a, (((cd)e)a)b, (((de)a)b)c, (((ea)b)c)d, a(b(c(de)))\}.$$

Based on result we claim that the number of base elements of $P_5(\mathfrak{M})$ is equal to 6. $\square$

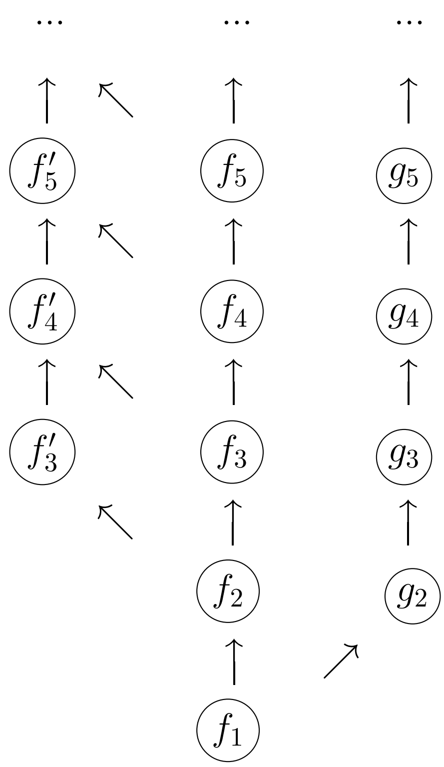
This implies the number of base elements of $P_n(\mathfrak{M})$

n	1	2	3	4	5	6	...	n	...
$dim(P_n(\mathfrak{M}))$	1	2	4	5	6	7	...	$n+1$	...

Theorem 3.8. Let a linearization of f generates an irreducible S_n-submodule of $P_n(\mathfrak{M})$. Then the consequences of higher degrees from the f are equivalent to the following identities

(a) f_{n+1} if $f = f_n,\ n \geq 1$;
(b) f'_{n+1} if $f = f_n,\ n \geq 1$;
(c) f'_{n+1} if $f = f'_n,\ n \geq 1$;
(d) g_{n+1} if $f = g_n,\ n \geq 1$.

This theorem can be illustrated in the following lattice



Proof. Notice that $f_1 = f'_1 = g_1$ and $f_2 = f'_2$.

(a) For $n = 1:\ \ f_1 = x = 0$ implies $f_2 = xx = 0 \cdot x = 0$.

For $n = 2:\ \ f_2 = xx = 0$ implies $f_3 = (xx)x = 0 \cdot x = 0$.

For $n = k$ we have $f_k = 0$ implies $f_{k+1} =$ implies $f_k \cdot x = 0 \cdot x = 0$.

(b) For $n = 2$: $f_2 = xx = 0$ implies $f_3' = x(xx) = x \cdot 0 = 0$.

For $n = 3$: $f_3 = (xx)x = 0$ implies $f_4' = x(x(xx)) = x((xx)x) = x \cdot f_3 = x \cdot 0 = 0$.

For $n = k$ we have $f_k = 0$ implies $f_{k+1}' = x \cdot f_k = x \cdot 0 = 0$.

(c) For $n = 3$: $f_3' = x(xx) = 0$ implies $f_4' = x(x(xx)) = x \cdot f_3' = x \cdot 0 = 0$.

For $n = 4$: $f_4' = x(x(xx)) = 0$ implies $f_5' = x(x(x(xx))) = x \cdot f_4' = x \cdot 0 = 0$.

For $n = k$ we have $f_k' = 0$ implies $f_{k+1}' = x \cdot f_k' = x \cdot 0 = 0$.

(d) $f_1 = x = 0$ implies $g_2 = xy - yx = 0 \cdot y - y \cdot 0 = 0$

For $n = 2$: $g_2 = xy - yx = 0$ implies $g_3 = (xy)x - (yx)x = (xy - yx)x = g_2 \cdot x = 0 \cdot x = 0$.

For $n = 3$: $g_3 = (xy - yx)x = 0$ implies $g_4 = ((xy)x)x - ((yx)x)x = ((xy - yx)x)x = g_3 \cdot x = 0 \cdot x = 0$

For $n = k$ we have $g_k = 0$ implies $g_{k+1} = g_k \cdot x = 0 \cdot x = 0$.

□

4 Conclusion

The main task of this thesis was to classify all subvarieties of the variety of bicommutative algebras defined by the following identity

$$\gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + b(ac)] = 0.$$

We have built bases by constructing theorems and proved theorems by using

- The methods of linear algebra;
- The methods of the representation theory of groups.

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