Alexandria Engineering Journal (2022) 61, 4757–4763

HOSTED BY

ELSEVIER

Alexandria University

Alexandria Engineering Journal

www.elsevier.com/locate/aej
www.sciencedirect.com

Linear differential equations with variable coefficients and Mittag-Leffler kernels

Arran Fernandez [a,*], **Joel E. Restrepo** [b,c], **Durvudkhan Suragan** [b]

[a] *Department of Mathematics, Eastern Mediterranean University, Northern Cyprus, via Mersin-10, Turkey*
[b] *Department of Mathematics, Nazarbayev University, Kazakhstan*
[c] *Department of Mathematics, University of Antioquia, Colombia*

Received 15 December 2020; revised 29 July 2021; accepted 13 October 2021
Available online 27 October 2021

KEYWORDS

Fractional differential equations;
Atangana–Baleanu fractional calculus;
Differential equations with variable coefficients;
Series solutions;
Analytical solutions

2010 MATHEMATICS SUBJECT CLASSIFICATION

34A08;
26A33

Abstract Fractional differential equations with constant coefficients can be readily handled by a number of methods, but those with variable coefficients are much more challenging. Recently, a method has appeared in the literature for solving fractional differential equations with variable coefficients, the solution being in the form of an infinite series of iterated fractional integrals. In the current work, we consider fractional differential equations with Atangana–Baleanu integro-differential operators and continuous variable coefficients, and establish analytical solutions for such equations. The representation of the solution is given by a uniformly convergent infinite series involving Atangana–Baleanu operators. To the best of our knowledge, this is the first time that explicit analytical solutions have been given for such general Atangana–Baleanu differential equations with variable coefficients. The corresponding results for fractional differential equations with constant coefficients are also given.

1. Introduction

Much of applied mathematics is concerned with differential equations and the various methods for solving them. It is well-known that linear differential equations are simpler than nonlinear ones, and it can also be observed that, among linear differential equations, those with constant coefficients are easier to handle than those with variable coefficients. For example, various transform methods, such as Laplace and Fourier analysis, are most useful when applied to differential equations with constant coefficients. For differential equations with variable coefficients, it often happens that more advanced methods must be employed [3,4,10,18,32].

An emerging field of study in mathematics and the sciences, especially popular in the last few decades, is the use of fractional derivatives and integrals to formulate and solve fractional differential equations. Starting in the 1970s, various textbooks have been written on the development of fractional calculus [23,25,35], the study of fractional differential equations [9,19,29], and their applications in fields of science including dynamical systems [36], continuum mechanics [8],

* Corresponding author.
E-mail addresses: arran.fernandez@emu.edu.tr (A. Fernandez), joel.restrepo@nu.edu.kz, cocojoel89@yahoo.es (J.E. Restrepo), durvudkhan.suragan@nu.edu.kz (D. Suragan).

Peer review under responsibility of Faculty of Engineering, Alexandria University.

https://doi.org/10.1016/j.aej.2021.10.028
1110-0168

viscoelastic flow [21], and various other topics in physics and engineering [14,38].

In the second half of the 2010s, some researchers have focused a great deal of attention on newly developing operators of fractional calculus, which in some cases have been more effective in modelling than the classical operators of Riemann–Liouville and Caputo [7,15,16]. These include operators with non-singular kernels, and in particular the so-called Atangana–Baleanu model [5,6] has attracted immense attention [37] since its creation in 2016.

The Atangana–Baleanu (AB) operators were defined using as a kernel the Mittag–Leffler function, a function of immense importance in fractional calculus [13]. As well as the study of these operators from the pure mathematical viewpoint [1,11,17], fractional differential equations and systems using the AB operators have been widely used in modelling of various phenomena including in biology, engineering, fluid dynamics, etc. [25,34,27]. The mathematical analysis of such equations has proceeded in a number of directions, including various numerical methods [29,33] and analytical ones such as fixed point theorems [30] and transform methods [23], but the work has mostly focused on specific equations which are required for particular applications. Here we shall study and solve a general class of equations, with arbitrary continuous variable coefficients. Our results will include as special cases many particular equations which will have important applications for the advancement of science.

In the current paper, we shall study fractional differential equations with Caputo-type AB operators and continuous variable coefficients. We shall follow the approach of successive approximations leading to an infinite series solution, which has been applied for fractional differential equations with classical Riemann–Liouville [20] and Caputo [26] derivatives and also for those with Caputo derivatives with respect to monotone functions [31]. The same method will work for AB fractional differential equations, again giving rise to solutions in the form of infinite series.

This paper is organised as follows. In Section 2, we give some definitions and basic results on fractional calculus and AB integro-differential operators, including proving a formula for the ABC derivative which is valid for any continuous function without any differentiability assumptions. Section 3 is devoted to the main results of the paper: namely, finding explicitly the unique continuous solution of the considered AB type differential equation. Some conclusions can be found in Section 4.

2. Preliminaries

In this section we give definitions and auxiliary results on integro-differential operators of Riemann–Liouville and Atangana–Baleanu type. We first need to recall the Riemann–Liouville fractional integral.

Definition 2.1. [19,35] Let $\alpha > 0$ and f an integrable function defined on $[a, b]$. The left-sided Riemann–Liouville fractional integral of f is defined as

$$ {}^{RL}I^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-u)^{\alpha-1} f(u)\,\mathrm{d}u, \qquad a < t < b. \tag{2.1}$$

This integral operator has a semigroup property in α: namely, ${}^{RL}I^{\alpha}_{a+}\,{}^{RL}I^{\beta}_{a+}f(t) = {}^{RL}I^{\alpha+\beta}_{a+}f(t)$ for any $\alpha, \beta > 0$ and any integrable function f.

Next, we recall the function $E_{\alpha,\beta}$, called the Wiman function or two-parameter Mittag–Leffler function:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\alpha k + \beta)}, \quad z, \beta, \alpha \in \mathbb{C}, \operatorname{Re}\alpha > 0,$$

where this series is locally uniformly convergent for all $z \in \mathbb{C}$ under the condition $\operatorname{Re}\alpha > 0$. For $\beta = 1$ we recover the original Mittag–Leffler function $E_{\alpha}(z) = E_{\alpha,1}(z)$.

Definition 2.2. [5,6] The ABC fractional derivative is defined by

$$ {}^{ABC}D^{\alpha}_{a+}f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t f'(u) E_{\alpha}\Big(\frac{-\alpha}{1-\alpha}(t-u)^{\alpha}\Big)\,\mathrm{d}u, \tag{2.2}$$

where $0 < \alpha < 1, a < t < b$, $f \in AC[a, b]$, and $B(\alpha)$ a positive real multiplier satisfying $B(0) = B(1) = 1$.

The requirement that $f \in AC[a, b]$ implies that f is differentiable a.e. and $f' \in L^1(a, b)$, which is sufficient for existence of the ABC derivative by [6, Definition 2.2]. In fact, however, the ABC derivative can be defined for any L^1 function f, just like its partner the ABR derivative, by means of a different formula. We state the result as follows.

Lemma 2.3. *The ABC derivative can be written in the following form for $f \in AC[a, b]$, and the right-hand side is well-defined for all $f \in L^1[a, b]$:*

$$ {}^{ABC}D^{\alpha}_{a+}f(t) = \frac{B(\alpha)}{1-\alpha} \Big(f(t) - f(a) E_{\alpha}\Big(\frac{-\alpha}{1-\alpha}(t-a)^{\alpha}\Big) - \frac{\alpha}{1-\alpha} \int_a^t (t-u)^{\alpha-1} f(u) E_{\alpha,\alpha}\Big(\frac{-\alpha}{1-\alpha}(t-u)^{\alpha}\Big)\,\mathrm{d}u \Big). \tag{2.3}$$

Therefore, the formula (2.3) can be used as the definition of the ABC derivative for any L^1 function f.

Proof. The identity follows from integrating by parts in (2.2). The right-hand side is well-defined for all L^1 functions f by the same argument as that used in [6, Lemma 2.1] for the ABR derivative. In fact, the expression (2.3) shows that the ABC derivative is identical to the ABR derivative minus an initial value term ($f(a)$ times a Mittag–Leffler function). Since the right-hand side of (2.3) gives an operator defined for all $f \in L^1[a, b]$, which is identical to the ABC derivative on the subspace $AC[a, b] \subset L^1[a, b]$, it is natural to define the ABC derivative on the larger space $L^1[a, b]$ using this formula. □

Note that integration by parts can only be applied to the Caputo-type AB operator in this way because it is a non-singular operator [12]. This non-singularity enables us to extend the definition of the ABC derivative to all L^1 functions. From now on, we shall treat (2.3) as an equivalent, alternative, way of defining the ABC derivative.

Lemma 2.4. *Let $f \in C[a,b]$ be a continuous function.*

(1) The ABC derivative ${}^{ABC}D^{\alpha}_{a+}f(t)$ is also continuous on $[a,b]$.
(2) $\lim_{t\to+a}{}^{ABC}D^{\alpha}_{a+}f(t) = 0$.

Proof. These facts follow from the new expression (2.3) for the ABC derivative, as well as the limiting properties of the kernel function. □

Definition 2.5. [5,6] The AB fractional integral is defined by

$$ {}^{AB}I^{\alpha}_{a+}f(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\,{}^{RL}I^{\alpha}_{a+}f(t), \tag{2.4}$$

where $0<\alpha<1, a<t<b, f \in L^1(a,b)$, and $B(\alpha)$ is as in Definition 2.2.

Lemma 2.6. [6, Theorem 2.2] *The following alternative representation of the ABC fractional derivative is useful for proving various properties:*

$$ {}^{ABC}D^{\alpha}_{a+}f(t) = \frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{+\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n {}^{RL}I^{\alpha n+1}_{a+}f'(t)$$

$$ = \frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{+\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n {}^{RL}I^{\alpha n}_{a+}(f(t)-f(a)), \tag{2.5}$$

where the series above are uniformly convergent on $[a,b], 0<\alpha<1$, and for the first series f should be in $AC[a,b]$, while for the second series (2.5) any continuous function f is acceptable.

We finish this subsection by proving a useful result, the inversion property for the ABC derivative and AB integral.

Lemma 2.7. *Let f be a continuous function on $[a,b]$. Then the following relations hold:*

$$ {}^{AB}I^{\alpha}_{a+}\left({}^{ABC}D^{\alpha}_{a+}f(t)\right) = f(t) - f(a), \quad t\in(a,b], \tag{2.6}$$

$$ {}^{ABC}D^{\alpha}_{a+}\left({}^{AB}I^{\alpha}_{a+}f(t)\right) = f(t) - f(a)E_{\alpha}\left(\frac{-\alpha}{1-\alpha}t^{\alpha}\right), \quad t\in(a,b]. \tag{2.7}$$

Proof. The first result (2.6) was proved in [6, Corollary 2.3] for $f \in AC[a,b]$, and for continuous f it can be proved easily using the formula (2.5). For the second one, by (2.5) we have

$$ {}^{ABC}D^{\alpha}_{a+}\left({}^{AB}I^{\alpha}_{a+}f(t)\right) = \frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{+\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n {}^{RL}I^{\alpha n}_{a+}\left({}^{AB}I^{\alpha}_{a+}f(t) - {}^{AB}I^{\alpha}_{a+}f(a)\right).$$

Since $f(t)$ is continuous and therefore bounded near $t=a$, we have ${}^{RL}I^{\alpha}_{a+}f(a) = 0$ and therefore ${}^{AB}I^{\alpha}_{a+}f(a) = \frac{1-\alpha}{B(\alpha)}f(a)$. So the right-hand side of the above expression equals

$$\begin{aligned}
&\tfrac{B(\alpha)}{1-\alpha}\sum_{n=0}^{+\infty}\left(\tfrac{-\alpha}{1-\alpha}\right)^n {}^{RL}I^{\alpha n}_{a+}\left(\tfrac{1-\alpha}{B(\alpha)}f(t) + \tfrac{\alpha}{B(\alpha)}\,{}^{RL}I^{\alpha}_{a+}f(t) - \tfrac{1-\alpha}{B(\alpha)}f(a)\right)\\
&= \sum_{n=0}^{+\infty}\left(\tfrac{-\alpha}{1-\alpha}\right)^n {}^{RL}I^{\alpha n}_{a+}f(t) + \tfrac{\alpha}{1-\alpha}\sum_{n=0}^{+\infty}\left(\tfrac{-\alpha}{1-\alpha}\right)^n {}^{RL}I^{\alpha n+\alpha}_{a+}f(t)\\
&\quad -\sum_{n=0}^{+\infty}\left(\tfrac{-\alpha}{1-\alpha}\right)^n \tfrac{t^{n\alpha}}{\Gamma(n\alpha+1)}f(a)\\
&= \sum_{n=0}^{+\infty}\left(\tfrac{-\alpha}{1-\alpha}\right)^n {}^{RL}I^{\alpha n}_{a+}f(t) - \sum_{n=1}^{+\infty}\left(\tfrac{-\alpha}{1-\alpha}\right)^n {}^{RL}I^{\alpha n}_{a+}f(t) - E_{\alpha}\left(\tfrac{-\alpha}{1-\alpha}t^{\alpha}\right)f(a)\\
&= f(t) - E_{\alpha}\left(\tfrac{-\alpha}{1-\alpha}t^{\alpha}\right)f(a).
\end{aligned}$$

Thus, both inversion relations are proved. □

3. Main results

We will study the following differential equation with continuous variable coefficients and AB operators:

$$ {}^{ABC}D^{\beta_0}_{0+}x(t) + \sum_{i=1}^{m} d_i(t)\,{}^{ABC}D^{\beta_i}_{0+}x(t) = h(t), \quad t\in[0,T], \tag{3.1}$$

under either the homogeneous initial condition

$$ x(0^+) = 0, \tag{3.2}$$

or the general initial condition

$$ x(0^+) = c_0 \in \mathbb{R}, \tag{3.3}$$

where $m\in\mathbb{N}$ and $1>\beta_0>\beta_1>\ldots>\beta_m>0$ and x,h,d_i $(i=1,\ldots,m)$ are continuous functions on $[0,T]$. For consistency of the problem, the function h must satisfy $h(0)=0$.

The following theorem establishes existence and uniqueness of a continuous solution of the Eq. (3.1) with the homogeneous initial condition (3.2) (later we will extend this result to the same equation with general initial conditions).

Theorem 3.1. *Let h, d_i $(i=1,\ldots,m)$ be continuous functions on $[0,T]$, satisfying $h(0)=0$ and also the following bound on the d_i functions:*

$$ \frac{1-\beta_0}{|B(\beta_0)|}\sum_{i=1}^{m}\frac{|B(\beta_i)|}{1-\beta_i}\|d_i\|_{\infty} \leqslant C < 1. \tag{3.4}$$

Then the initial value problem 3.1,3.2 has a unique continuous solution $x(t)$, and it is given by the uniform limit $x(t) = \lim_{n\to+\infty}x_n(t)$ of the following sequence:

$$\begin{cases} x_0(t) &= {}^{AB}I^{\beta_0}_{0+}h(t),\\ x_n(t) &= x_0(t) - {}^{AB}I^{\beta_0}_{0+}\sum_{i=1}^{m} d_i(t)\,{}^{ABC}D^{\beta_i}_{0+}x_{n-1}(t), \quad n=1,2,\ldots. \end{cases} \tag{3.5}$$

Proof. Firstly, we prove that the initial value problem (3.1) and (3.2) can be seen equivalently as an integral equation.

Let us suppose that $x(t)$ is a continuous function on $[a,b]$ satisfying (3.1) and (3.2). Define $w(t) = {}^{ABC}D^{\beta_0}_{0+}x(t)$. By Lemma 2.4, we have $w\in C[0,T]$ and $w(0)=0$. By Lemma 2.6 and the initial condition (3.2), we have ${}^{AB}I^{\beta_0}_{0+}w(t) = x(t)$. Hence, Eq. (3.1) becomes

$$w(t)+\sum_{i=1}^{m}d_i(t)\,{}^{ABC}D_{0+}^{\beta_i}\,{}^{AB}I_{0+}^{\beta_0}w(t)=h(t). \tag{3.6}$$

We have then proved that $x\in C[a,b]$ is a solution of the initial value problem 3.1,3.2, then $w(t)={}^{ABC}D_{0+}^{\beta_0}x(t)$ gives a solution $w\in C[0,T]$ of the integral Eq. (3.6). Note that $w(0)=h(0)=0$ since ABC derivatives are zero at the initial point.

Let us now prove the converse statement. Assume that $w\in C[0,T]$ is a solution of (3.6) such that $w(0)=0$. Applying the operator ${}^{AB}I_{0+}^{\beta_0}$ to expression (3.6), we obtain

$${}^{AB}I_{0+}^{\beta_0}w(t)+{}^{AB}I_{0+}^{\beta_0}\sum_{i=1}^{m}d_i(t)\,{}^{ABC}D_{0+}^{\beta_i}\,{}^{AB}I_{0+}^{\beta_0}w(t)={}^{AB}I_{0+}^{\beta_0}h(t).$$

Define $x(t)={}^{AB}I_{0+}^{\beta_0}w(t)$, which is also continuous on $[0,T]$. We have

$$x(t)+{}^{AB}I_{0+}^{\beta_0}\sum_{i=1}^{m}d_i(t)\,{}^{ABC}D_{0+}^{\beta_i}x(t)={}^{AB}I_{0+}^{\beta_0}h(t),$$

where all the functions in this expression are continuous on $[0,T]$. Applying the operator ${}^{ABC}D_{0+}^{\beta_0}$, we have

$${}^{ABC}D_{0+}^{\beta_0}x(t)+{}^{ABC}D_{0+}^{\beta_0}\,{}^{AB}I_{0+}^{\beta_0}\sum_{i=1}^{m}d_i(t)\,{}^{ABC}D_{0+}^{\beta_i}x(t)$$
$$={}^{ABC}D_{0+}^{\beta_0}\,{}^{AB}I_{0+}^{\beta_0}h(t).$$

Using Lemma 2.7, and the fact that both h and all ABC derivatives are zero at the initial point $t=0$, it follows that

$${}^{ABC}D_{0+}^{\beta_0}x(t)+\sum_{i=1}^{m}d_i(t)\,{}^{ABC}D_{0+}^{\beta_i}x(t)=h(t).$$

Now we have shown that x satisfies (3.1), and it remains to check the initial condition (3.2).

Since $x(t)={}^{AB}I_{0+}^{\beta_0}w(t)$ is a linear combination of $w(t)$ and ${}^{RL}I_{0+}^{\beta_0}w(t)$, it must be zero as $t\to 0^+$, since $w(0)=0$ and w is continuous therefore bounded near $t=0$. So we have the desired initial condition (3.2).

Then the solution $w(t)\in C[0,T]$ of (3.6) implies that $x(t)=I_{0+}^{\beta_0}w(t)\in C[0,T]$ is the solution of problem (3.1) satisfying condition (3.2).

Now we have proved that the initial value problem 3.1,3.2 is equivalent to the integral Eq. (3.6) with the initial condition $w(0)=0$, via the substitutions $w(t)={}^{ABC}D_{0+}^{\beta_0}x(t)$ and $x(t)=I_{0+}^{\beta_0}w(t)$, both problems being considered and solved for continuous functions x,w. It remains to prove the existence and uniqueness of a continuous solution of Eq. (3.6).

Let us define the operator T by

$$Tw(t):=h(t)-\sum_{i=1}^{m}d_i(t)\,{}^{ABC}D_{0+}^{\beta_i}\,{}^{AB}I_{0+}^{\beta_0}w(t).$$

Then the Eq. (3.6) is equivalent to $Tw(t)=w(t)$, so we consider T as an operator mapping the space $C_0[0,T]$ of continuous functions w with $w(0)=0$ to itself. Now we use an equivalent norm to the supremum norm on $C_0[0,T]$, namely the following:

$$\|z\|_k:=\sup_{t\in[0,T]}\{e^{-kt}|z(t)|\},$$

for some $k\in\mathbb{R}^+$ to be fixed later. Using the series representation (2.5) of the ABC derivative, and taking into account the fact that $w(0)=0$ and therefore $\lim_{t\to 0}{}^{AB}I_{0+}^{\beta_0}w(t)=0$, we get for any $w_1,w_2\in C_0[0,T]$ and any $t\in[0,T]$ that

$$\begin{aligned}|Tw_1(t)-Tw_2(t)| &\leqslant \sum_{i=1}^{m}\|d_i\|_\infty\left|{}^{ABC}D_{0+}^{\beta_i}\,{}^{AB}I_{0+}^{\beta_0}w_1(t)-{}^{ABC}D_{0+}^{\beta_i}\,{}^{AB}I_{0+}^{\beta_0}w_2(t)\right|\\ &=\sum_{i=1}^{m}\|d_i\|_\infty\left|\tfrac{B(\beta_i)}{1-\beta_i}\sum_{n=0}^{+\infty}\left(\tfrac{-\beta_i}{1-\beta_i}\right)^n {}^{RL}I_{a+}^{\beta_i n}\,{}^{AB}I_{0+}^{\beta_0}(w_1(t)-w_2(t))\right|\\ &\leqslant \|w_1-w_2\|_k(J_1(t)+J_2(t)),\end{aligned}$$

where

$$J_1(t)=\frac{1-\beta_0}{|B(\beta_0)|}\sum_{i=1}^{m}\|d_i\|_\infty\frac{|B(\beta_i)|}{1-\beta_i}\sum_{n=0}^{+\infty}\left(\frac{\beta_i}{1-\beta_i}\right)^n {}^{RL}I_{a+}^{\beta_i n}(e^{kt})$$

and

$$J_2(t)=\frac{\beta_0}{|B(\beta_0)|}\sum_{i=1}^{m}\|d_i\|_\infty\frac{|B(\beta_i)|}{1-\beta_i}\sum_{n=0}^{+\infty}\left(\frac{\beta_i}{1-\beta_i}\right)^n {}^{RL}I_{a+}^{\beta_i n+\beta_0}(e^{kt}).$$

Let us now prove that, for an appropriate choice of $k\in\mathbb{R}_+$, there exist positive constants $C_1,C_2<1$ such that

$$|J_1(t)|<C_1e^{kt},\quad |J_2(t)|<C_2e^{kt},\quad \text{forany}\quad t\in[0,T].$$

We first recall the following estimate from [32, Remark 3.2]:

$$I_{0+}^{\lambda}e^{pt}\leqslant\frac{e^{pt}}{p^{\lambda}},\quad t,\lambda>0,\ p\in\mathbb{R}_+. \tag{3.7}$$

For the case of J_1, we must treat the $n=0$ term of the inner series separately and use (3.4), while the series over $n\geqslant 1$ is treated using (3.7):

$$\begin{aligned}J_1(t)&=\tfrac{1-\beta_0}{|B(\beta_0)|}\sum_{i=1}^{m}\|d_i\|_\infty\tfrac{|B(\beta_i)|}{1-\beta_i}\left(e^{kt}+\sum_{n=1}^{+\infty}\left(\tfrac{\beta_i}{1-\beta_i}\right)^n {}^{RL}I_{a+}^{\beta_i n}(e^{kt})\right)\\ &\leqslant Ce^{kt}+e^{kt}\tfrac{1-\beta_0}{|B(\beta_0)|}\sum_{i=1}^{m}\|d_i\|_\infty\tfrac{|B(\beta_i)|}{1-\beta_i}\sum_{n=1}^{+\infty}\left(\tfrac{\beta_i}{1-\beta_i}\right)^n\tfrac{1}{k^{\beta_i n}}\\ &=Ce^{kt}+e^{kt}\tfrac{1-\beta_0}{|B(\beta_0)|}\sum_{i=1}^{m}\|d_i\|_\infty\tfrac{\beta_i|B(\beta_i)|}{(1-\beta_i)^2}\cdot\tfrac{k^{-\beta_i}}{1-\frac{\beta_i}{1-\beta_i}k^{-\beta_i}},\end{aligned}$$

assuming $k\in\mathbb{R}_+$ is large enough so that $\left|\frac{\beta_i}{1-\beta_i}k^{-\beta_i}\right|<1$. The freedom of the choice of $k\in\mathbb{R}_+$ enables us to guarantee that there exists sufficiently large k such that $|J_1(t)|<(C+C_1)e^{kt}$ for any $t\in[0,T]$, where $C_1\in(0,1)$ is a fixed constant satisfying $C_1<\frac{1-C}{2}$. The case of J_2 is similar, except that the $n=0$ term does not merit separate consideration, since $\beta_i n+\beta_0$ is positive for all values of $n\geqslant 0$. Therefore, we can guarantee $|J_2(t)|<C_2e^{kt}$ for any $t\in[a,b]$, where $C_2\in(0,1)$ is a fixed constant satisfying $C_2<\frac{1-C}{2}$. We then get a constant $C\prime=C+C_1+C_2\in(0,1)$ such that

$$e^{-kt}|Tw_1(t)-Tw_2(t)|\leqslant C\prime\|w_1-w_2\|_k,$$

which means $\|Tw_1-Tw_2\|_k\leqslant C\|w_1-w_2\|_k$. Thus, the operator T is contractive with respect to the norm $\|\cdot\|_k$, and hence it is contractive with respect to the supremum norm $\|\cdot\|_\infty$. By the Banach fixed point theorem, it follows that the integral Eq. (3.6) has a unique solution in $C_0[0,T]$ with respect to $\|\cdot\|_\infty$. Also, the sequence $\{w_n(t)\}_{n\geqslant 0}$ defined recursively by $w_0=h\in C_0[0,T]$ and

$$w_n(t) = Tw_{n-1}(t) = h(t) - \sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+}\, {}^{AB}I^{\beta_0}_{0+} w_{n-1}(t)$$

converges uniformly to the fixed point $w(t)$. Defining $x_n(t) = {}^{AB}I^{\beta_0}_{0+} w_n(t)$ and making use of the fact that fractional integrals preserve uniform convergence, we obtain the sequence of functions (3.5) converging uniformly to the unique continuous solution $x(t)$ of 3.1,3.2, which completes the proof. □

We now establish the main result of the paper on the explicit representation of a continuous solution of Eq. (3.1).

Theorem 3.2. *Let h, d_i $(i = 1, \ldots, m)$ be continuous functions on $[0, T]$ satisfying $h(0) = 0$ and the bound assumption (3.4). Then the initial value problem 3.1,3.2 has a unique continuous solution $x(t)$ and it is given by the uniformly convergent series formula*

$$x(t) = \sum_{k=0}^{+\infty} (-1)^k\, {}^{AB}I^{\beta_0}_{0+} \left(\sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+}\, {}^{AB}I^{\beta_0}_{0+} \right)^k h(t). \tag{3.8}$$

Proof. From Theorem 3.1 follows the existence and uniqueness of a continuous solution of the initial value problem 3.1,3.2. To find the explicit solution, we use the method of successive approximations based on the recursively defined sequence of functions (3.5). The first approximation $x_1 \in C[0, T]$ is

$$\begin{aligned} x_1(t) &= {}^{AB}I^{\beta_0}_{0+} h(t) - {}^{AB}I^{\beta_0}_{0+} \sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+}\, {}^{AB}I^{\beta_0}_{0+} h(t) \\ &= \sum_{k=0}^{1} (-1)^k\, {}^{AB}I^{\beta_0}_{0+} \left(\sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+}\, {}^{AB}I^{\beta_0}_{0+} \right)^k h(t), \end{aligned}$$

The second approximation $x_2 \in C[0, T]$ is

$$\begin{aligned} x_2(t) &= {}^{AB}I^{\beta_0}_{0+} h(t) - {}^{AB}I^{\beta_0}_{0+} \sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+} x_1(t) \\ &= {}^{AB}I^{\beta_0}_{0+} h(t) - {}^{AB}I^{\beta_0}_{0+} \sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+} \Big({}^{AB}I^{\beta_0}_{0+} h(t) \\ &\quad - {}^{AB}I^{\beta_0}_{0+} \sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+}\, {}^{AB}I^{\beta_0}_{0+} h(t) \Big) \\ &= {}^{AB}I^{\beta_0}_{0+} h(t) - {}^{AB}I^{\beta_0}_{0+} \sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+}\, {}^{AB}I^{\beta_0}_{0+} h(t) \\ &\quad + {}^{AB}I^{\beta_0}_{0+} \left(\sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+}\, {}^{AB}I^{\beta_0}_{0+} \right)^2 h(t) \\ &= \sum_{k=0}^{2} (-1)^k\, {}^{AB}I^{\beta_0}_{0+} \left(\sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+}\, {}^{AB}I^{\beta_0}_{0+} \right)^k h(t). \end{aligned}$$

Now we prove by induction that for all $n \in \mathbb{N}$ the nth approximation function $x_n \in C[0, T]$ is given by

$$x_n(t) = \sum_{k=0}^{n} (-1)^k\, {}^{AB}I^{\beta_0}_{0+} \left(\sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+}\, {}^{AB}I^{\beta_0}_{0+} \right)^k h(t). \tag{3.9}$$

Let us assume that x_n can be written in the form of (3.9), and show that the corresponding formula also holds for x_{n+1}. We have

$$\begin{aligned} x_{n+1}(t) &= {}^{AB}I^{\beta_0}_{0+} h(t) - {}^{AB}I^{\alpha_0}_{0+} \sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+} x_n(t) \\ &= {}^{AB}I^{\beta_0}_{0+} h(t) \\ &\quad - {}^{AB}I^{\beta_0}_{0+} \sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+} \left(\sum_{k=0}^{n} (-1)^k\, {}^{AB}I^{\beta_0}_{0+} \left(\sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+}\, {}^{AB}I^{\beta_0}_{0+} \right)^k h(t) \right) \\ &= {}^{AB}I^{\beta_0}_{0+} h(t) + \sum_{k=0}^{n} (-1)^{k+1}\, {}^{AB}I^{\beta_0}_{0+} \left(\sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+}\, {}^{AB}I^{\beta_0}_{0+} \right)^{k+1} h(t) \\ &= \sum_{k=0}^{n+1} (-1)^k\, {}^{AB}I^{\beta_0}_{0+} \left(\sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+}\, {}^{AB}I^{\beta_0}_{0+} \right)^k h(t). \end{aligned}$$

By the principle of induction, we have shown that the *nth* approximation x_n to the solution is given by (3.9) for all $n \in \mathbb{N}$. Since the sequence of functions $\{x_n(t)\}_{n \geqslant 0}$ is uniformly convergent to $x(t)$, we obtain

$$\begin{aligned} x(t) &= \lim_{n \to +\infty} x^n(t) \\ &= \sum_{k=0}^{+\infty} (-1)^k\, {}^{AB}I^{\beta_0}_{0+} \left(\sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+}\, {}^{AB}I^{\beta_0}_{0+} \right)^k h(t), \end{aligned}$$

where the infinite series over k is uniformly convergent on $[0, T]$. □

The above work was all done under the assumption of the homogeneous initial condition (3.2). For the general initial condition $x(t) = c_0 \in \mathbb{R}$, the following result is an easy consequence of Theorem 3.2, since the ABC derivative of a constant is zero.

Theorem 3.3. *Let h, d_i $(i = 1, \ldots, m)$ be continuous functions on $[0, T]$ satisfying $h(0) = 0$ and the bound assumption (3.4). Then the initial value problem given by (3.1) and the general initial condition (3.3) has a unique continuous solution $x(t)$ and it is given by the uniformly convergent series formula*

$$x(t) = c_0 + \sum_{k=0}^{+\infty} (-1)^k\, {}^{AB}I^{\beta_0}_{0+} \left(\sum_{i=1}^{m} d_i(t)\, {}^{ABC}D^{\beta_i}_{0+}\, {}^{AB}I^{\beta_0}_{0+} \right)^k h(t). \tag{3.10}$$

As a consequence of the above results on AB fractional differential equations with variable coefficients, we also obtain the following result for the solution in the simpler case of constant coefficients.

Corollary 3.4. Let h be a continuous function on $[0, T]$ satisfying $h(0) = 0$, and let $d_1, \ldots, d_m$ and $1 > \beta_0 > \beta_1 > \ldots > \beta_m > 0$ be constants such that

$$\frac{1 - \beta_0}{|B(\beta_0)|} \sum_{i=1}^{m} \frac{|B(\beta_i)|}{1 - \beta_i} |d_i| < 1.$$

Then the initial value problem given by

$$\begin{aligned} &{}^{ABC}D^{\beta_0}_{0+} x(t) + \sum_{i=1}^{m} d_i\, {}^{ABC}D^{\beta_i}_{0+} x(t) = h(t), \quad t \in [0, T], \\ &x(0^+) = c_0 \in \mathbb{R}, \end{aligned}$$

has a unique continuous solution $x(t)$ and it is given by the uniformly convergent series formula

$$x(t) = c_0 + \sum_{k=0}^{+\infty} \sum_{i=1}^{m} (-d_i)^k \left({}^{AB}I^{\beta_0}_{0+}\, {}^{ABC}D^{\beta_i}_{0+} \right)^k\, {}^{AB}I^{\beta_0}_{0+} h(t). \tag{3.11}$$

Remark 3.5. Note that the above result for constant-coefficient AB fractional differential equations cannot be expressed in such a neat form as the corresponding results for Riemann–Liouville or Caputo fractional differential equations [20,26]. This is due to the lack of semigroup property for AB operators [6]: a composition of arbitrarily many AB operators of independent orders, even with constant coefficients and no functions intervening, even assuming homogeneous initial conditions, cannot be written as a single simple integral operator of AB type. Even if we write the AB operators in terms of Prabhakar operators [12], we are unable to make use of the Prabhakar semigroup property since the orders β_i, β_0 are different. For this reason, we have not illustrated our results with a specific example, since even in the simplest case of constant coefficients and an elementary function $h(t)$, the formula for the solution function would still be an infinite series without a closed form. We expect that future works from the numerical point of view will be able to make use of our formula for efficiently constructing approximate solutions to AB differential equations of the type considered here, even if there is no simple closed form for the exact solution.

4. Conclusions

In this work, we have considered fractional differential equations with Atangana–Baleanu operators and variable coefficients. We used the Banach fixed point theorem to establish the existence and uniqueness of solutions to such equations, and also used the method of successive approximations to find an explicit expression for the solution in terms of a uniformly convergent infinite series. To the best of our knowledge, this is the first time that such a general fractional differential equation has been solved in the Atangana–Baleanu model by a constructive analytical method yielding an explicit solution function.

Series solutions are a tried and tested method for solving fractional differential equations [2,6], and the solution established here is exact and analytic, albeit perhaps difficult to calculate numerically in specific examples. In future work, we plan to improve the form of this solution by rewriting it in a way that is easier to calculate and therefore more applicable in the various real-world problems that arise involving AB operators and differential equations. Other directions of future research will include extending the work of this paper to the more general Prabhakar operators [12], and potentially also extending to abstract Cauchy problems in a Banach space setting.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The second and third authors were supported by the Nazarbayev University Program 091019CRP2120. No new data was collected or generated during the course of this research. The second author thanks Colciencias and Universidad de Antioquia (Convocatoria 848 - Programa de estancias postdoctorales 2019) for their support.

References

[1] T. Abdeljawad, D. Baleanu, Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel, J. Nonlinear Sci. Appl. 10 (2017) 1098–1107.
[2] C.N. Angstmann, B.I. Henry, Generalized fractional power series solutions for fractional differential equations, Appl. Math. Lett. 102 (2020) 106107.
[3] T.M. Atanacković, B. Stanković, Linear fractional differential equation with variable coefficients I. Bull. de l Acad, Serbe Sci. Arts, Cl. Math. 38 (2013) 27–42.
[4] T.M. Atanacković, B. Stanković, Linear fractional differential equation with variable coefficients II. Bull. de l Acad, Serbe Sci. Arts, Cl. Math. 39 (2014) 53–78.
[5] A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, Therm. Sci. 20 (2) (2016) 763–769.
[6] D. Baleanu, A. Fernandez, On some new properties of fractional derivatives with Mittag-Leffler kernel, Commun. Nonlinear Sci. Numer. Simul. 59 (2018) 444–462.
[7] E. Bas, B. Acay, R. Özarslan, Fractional models with singular and non-singular kernels for energy efficient buildings, Chaos 29 (2) (2019) 023110.
[8] A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York, 1997.
[9] K. Diethelm, The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type, Springer, Heidelberg, 2010.
[10] M.M. Dzhrbashyan, A.B. Nersessyan, Fractional derivatives and Cauchy problem for differential equations of fractional order, Izv. AN Arm. SSR. Mat. 3 (1968).
[11] A. Fernandez, A complex analysis approach to Atangana-Baleanu fractional calculus, Math. Meth. Appl. Sci. 44 (10) (2021) 8070–8087.
[12] A. Fernandez, D. Baleanu, Classes of Operators in Fractional Calculus: A Case Study, Math. Meth. Appl. Sci. 44 (11) (2021) 9143–9162.
[13] R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogosin. Mittag–Leffler Functions, Related Topics and Applications. Springer Monographs in Mathematics, Springer, New York (2014).
[14] R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[15] A. Jajarmi, S. Arshad, D. Baleanu, A new fractional modelling and control strategy for the outbreak of dengue fever, Physica A: Stat. Mech. Appl. 535 (2019) 122524.
[16] A. Jajarmi, A. Yusuf, D. Baleanu, M. Inc, A new fractional HRSV model and its optimal control: A non-singular operator approach, Physica A: Stat. Mech. Appl. 547 (2020) 123860.
[17] F. Jarad, T. Abdeljawad, Z. Hammouch, On a class of ordinary differential equations in the frame of Atangana-Baleanu fractional derivative, Chaos Solitons Fractals 117 (2018) 16–20.
[18] A.A. Kilbas, M. Rivero, L. Rodríguez-Germa, J.J. Trujillo, α–Analytic solutions of some linear fractional differential equations with variable coefficients, Appl. Math. Comput. 187 (9) (2007) 239–249.
[19] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier Science B.V., Amsterdam (2006).
[20] M.-H. Kim, H.-C.O. Explicit representation of Green's function for linear fractional differential operator with variable coefficients. J. Fract. Calc. Appl. 5(1) (2014), 26–36.

[21] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity, Imperial College Press, London, 2010.
[22] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley, New York, 1993.
[23] I.A. Mirza, M.S. Akram, N.A. Shah, W. Imtiaz, J.D. Chung, Analytical solutions to the advection-diffusion equation with Atangana-Baleanu time-fractional derivative and a concentrated loading, Alexandria Eng. J. 60 (1) (2021) 1199–1208.
[24] P.A. Naik, M. Yavuz, J. Zu, The role of prostitution on HIV transmission with memory: A modeling approach, Alexandria Eng. J. 59 (4) (2020) 2513–2531.
[25] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.
[26] S. Pak, H. Choi, K. Sin, K. Ri, Analytical solutions of linear inhomogeneous fractional differential equation with continuous variable coefficients, Adv. Differ. Equ. 2019 (2019) 256.
[27] S.K. Panda, C. Ravichandran, B. Hazarika, Results on system of Atangana-Baleanu fractional order Willis aneurysm and nonlinear singularly perturbed boundary value problems, Chaos Solitons Fractals 142 (2021) 110390.
[28] C. Phang, Y.T. Toh, F.S.M. Nasrudin, An Operational Matrix Method Based on Poly-Bernoulli Polynomials for Solving Fractional Delay Differential Equations, Computation 8 (3) (2020) 82.
[29] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[30] C. Ravichandran, K. Logeswari, F. Jarad, New results on existence in the framework of Atangana-Baleanu derivative for fractional integro-differential equations, Chaos Solitons Fractals 125 (2019) 194–200.
[31] J.E. Restrepo, M. Ruzhansky, D. Suragan, Explicit solutions for linear variable–coefficient fractional differential equations with respect to functions, Appl. Math. Comput. 403 (2021) 126177.
[32] M. Rivero, L. Rodrígez-Ciermá, J.J. Trujillo, Linear fractional differential equations with variable coefficients, Appl. Math. Lett. 21 (2008) 892–897.
[33] K.M. Saad, M.M. Khader, J.F. Gómez-Aguilar, D. Baleanu, Numerical solutions of the fractional Fisher's type equations with Atangana-Baleanu fractional derivative by using spectral collocation methods, Chaos 29 (2) (2019) 023116.
[34] S.T. Saeed, M.B. Riaz, D. Baleanu, K.A. Abro, A mathematical study of natural convection flow through a channel with non-singular kernels: an application to transport phenomena, Alexandria Eng. J. 59 (4) (2020) 2269–2281.
[35] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional integrals and derivatives, translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.
[36] V.E. Tarasov, Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media, Springer, New York, 2011.
[37] A. Templeton, A bibliometric analysis of Atangana-Baleanu operators in fractional calculus, Alexandria Eng. J. 59 (4) (2020) 2733–2738.
[38] V.V. Uchaikin. Fractional Derivatives for Physicists and Engineers: Vol. 1. Background and Theory; Vol 2. Application. Springer, (2013).