PENROSE INSTABILITY ANALYSIS IN THE HIROTA EQUATION

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Abstract. In this research, we study the Penrose instability analysis in the Hirota equation, which is a higher-order version of Nonlinear Schrödinger equation. We apply the Wigner function to Hirota equation in order to obtain Wigner-Hirota equation.
We take the ansatz as $W(x,k,t) = W_0(k) + \varepsilon e^{i(Kx-\Omega t)}$. For $W_0(k)$ we consider two different functions: Dirac-delta functional and Landau damping function. Finally, depending on the $Im(\Omega)$ we seek to find instability intervals for K.

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1. INTRODUCTION

Consider the Hirota equation (HE) introduced by R. Hirota in [10]

$$i\partial_t\psi + i\alpha|\psi|^2\partial_x\psi + \beta\partial_x^2\psi + i\gamma\partial_x^3\psi + \theta|\psi|^2\psi = 0, \tag{1}$$

where $\psi = \psi(x,t)$, $x,t \in \mathbb{R}$, and $\alpha, \beta, \theta, \gamma$ are positive constants. This equation has been used to solve problems in optics [6], oceanography. Observe that when we take in the HE (1) the parameters $\alpha = \gamma = 0$ and $\beta = 1$ we are reduced to the Nonlinear Schrödinger equation (NLSE) (2).

$$i\partial_t\psi + \partial_x^2\psi + \lambda|\psi|^2\psi = 0, \tag{2}$$

where $\psi = \psi(x,t)$, $x,t \in \mathbb{R}$, and $\lambda \in \mathbb{R}$ is a fixed constant. Moreover the choice $\beta = \theta = 0$, $\alpha = \gamma = 1$ and ψ a real function give us the mKdV (3).

$$\partial_t\psi + \partial_x^3\psi + \psi^2\partial_x\psi = 0, \tag{3}$$

where $\psi = \psi(x,t)$, $x,t \in \mathbb{R}$.

Recently, Hirota equation has been used to describe different wave phenomena, and certain results obtained in [1] show that the Hirota equation, being a modified version of Nonlinear Shrödinger equation, gives more accurate descriptions of phenomena in oceans, particularly, rogue waves [5]. In this research presented by A. Ankiewicz, M. Soto-Crespo and N. Akhmediev the goal was to find explicit shapes for the lower-order solutions. They concluded that modified Darboux transformation methods help to find rational solutions of lower orders, moreover, numerical analysis showed that second order rational solution is a good approximation of a rogue wave produced by Hirota equation.

Other results were obtained in [4]. In their research rogue waves phenomena is studied with the Hirota system, which consists of two coupled Hirota equations. They try to connect rogue waves with modulation instability and support it by examining the connection with numerical simulations.

In most papers Hirota equation appears with specific relation between constants, the most common is $\theta = 2\beta$ and $\alpha = 6\gamma$. However, in [10], R. Hirota, after whom the equation is named, proposed the most general form of the equation, considering coefficients as constants from real numbers space. In our paper we also use the most general case of Hirota equation, allowing coefficients to be positive constants.

Let us introduce the Wigner function (eq. (5)), which was introduced in [11], that we use to transform the Hirota equation into the Wigner-Hirota equation (10). It is important to emphasize that the latter equation represents an alternative description of the problem that is formally equivalent to the Hirota equation. This is in the Wigner function framework, which is widely used in optics to study the Linear Schrödinger equation. Thus, wave function approach and Wigner function approach are equivalent.

In order to perform the Penrose instability analysis, as a next step we propose the ansatz for the equations (10) as follows:

$$W(x,k,t) := W_0(k) + \varepsilon e^{i(Kx-\Omega t)}.$$

Thus, the ansatz consists of the single variable function $W_0(k)$, which is a solution to Wigner-Hirota equation, and oscillatory perturbation with parameters Ω and K. The idea of the Penrose instability analysis is to find interval for $K \in \mathbb{R}$ such that $Im(\Omega) > 0$, since then $e^{-i\Omega t}$ is a positive exponential, which allows the solution to diverge as $t \to \infty$, and this is what we mean by instability in our paper. This approach was introduced by O. Penrose in [12] in 1960.

As we insert the above ansatz to the equation we seek to find a relation between $W_0(k)$, Ω and K such that it is a solution to Wigner-Hirota equation. In order to simplify the problem, we linearize in ε, assuming that ε is small enough, and this results in the dispersion relation (13).

Similar procedure was used to study the instability in the NLSE in the paper [9] by B. Hall, M. Lisak, D. Anderson, R. Fedele, V. Semenov. They propose two different spectra for $W_0(k)$:

(1) $W_0(k) = \psi_0^2\delta(k)$, which is Dirac delta function for some constant ψ_0.

(2) $W_0(k) = \frac{\psi_0^2}{\pi}\frac{p_0}{k^2+p_0^2}$, the so called Landau damping function for some constant ψ_0 and $p_0 > 0$.

Both spectra represent different physical situations: The first spectra (1) is extensively used in the standard instability analysis and corresponds to a free plane wave, whereas the second spectra (2) can be interpreted as a the solution characterized by the Landau damping, where p_0 determines the strength of the damping.

Each spectra was inserted into the above mentioned dispersion relation resulting in an explicit relation between Ω and K. Similar work for NLSE was also done in [3] by A. Athanassoulis, G. Athanassoulis and T. Sapsis.

In our paper, we perform a similar analysis on Hirota equation. As Hirota equation is a higher order NLSE with additional nonlinearity we expect a more involved and interesting result.

So, for Dirac delta functional, we obtain an improved expression for Ω (eq. (18)), which we also call as instability growth rate

$$\Omega = i\beta K^2\Big(\frac{\theta\psi_0^2}{\pi\beta K} - 1\Big)^{1/2} - \gamma K^3 + \frac{\alpha\psi_0^2 K}{2\pi}.$$

However, the $Im(\Omega)$, which is the unstable part, is absolutely the same as in the NLSE case. We focus on $Im(\Omega)$, because from this we can find the instability interval for K, which is the main goal of our project. Thus, for the Dirac delta functional, instability interval for K is the following:

$$K \in \Bigg[-\sqrt{\frac{\psi_0^2\theta}{\pi\beta}}, \sqrt{\frac{\psi_0^2\theta}{\pi\beta}}\Bigg].$$

In order to compare Wigner function approach with the wave function approach, we tried to work directly with Hirota equation. However, we obtained a highly nonlinear equation (see (20)), which cannot be solved explicitly. While in the Wigner function approach we arrived at quadratic equation in Ω (see (19)), solution of which is trivial.

For Landau damping function, as a first step we consider the simpler Hirota equation with $\gamma = 0$ (see Subsection 3.2). The case $\gamma \neq 0$ is discussed in Section 5. Now, the instability growth rate Ω:

$$\Omega = \frac{\alpha\psi_0^2 K}{2\pi} + 2ip_0\beta K \pm i\beta K^2\sqrt{\frac{\psi_0^2}{\pi\beta K^2}(\theta - ip_0\alpha) - 1}.$$

In order to find the instability interval for K, we again impose $Im(\Omega) > 0$. However, in this expression we do not see the $Im(\Omega)$ clearly. Thus, using the method of polar representation of complex numbers we find that (see Theorem 3.9):

$$K \in \left[-\sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2 - 4p_0^2}\ ,\ \sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2 - 4p_0^2}\right].$$

In the case of NLSE (when $\alpha = 0$), the interval for K is:

$$K \in \left[-\sqrt{\frac{\psi_0^2\theta}{\pi\beta} - 4p_0^2}\ ,\ \sqrt{\frac{\psi_0^2\theta}{\pi\beta} - 4p_0^2}\right].$$

If we look at the instability intervals for K, in the case of Dirac delta function, Hirota equation is the same as NLSE. Thus, approximation with Dirac delta function does not show the influence of highly nonlinear term with α, whereas Landau damping function shows the role of this nonlinear term with α. This means that Landau damping function is a better approximation than Dirac delta function, as nonlinearity with α shows its influence only when we use Landau damping function. See more details in Section 4.

Moreover, our future plans (see Section 5 for more details) are to deal with $\gamma \neq 0$ case with Landau damping function and to investigate the emergence of rogue waves.

2. WIGNER DESCRIPTION

In this section our main goal is to apply the Wigner function to the Hirota equation. To do so, we firstly state Lemma 2.1, Lemma 2.2 and Lemma 2.3, which will be helpful to apply the transformation. Then, in Proposition 2.4 we use these lemmas to finally find the Wigner-Hirota equation.

The Fourier transform [8, p. 213] for an integrable function u on $\mathbb{R}$ is defined by

$$\widehat{u}(\xi) := \int_{\mathbb{R}} e^{-i\xi x} u(x)dx. \tag{4}$$

The Wigner function of two functions $u, v : \mathbb{R} \times (0,\infty) \longrightarrow \mathbb{C}$ is defined by

$$W[u,v](x,k,t) := \int_{\mathbb{R}} e^{-iky}\, u\Big(x + \frac{y}{2}, t\Big)\, \overline{v}\Big(x - \frac{y}{2}, t\Big) dy, \quad x, k \in \mathbb{R},\ t > 0. \tag{5}$$

Lemma 2.1. *For functions $u, v : \mathbb{R} \times (0,\infty) \longrightarrow \mathbb{C}$ we have that*

$$W[u,v] = \overline{W[v,u]} \quad \textit{and} \quad W[\partial_x u, v] = \Big(ik + \frac{1}{2}\partial_x\Big) W[u,v]. \tag{6}$$

Proof. We have that

$$\begin{aligned}\overline{W[v,u]} &= \overline{\int_{\mathbb{R}} e^{-iky} v(x+\frac{y}{2},t)\overline{u}(x-\frac{y}{2},t)dy} = \int_{-\infty}^{\infty} e^{iky} u(x-\frac{y}{2},t)\overline{v}(x+\frac{y}{2},t)dy \\ &= \int_{-\infty}^{\infty} e^{-iky} u(x+\frac{y}{2},t)\overline{v}(x-\frac{y}{2},t)dy = W[u,v].\end{aligned}$$

Moreover, note that

$$\begin{aligned}\partial_x W[u,v] &= W[\partial_x u, v] - 2W[u, \partial_y v] = W[\partial_x u, v] + 2W[\partial_y u, v] - 2(ik)W[u,v] \\ &= W[\partial_x u, v] + W[\partial_x u, v] - 2(ik)W[u,v] \\ &= 2W[\partial_x u, v] - 2(ik)W[u,v],\end{aligned}$$

which implies the second identity.

□

Lemma 2.2. *For $u : \mathbb{R} \times (0,\infty) \longrightarrow \mathbb{C}$ and $V : \mathbb{R} \times (0,\infty) \longrightarrow \mathbb{R}$ we have that*

$$\begin{aligned}W[Vu,u] \pm W[u,Vu] = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\lambda y} \Big(V(x+\frac{y}{2},t) \pm V(x-\frac{y}{2},t)\Big)\, dy \\ \times W[u,u](x,k-\lambda,t)\, d\lambda.\end{aligned} \tag{7}$$

Proof. Let's use definition of the Wigner function and the following property of the Fourier transform (see for example [8, Property 8, p. 223])

$$\widehat{fg}(\xi) = \frac{1}{2\pi} \widehat{f} * \widehat{g}(\xi), \tag{8}$$

where $*$ refers to the standard convolution, which is defined as

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

We obtain that

$$\begin{aligned}W[Vu,u] \pm W[u,Vu] &= \int_{\mathbb{R}} e^{-iky} V(x+\frac{y}{2},t)u(x+\frac{y}{2},t)\overline{u}(x-\frac{y}{2},t)dy \\ &\pm \int_{\mathbb{R}} e^{-iky} u(x+\frac{y}{2},t)\overline{V}(x-\frac{y}{2},t)\overline{u}(x-\frac{y}{2},t)dy \\ &= \int_{\mathbb{R}} e^{-iky} [V(x+\frac{y}{2},t) \pm V(x-\frac{y}{2},t)] u(x+\frac{y}{2},t)\overline{u}(x-\frac{y}{2},t)dy \\ &= \Big([V(x+\frac{\cdot}{2},t) \pm V(x-\frac{\cdot}{2},t)] u(x+\frac{\cdot}{2},t)\overline{u}(x-\frac{\cdot}{2},t)\Big)^{\wedge}(k) \\ &= \frac{1}{2\pi}\big(V(x+\frac{\cdot}{2},t) \pm V(x-\frac{\cdot}{2},t)\big)^{\wedge} * \big(u(x+\frac{\cdot}{2},t)\overline{u}(x-\frac{\cdot}{2},t)\big)^{\wedge}(k)\end{aligned}$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda y}\big[V(x+\frac{y}{2},t) \pm V(x-\frac{y}{2},t)\big]dy \int_{\mathbb{R}} e^{-i(k-\lambda)y} u(x+\frac{y}{2})\overline{u}(x-\frac{y}{2})dy\, d\lambda$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda y}\big[V(x+\frac{y}{2},t) \pm V(x-\frac{y}{2},t)\big]dy \times W[u,u](x,k-\lambda,t)d\lambda.$$

□

Lemma 2.3. *For function $u : \mathbb{R} \times (0,\infty) \longrightarrow \mathbb{C}$ we observe that*

$$\int_{\mathbb{R}} W[u,u](x,k,t)dk = 2\pi|u(x,t)|^2. \tag{9}$$

Proof. By (5) we have that

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-iky} u(x+y/2,t)\overline{u}(x-y/2,t)dkdy \\
&= \int_{\mathbb{R}} u(x+y/2,t)\overline{u}(x-y/2,t)\int_{\mathbb{R}} e^{-iky}dkdy \\
&= 2\pi\int_{\mathbb{R}} u(x+y/2,t)\overline{u}(x-y/2,t)\delta(y)dy \\
&= 2\pi u(x,t)\overline{u}(x,t) = 2\pi|u(x,t)|^2.
\end{aligned}$$

□

Finally, we can apply the Wigner function to the Hirota equation using lemmas stated above and obtain a new PDE in the proposition below.

Proposition 2.4. *Let u be a solution of the Hirota equation* (1) *and $W := W[u,u]$. Then, we obtain Wigner-Hirota equation for $W(x,k,t)$*

$$\begin{aligned}
\partial_t W &= \Big(-2\beta k + 3\gamma k^2\Big)\,\partial_x W - \frac{1}{4}\gamma\,\partial_x^3 W \\
&\quad + \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\Big(i\theta - i\alpha(k-\lambda)\Big)e^{-i\lambda y}\Big\{Z\Big(x+\frac{y}{2},t\Big) - Z\Big(x-\frac{y}{2},t\Big)\Big\}\,dy\,W(x,k-\lambda,t)d\lambda \\
&\quad - \frac{\alpha}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda y}\Big\{Z\Big(x+\frac{y}{2},t\Big) + Z\Big(x-\frac{y}{2},t\Big)\Big\}\,dy\,\partial_x W(x,k-\lambda,t)d\lambda
\end{aligned} \tag{10}$$

with

$$Z(x,t) := \frac{1}{2\pi}\int_{\mathbb{R}} W(x,k,t)\,dk. \tag{11}$$

Proof. We have that

$$\begin{aligned}
\partial_t W[u,u] &= W[\partial_t u,u] + W[u,\partial_t u] \\
&= W[-\alpha|u|^2\partial_x u + i\beta\partial_x^2 u - \gamma\partial_x^3 u + i\theta|u|^2 u, u] \\
&\quad + W[u, -\alpha|u|^2\partial_x u + i\beta\partial_x^2 u - \gamma\partial_x^3 u + i\theta|u|^2 u] \\
&= -\alpha\Big(W[|u|^2\partial_x u, u] + W[u, |u|^2\partial_x u]\Big) \\
&\quad + i\beta\Big(W[\partial_x^2 u, u] - W[u, \partial_x^2 u]\Big) \\
&\quad - \gamma\Big(W[\partial_x^3 u, u] + W[u, \partial_x^3 u]\Big) \\
&\quad + i\theta\Big(W[|u|^2 u, u] - W[u, |u|^2 u]\Big).
\end{aligned}$$

We treat each term separately. Using (6) repeatedly we deduce

$$\begin{aligned}
W[\partial_x^2 u, u] - W[u, \partial_x^2 u] &= W[\partial_x^2 u, u] - \overline{W[\partial_x^2 u, u]} \\
&= \Big(ik + \frac{1}{2}\partial_x\Big)^2 W[u,u] - \Big(-ik + \frac{1}{2}\partial_x\Big)^2\overline{W[u,u]} \\
&= \Big[\Big(ik + \frac{1}{2}\partial_x\Big)^2 - \Big(-ik + \frac{1}{2}\partial_x\Big)^2\Big]W[u,u]
\end{aligned}$$

$$= 4(ik)\Big(\frac{1}{2}\partial_x\Big)W[u,u] = 2ik\,\partial_x W[u,u],$$

and similarly,

$$\begin{aligned}
W[\partial_x^3 u, u] + W[u, \partial_x^3 u] &= W[\partial_x^3 u, u] + \overline{W[\partial_x^3 u, u]} \\
&= \Big[\Big(ik + \frac{1}{2}\partial_x\Big)^3 + \Big(-ik + \frac{1}{2}\partial_x\Big)^3\Big]W[u,u] \\
&= \Big[6(ik)^2\Big(\frac{1}{2}\partial_x\Big) + 2\Big(\frac{1}{2}\partial_x\Big)^3\Big]W[u,u] \\
&= \Big(-3k^2\partial_x + \frac{1}{4}\partial_x{}^3\Big)W[u,u].
\end{aligned}$$

On the other hand, an application of (7), taken with $V = |u|^2$, give us

$$\begin{aligned}
W[|u|^2u, u] - W[u, |u|^2u] &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda y}\Big\{|u(x+y/2,t)|^2 - |u(x-y/2,t)|^2\Big\}\,dy \\
&\qquad\qquad \times W[u,u](x,k-\lambda,t)\,d\lambda \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda y}\Big\{Z(x+y/2,t) - Z(x-y/2,t)\Big\}\,dy \\
&\qquad\qquad \times W[u,u](x,k-\lambda,t)\,d\lambda,
\end{aligned}$$

where

$$Z(x,t) := \frac{1}{2\pi}\int_{\mathbb{R}} W[u,u](x,k,t)\,dk = |u(x,t)|^2$$

and we used the marginal property (9).

On the other hand, we can also write

$$\begin{aligned}
&W[|u|^2\partial_x u, u] + W[u, |u|^2\partial_x u] \\
&\quad = \int_{\mathbb{R}} e^{-iky}\Big|u\Big(x+\frac{y}{2},t\Big)\Big|^2\,\partial_x u\Big(x+\frac{y}{2},t\Big)\,\overline{u}\Big(x-\frac{y}{2},t\Big)dy \\
&\qquad + \int_{\mathbb{R}} e^{-iky}\Big|u\Big(x-\frac{y}{2},t\Big)\Big|^2\,u\Big(x+\frac{y}{2},t\Big)\partial_x\overline{u}\Big(x-\frac{y}{2},t\Big)\,dy.
\end{aligned}$$

Here, by Fourier transform definition (4) the above expression becomes

$$\begin{aligned}
&\Big(\Big|u\Big(x+\frac{y}{2},t\Big)\Big|^2\,\partial_x u\Big(x+\frac{y}{2},t\Big)\,\overline{u}\Big(x-\frac{y}{2},t\Big)\Big)^{\wedge}(\xi) \\
&\quad + \Big(\Big|u\Big(x-\frac{y}{2},t\Big)\Big|^2\,u\Big(x+\frac{y}{2},t\Big)\,\partial_x\overline{u}\Big(x-\frac{y}{2},t\Big)\Big)^{\wedge}(\xi).
\end{aligned}$$

Applying Fourier transform property (8) we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\Big(\Big|u\Big(x+\frac{y}{2},t\Big)\Big|^2\Big)^{\wedge} * \Big(\partial_x u\Big(x+\frac{y}{2},t\Big)\,\overline{u}\Big(x-\frac{y}{2},t\Big)\Big)\Big)^{\wedge}(\xi) \\
&\quad + \frac{1}{2\pi}\Big(\Big|u\Big(x-\frac{y}{2},t\Big)\Big|^2\Big)^{\wedge} * \Big(u\Big(x+\frac{y}{2},t\Big)\,\partial_x\,\overline{u}\Big(x-\frac{y}{2},t\Big)\Big)^{\wedge}(\xi).
\end{aligned}$$

Again, by (4) the above is transformed into

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda y}\Big|u\Big(x+\frac{y}{2},t\Big)\Big|^2 dy\int_{\mathbb{R}} e^{-i(k-\lambda)}\Big(\partial_x u\Big(x+\frac{y}{2},t\Big)\,\overline{u}\Big(x-\frac{y}{2},t\Big)\Big)dyd\lambda \\
&\quad + \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda y}\Big|u\Big(x-\frac{y}{2},t\Big)\Big|^2 dy\int_{\mathbb{R}} e^{-i(k-\lambda)}\Big(u\Big(x+\frac{y}{2},t\Big)\,\partial_x\overline{u}\Big(x-\frac{y}{2},t\Big)\Big)dyd\lambda.
\end{aligned}$$

Here, by Fubini's theorem [13, pp 164-165] and Wigner function definition (5) we simplify further

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda y}\Big|u\Big(x+\frac{y}{2},t\Big)\Big|^2\;dy\,W[\partial_x u, u](x,k-\lambda,t)d\lambda \\
&\quad + \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda y}\Big|u\Big(x-\frac{y}{2},t\Big)\Big|^2\;dy\,W[u, \partial_x u](x,k-\lambda,t)d\lambda.
\end{aligned}$$

Using properties from Lemma 2.1 (6) we find that

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda y}\left|u\Big(x+\frac{y}{2},t\Big)\right|^2 \; dy \\
&\qquad\qquad \times \Big(i(k-\lambda)+\frac{1}{2}\partial_x\Big)W[u,u](x,k-\lambda,t)d\lambda \\
&\qquad + \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda y}\left|u\Big(x-\frac{y}{2},t\Big)\right|^2 \; dy \\
&\qquad\qquad \times \Big(-i(k-\lambda)+\frac{1}{2}\partial_x\Big)W[u,u](x,k-\lambda,t)d\lambda.
\end{aligned}
$$

Using (11) we have that

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda y}\, Z\Big(x+\frac{y}{2},t\Big)\, dy \\
&\qquad\qquad \times \Big(i(k-\lambda)+\frac{1}{2}\partial_x\Big)W[u,u](x,k-\lambda,t)d\lambda \\
&\qquad + \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda y}\, Z\Big(x-\frac{y}{2},t\Big)\, dy \\
&\qquad\qquad \times \Big(-i(k-\lambda)+\frac{1}{2}\partial_x\Big)W[u,u](x,k-\lambda,t)d\lambda.
\end{aligned}
$$

Collect the terms with $W[u,u]$ and $\partial_x W[u,u]$ separately to obtain the following

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} i(k-\lambda)e^{-i\lambda y}\Big\{Z\Big(x+\frac{y}{2},t\Big)-Z\Big(x-\frac{y}{2},t\Big)\Big\}\,dy\, W[u,u](x,k-\lambda,t)d\lambda \\
&\qquad + \frac{1}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda y}\Big\{Z\Big(x+\frac{y}{2},t\Big)+Z\Big(x-\frac{y}{2},t\Big)\Big\}\,dy\, \partial_x W[u,u](x,k-\lambda,t)d\lambda.
\end{aligned}
$$

In conclusion, calling $W(x,k,t) := W[u,u](x,k,t)$, by summing up all the obtained terms, we arrive at

$$
\begin{aligned}
\partial_t W = &\Big(-2\beta k + 3\gamma k^2\Big)\,\partial_x W - \frac{1}{4}\gamma\,\partial_x^3 W \\
&+ \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\Big(i\theta - i\alpha(k-\lambda)\Big)e^{-i\lambda y}\Big\{Z\Big(x+\frac{y}{2},t\Big)-Z\Big(x-\frac{y}{2},t\Big)\Big\}\,dy\, W(x,k-\lambda,t)d\lambda \\
&- \frac{\alpha}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda y}\Big\{Z\Big(x+\frac{y}{2},t\Big)+Z\Big(x-\frac{y}{2},t\Big)\Big\}\,dy\, \partial_x W(x,k-\lambda,t)d\lambda
\end{aligned}
$$

with

$$
Z(x,t) := \frac{1}{2\pi}\int_{\mathbb{R}} W(x,k,t)\,dk,
$$

as we wanted. $\square$

3. PENROSE INSTABILITY ANALYSIS

In this section we aim to perform Penrose instability analysis. The idea of the Penrose instability analysis in our paper is as follows: we aim to find instability intervals for K, such that $Im(\Omega) > 0$, since for these conditions our ansatz, which we introduce in Proposition 3.1, diverges, as $t \to \infty$. In other words, oscillations grow exponentially and waves get bigger as time passes, so that the amplitude of the wave goes to ∞ as $t \to \infty$.

Thus, we follow the process:

- Propose and insert a general ansatz $W_0(k)$ to the Wigner-Hirota equation linearizing in ε, which results in a dispersion relation (13) (see Proposition 3.1).
- Insert the Dirac delta function $W_0(k) = \psi_0^2\delta(k)$ into the dispersion relation, find explicit formula for instability growth rate Ω and then find instability intervals for K (see Theorem 3.2).
- Show why Wigner function is better method than directly working with Hirota equation (see Remark 3.3).
- Then we insert Landau damping function $W_0(k) = \frac{\psi_0^2}{\pi}\frac{p_0}{k^2+p_0^2}$ into the dispersion relation (with $\gamma = 0$) and as before we seek for Ω and instability interval for K (see Theorem 3.9).

So, in the proposition below we suggest an ansatz for Wigner-Hirota equation in such a way that $W_0(k)$ is a solution that we begin with and the rest is a plane wave perturbation. We assume that ε is small enough so ε^2 can be neglected, allowing the ansatz to approximate the solution of the equation.

Proposition 3.1. *If we introduce the ansatz as*

$$W(x,k,t) := \Big(W_0(k) + \varepsilon\, e^{i(Kx-\Omega t)}\Big), \tag{12}$$

in equation (10) *and linearize in* ε, *then we obtain the dispersion relation:*

$$1 + \frac{1}{2\pi}\int_{\mathbb{R}} \frac{(\alpha k-\theta)\big(W_0(k+K/2) - W_0(k-K/2)\big) + \alpha\frac{K}{2}\big(W_0(k+K/2) + W_0(k-K/2)\big)}{\Omega + \frac{1}{4}\gamma K^3 - \frac{1}{2\pi}\alpha K\psi_0^2 + \big(-2\beta k + 3\gamma k^2\big)K} dk = 0. \tag{13}$$

Proof. Let's introduce bounds for k in such a way that the ansatz is now:

$$W(x,k,t) := \Big(W_0(k) + \varepsilon\, e^{i(Kx-\Omega t)}\Big)\mathbb{1}_{[-M,M]}(k) \tag{14}$$

Substituting (14) in (10) we get, for $-M \le k \le M$,

$$\begin{aligned}
-i\varepsilon\,\Omega\, e^{i(Kx-\Omega t)} &= \Big(-2\beta k + 3\gamma k^2\Big) i\varepsilon\, K\, e^{i(Kx-\Omega t)} + \frac{1}{4}\gamma\, i\varepsilon\, K^3\, e^{i(Kx-\Omega t)} \\
&+ \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \Big(i\theta - i\alpha(k-\lambda)\Big)e^{-i\lambda y}\Big\{Z\Big(x+\frac{y}{2},t\Big) - Z\Big(x-\frac{y}{2},t\Big)\Big\}\, dy \\
&\qquad \times \Big(W_0(k-\lambda) + \varepsilon\, e^{i(Kx-\Omega t)}\Big) d\lambda \\
&- \frac{\alpha}{4\pi} i\varepsilon\, K\, e^{i(Kx-\Omega t)} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda y}\Big\{Z\Big(x+\frac{y}{2},t\Big) + Z\Big(x-\frac{y}{2},t\Big)\Big\}\, dy\, d\lambda,
\end{aligned}$$

where

$$Z(x,t) := \frac{1}{2\pi}\int_{-M}^{M} W_0(k)\, dk + \frac{M}{\pi}\varepsilon\, e^{i(Kx-\Omega t)}$$

and we define

$$\psi_0^2 := \int_{-M}^{M} W_0(k)\, dk.$$

Linearizing in ε, one gets

$$-\Omega = \Big(-2\beta k + 3\gamma k^2\Big) K + \frac{1}{4}\gamma\, K^3$$

$$
\begin{aligned}
&+\frac{Mi}{\pi^2}\int_{\mathbb{R}}\Big(\theta-\alpha(k-\lambda)\Big)\Big\{\int_{\mathbb{R}}e^{-i\lambda y}\,\sin\left(\frac{Ky}{2}\right)dy\Big\}\,W_0(k-\lambda)d\lambda \\
&-\frac{1}{2\pi}\alpha\,K\,\psi_0^2,
\end{aligned}
\tag{15}
$$

In order to show the above, we need further properties

$$
\int_{\mathbb{R}}\int_{\mathbb{R}}e^{-i\lambda y}\;dy\,d\lambda=2\pi\int_{\mathbb{R}}\delta(\lambda)d\lambda=2\pi,
\tag{16}
$$

as stated in [2, p. 86]. Moreover,

$$
Z\Big(x+\frac{y}{2},t\Big)-Z\Big(x-\frac{y}{2},t\Big)=\frac{1}{\pi}i\varepsilon 2Me^{i(Kx-\Omega t)}\sin\left(\frac{Ky}{2}\right),
$$

$$
\int_{\mathbb{R}}e^{-i\lambda y}\,\sin\left(\frac{Ky}{2}\right)dy=i\pi\Big(\delta\Big(\lambda+\frac{K}{2}\Big)-\delta\Big(\lambda-\frac{K}{2}\Big)\Big)
$$

and

$$
\begin{aligned}
&\frac{Mi}{\pi^2}\int_{\mathbb{R}}\Big(\theta-\alpha(k-\lambda)\Big)\Big\{\int_{\mathbb{R}}e^{-i\lambda y}\,\sin\left(\frac{Ky}{2}\right)dy\Big\}\,W_0(k-\lambda)d\lambda \\
&=\frac{Mi}{\pi^2}\int_{\mathbb{R}}\big(\theta-\alpha(k-\lambda)\big)i\pi\big(\delta(\lambda+K/2)-\delta\ (\lambda-K/2)\big)\,W_0(k-\lambda)d\lambda \\
&=-\frac{M}{\pi}\int_{\mathbb{R}}\big(\theta-\alpha(k-\lambda)\big)\big(\delta(\lambda+K/2)-\delta(\lambda-K/2)\big)W_0(k-\lambda)d\lambda \\
&=\frac{M}{\pi}\int_{\mathbb{R}}\big(\alpha(k-\lambda)-\theta\big)\big(\delta(\lambda+K/2)-\delta(\lambda-K/2)\big)W_0(k-\lambda)d\lambda \\
&=\frac{M}{\pi}\big(\alpha(k+K/2)-\theta\big)W_0(k+K/2)-\frac{M}{\pi}\big(\alpha(k-K/2)-\theta\big)W_0(k-K/2) \\
&=\frac{M}{\pi}(\alpha k-\theta)\big(W_0(k+K/2)-W_0(k-K/2)\big)+\frac{M}{\pi}\frac{\alpha K}{2}\big(W_0(k+K/2)+W_0(k-K/2)\big).
\end{aligned}
$$

Finally, integrating in (15) we get

$$
1+\frac{1}{2\pi}\int_{-M}^{M}\frac{(\alpha k-\theta)\big(W_0(k+K/2)-W_0(k-K/2)\big)+\alpha\frac{K}{2}\big(W_0(k+K/2)+W_0(k-K/2)\big)}{\Omega+\frac{1}{4}\gamma\,K^3-\frac{1}{2\pi}\alpha K\psi_0^2+\big(-2\beta k+3\gamma k^2\big)\,K}dk=0.
$$

Therefore, letting $M\to\infty$, we conclude (13). □

In particular, when $\alpha=\gamma=0$, $\beta=\beta'/2$ we recover the dispersion relation for NLSE

$$
1+\frac{\theta}{2\pi}\int_{\mathbb{R}}\frac{W_0\Big(k+\frac{K}{2}\Big)-W_0\Big(k-\frac{K}{2}\Big)}{\beta' kK-\Omega}dk=0.
$$

which was established in [9, eq. (10)].

3.1. **Dirac delta.** In this subsection we suggest that $W_0(k)=\psi_0^2\delta(k)$ in (13) and find a relation between Ω and K. Also, we look for instability interval for K such that $Im(\Omega)>0$, which guarantees the instability of the solution $W(x,k,t)$ in (12).

Theorem 3.2. *If we start with the solution for Hirota equation*

$$
\psi(t):=\frac{\psi_0}{\sqrt{2\pi}}e^{i\theta\frac{\psi_0^2}{2\pi}t},
$$

then we have that

$$
W[\psi,\psi](k)=\psi_0^2\,\delta(k).
\tag{17}
$$

Moreover, after inserting this into dispersion relation (13)*, we obtain the instability growth rate for* Ω

$$\Omega = i\beta K^2\Big(\frac{\theta\psi_0^2}{\pi\beta K^2} - 1\Big)^{1/2} - \gamma K^3 + \frac{\alpha\psi_0^2 K}{2\pi}. \tag{18}$$

Furthermore, modulation instability is experienced when K *is in the interval:*

$$K \in \Big[-\sqrt{\frac{\theta\psi_0^2}{\pi\beta}}, \sqrt{\frac{\theta\psi_0^2}{\pi\beta}}\Big].$$

Proof. So, let us prove (17) by using the definition of the Wigner function

$$\begin{aligned} W[\psi,\psi](k) &= \int_{\mathbb{R}} e^{-iky}\,\psi(x+y/2,t)\,\bar{\psi}(x-y/2,t)\,dy \\ &= \int_{\mathbb{R}} e^{-iky}\,\frac{\psi_0}{\sqrt{2\pi}}e^{i\theta\psi_0^2 t}\,\frac{\psi_0}{\sqrt{2\pi}}e^{-i\theta\psi_0^2 t}\,dy \\ &= \psi_0^2\delta(k). \end{aligned}$$

Now, introducing (17) in (13), we obtain the following relation

$$1+\frac{\psi_0^2}{2\pi}\int_{\mathbb{R}} \frac{(\alpha k-\theta)\big[\delta(k+K/2)-\delta(k-K/2)\big]+\alpha K/2\big[\delta(k+K/2)+\delta(k-K/2)\big]}{\Omega+\gamma K^3/4-\alpha\psi_0^2K/2\pi+(-2\beta k+3\gamma k^2)K}\,dk = 0.$$

Separating integral into two parts, each containing a delta function will lead us to

$$\begin{aligned} 1&+\frac{\psi_0^2}{2\pi}\int_{\mathbb{R}}\Big[\frac{\delta(k+K/2)[\alpha k-\theta+\alpha K/2]}{\Omega+\gamma K^3/4-\alpha\psi_0^2K/2\pi+(-2\beta k+3\gamma k^2)K}\Big]dk \\ &-\frac{\psi_0^2}{2\pi}\int_{\mathbb{R}}\Big[\frac{\delta(k-K/2)[\alpha k-\theta-\alpha K/2]}{\Omega+\gamma K^3/4-\alpha\psi_0^2K/2\pi+(-2\beta k+3\gamma k^2)K}\Big]dk = 0. \end{aligned}$$

Using (16), we obtain

$$1+\frac{\psi_0^2}{2\pi}\Big[-\frac{\theta}{\Omega+\gamma K^3+\beta K^2-\alpha\psi_0^2K/2\pi}+\frac{\theta}{\Omega+\gamma K^3-\beta K^2-\alpha\psi_0^2K/2\pi}\Big] = 0.$$

By simple algebraic operation we have that

$$\begin{aligned} &\frac{2\theta\beta K^2}{(\Omega+\gamma K^3-\alpha\psi_0^2K/2\pi+\beta K^2)(\Omega+\gamma K^3-\alpha\psi_0^2K/2\pi-\beta K^2)} = -\frac{2\pi}{\psi_0^2}, \\ &\frac{\theta\beta K^2}{(\Omega+\gamma K^3-\alpha\psi_0^2K/2\pi)^2-(\beta K^2)^2} = -\frac{\pi}{\psi_0^2}, \\ &(\theta\beta K^2\psi_0^2)/\pi = -(\Omega+\gamma K^3-\alpha\psi_0^2K/2\pi)^2+(\beta K^2)^2, \\ &\Omega+\gamma K^3-\alpha\psi_0^2K/2\pi = \pm\sqrt{(\beta K^2)^2-(\theta\beta K^2\psi_0^2)/\pi}, \\ &\Omega = \pm i\beta K^2\Big(\frac{\psi_0^2\theta}{\pi\beta K^2}-1\Big)^{1/2} - \gamma K^3 + \frac{\alpha\psi_0^2K}{2\pi}. \end{aligned} \tag{19}$$

So, we obtained a relation for Ω, which we call instability growth rate.

Further, to find instability interval for K, we impose $Im(\Omega) > 0$, which guarantees unstable waves. In order to satisfy this condition, we need to have the expression under the squared root to be positive, that is,

$$\frac{\psi_0^2\theta}{\pi\beta K^2} - 1 > 0,$$

which implies

$$K \in \Big[-\sqrt{\frac{\psi_0^2\theta}{\pi\beta}}, \sqrt{\frac{\psi_0^2\theta}{\pi\beta}}\Big].$$

□

In particular, when $\alpha=\gamma=0$ and $\beta=\beta'/2$, we obtain a similar relation as in [9, eq. 9]

Let us find the instability growth rate (18) working directly with the Hirota equation. In order to do that, insert the following ansatz in (1)

$$\psi(x,t) := \frac{\psi_0}{\sqrt{2\pi}} e^{i\theta \frac{\psi_0^2}{2\pi} t}\big(1+\varepsilon\, e^{i(Kx-\Omega t)}\big).$$

Linearizing in ε, we claim that a highly nonlinear equation in Ω will be obtained

$$-\Omega - \frac{\psi_0^2}{2\pi} + \frac{\alpha\psi_0^2 K}{2\pi} - \beta K^2 + \gamma K^3 + \frac{\theta\psi_0^2}{\pi} + \frac{\theta\psi_0^2}{2\pi} e^{-2iKx} e^{2iRe(\Omega)t} = 0. \tag{20}$$

Indeed, let's verify (20) by inserting the ansatz directly to Hirota equation.

Note that we linearize in ε, as before. So, all the higher order terms in ε we do not consider. Firstly, simplify the term $|\psi|^2$ as

$$\begin{aligned}
|\psi|^2 &= \psi * \overline{\psi} \\
&= \left(\frac{\psi_0}{\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} + \frac{\varepsilon\psi_0}{\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} e^{i(Kx-\Omega t)}\right)\left(\frac{\psi_0}{\sqrt{2\pi}} e^{-i\frac{\theta\psi_0^2}{2\pi}t} + \frac{\varepsilon\psi_0}{\sqrt{2\pi}} e^{-i\frac{\theta\psi_0^2}{2\pi}t} e^{-i(Kx-\overline{\Omega}t)}\right) \\
&= \frac{\psi_0^2}{2\pi} + \frac{\varepsilon\psi_0^2}{2\pi} e^{-i(Kx-\overline{\Omega}t)} + \frac{\varepsilon\psi_0^2}{2\pi} e^{i(Kx-\Omega t)}.
\end{aligned}$$

Now, let us calculate all the needed partial derivatives of $\psi(x,t)$:

$$\begin{aligned}
\partial_t\psi &= \frac{i\theta\psi_0^3}{2\pi\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} + i\frac{\varepsilon\psi_0}{\sqrt{2\pi}}\Big(\frac{\theta\psi_0^2}{pi} - \Omega\Big) e^{i\frac{\theta\psi_0^2}{2\pi}t} e^{i(Kx-\Omega t)}, \\
\partial_x\psi &= iK\frac{\varepsilon\psi_0}{\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} e^{i(Kx-\Omega t)}, \\
\partial_x^2\psi &= -K^2\frac{\varepsilon\psi_0}{\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} e^{i(Kx-\Omega t)}, \\
\partial_x^3\psi &= -iK^3\frac{\varepsilon\psi_0}{\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} e^{i(Kx-\Omega t)}.
\end{aligned}$$

Moreover, the remaining terms of the Hirota equation are given by

$$\begin{aligned}
|\psi|^2\partial_x\psi &= \left(\frac{\psi_0^2}{2\pi} + \frac{\varepsilon\psi_0^2}{2\pi} e^{-i(Kx-\overline{\Omega}t)} + \frac{\varepsilon\psi_0^2}{2\pi} e^{i(Kx-\Omega t)}\right)\left(iK\frac{\varepsilon\psi_0}{\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} e^{i(Kx-\Omega t)}\right) \\
&= iK\frac{\varepsilon\psi_0^3}{2\pi\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} e^{i(Kx-\Omega t)},
\end{aligned}$$

$$\begin{aligned}
|\psi|^2\psi &= \left(\frac{\psi_0^2}{2\pi} + \frac{\varepsilon\psi_0^2}{2\pi} e^{-i(Kx-\overline{\Omega}t)} + \frac{\varepsilon\psi_0^2}{2\pi} e^{i(Kx-\Omega t)}\right)\left(\frac{\psi_0}{\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} + \frac{\varepsilon\psi_0^3}{2\pi\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} e^{i(Kx-\Omega t)}\right) \\
&= \frac{\psi_0^3}{2\pi\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} + \frac{\varepsilon\psi_0^3}{\pi\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} e^{i(Kx-\Omega t)} + \frac{\varepsilon\psi_0^3}{2\pi\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} e^{-i(Kx-\overline{\Omega}t)}.
\end{aligned}$$

Finally, collecting all the terms with corresponding coefficients, we obtain a relation:

$$\begin{aligned}
&-\frac{\theta\psi_0^3}{2\pi\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} - \frac{\varepsilon\psi_0}{\sqrt{2\pi}}\Big(\frac{\theta\psi_0^2}{2\pi} + \Omega\Big) e^{i\frac{\theta\psi_0^2}{2\pi}t} e^{i(Kx-\Omega t)} - \alpha K\frac{\varepsilon\psi_0^3}{2\pi\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} e^{i(Kx-\Omega t)} \\
&\qquad - \beta K^2\frac{\varepsilon\psi_0}{\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} e^{i(Kx-\Omega t)} + \gamma K^3\frac{\varepsilon\psi_0}{\sqrt{2\pi}} e^{i\frac{\theta\psi_0^2}{2\pi}t} e^{i(Kx-\Omega t)}
\end{aligned}$$

$$+\frac{\theta\psi_0^3}{2\pi\sqrt{2\pi}}e^{i\frac{\theta\psi_0^2}{2\pi}t}+\frac{\varepsilon\psi_0^3}{\pi\sqrt{2\pi}}e^{i\frac{\theta\psi_0^2}{2\pi}t}e^{i(Kx-\Omega t)}+\frac{\varepsilon\psi_0^3}{2\pi\sqrt{2\pi}}e^{i\frac{\theta\psi_0^2}{2\pi}t}e^{-i(Kx-\overline{\Omega}t)}=0.$$

Cancelling the term $\varepsilon e^{i\frac{\theta\psi_0^2}{2\pi}t}e^{i(Kx-\Omega t)}$, we conclude:

$$-\Omega-\frac{\psi_0^2}{2\pi}+\frac{\alpha\psi_0^2K}{2\pi}-\beta K^2+\gamma K^3+\frac{\theta\psi_0^2}{\pi}+\frac{\theta\psi_0^2}{2\pi}e^{-2iKx}e^{2iRe(\Omega)t}=0,$$

as we claimed.

Remark 3.3. *The equation* (20) *that we obtain here is highly nonlinear in* Ω*, so we cannot find* Ω *explicitly. However, if we use the Wigner function, as we did before, we only deal with the quadratic nonlinear equation* (19)*, where it is trivial to find* Ω*. This justifies the use of the Wigner function over an option of working directly with Hirota equation.*

3.2. **Landau damping.** As a first step to deal with the Landau damping function, for simplicity let us work with Hirota equation with $\gamma=0$. In this subsection we aim to find instability growth rate for Ω and instability intervals for K. In order to do so, we follow the involved process:

- Firstly, we show that a solution to Hirota equation under the Wigner function is a Landau damping function in Lemma 3.4.
- Then, we insert the Landau damping function into the dispersion relation (21), where $\gamma=0$ (see (21)).
- Further, in Lemma 3.5 we showed that we need to use Cauchy Residue's Theorem to solve the integral in (21).
- After that, Proposition 3.6 gives the instability growth rate for Ω in the case $K<0$.
- To find instability interval for $K<0$ we used Proposition 3.7.
- Finally, Lemma 3.8 shows the symmetry in the instability interval for $K>0$, thus we obtained the whole interval for K. In the end we state all the results in a Theorem 3.9

In the lemma below we again start with a solution of Hirota equation and transform via Wigner definition. Note that in contrast with the Dirac delta case the solution also depends on the x-variable.

Lemma 3.4. *Let* ψ *be the incoherent wave and define it as:*

$$\psi(x,t):=\frac{\psi_0}{\sqrt{2\pi}}e^{i\theta\frac{\psi_0^2}{2\pi}t+i\phi(x)},$$

where ϕ *is chosen in such a way that*

$$\psi(x+y/2,t)\overline{\psi}(x-y/2,t)={\psi_0}^2e^{-p_0|y|},$$

for some $p_0>0$ *and* $\psi_0\in R$*, as it was suggested in* [9, eq. (12)]*. Then the Wigner function of this wave is the Landau damping function*

$$W_0(k):=\frac{\psi_0^2}{\pi}\frac{p_0}{k^2+p_0^2}.$$

Proof. Observe that

$$\begin{aligned}W[\psi,\psi](k)&=\int_{\mathbb{R}}e^{-iky}\psi(x+y/2,t)\overline{\psi}(x-y/2,t)dy\\&=\int_{\mathbb{R}}e^{-iky}\frac{|\psi_0^2|}{2\pi}e^{-p_0|y|}dy\\&=\frac{\psi_0^2}{2\pi}\int_{\mathbb{R}}e^{-iky-p_0|y|}dy\\&=\frac{\psi_0^2}{2\pi}\Big(\int_{-\infty}^0e^{-iky+p_0y}\,dy+\int_0^\infty e^{-iky-p_0y}\,dy\Big)\end{aligned}$$

$$= \frac{\psi_0^2}{2\pi}\Big(\int_{-\infty}^{0} e^{y(p_0-ik)}\,dy + \int_0^{\infty} e^{-y(p_0+ik)}\,dy\Big)$$
$$= \frac{\psi_0^2}{2\pi}\Big(\frac{1}{p_0-ik} + \frac{1}{p_0+ik}\Big)$$
$$= \frac{\psi_0^2}{\pi}\frac{p_0}{k^2+p_0^2}.$$

□

The dispersion relation with $\gamma = 0$ is given by

$$1 + \frac{1}{2\pi}\int_{\mathbb{R}} \frac{(\alpha k - \theta + \alpha K/2)W_0(k+K/2) + (\alpha K/2 - \alpha k + \theta)W_0(k-K/2)}{\Omega - \alpha K\psi_0^2/2\pi - 2\beta K k}dk = 0. \tag{21}$$

We insert a Landau damping function and obtain

$$1 + \frac{\psi_0^2}{2\pi^2}\int_{\mathbb{R}} \frac{\big(\alpha k - \theta + \alpha K/2\big)\frac{p_0}{(k+K/2)^2+p_0^2} + \big(\alpha K/2 - \alpha k + \theta\big)\frac{p_0}{(k-K/2)^2+p_0^2}}{\Omega - \alpha K\psi_0^2/2\pi - 2\beta K k}dk = 0.$$

After simple algebraic manipulations we get

$$1+\frac{\psi_0^2 p_0}{4\pi^2\beta}\int_{\mathbb{R}} \frac{\alpha k^2 - 2\theta k - \alpha(p_0^2 + K^2/4)}{\big(k-C\big)\big(k-(ip_0+\frac{K}{2})\big)\big(k-(ip_0-\frac{K}{2})\big)\big(k-(-ip_0+\frac{K}{2})\big)\big(k-(-ip_0-\frac{K}{2})\big)}dk = 0, \tag{22}$$

where $C := \frac{\Omega}{2\beta K} - \frac{\alpha\psi_0^2}{4\beta\pi}$.

In the Lemma below, we state the Cauchy Residue theorem can be used to solve the above intergal.

Lemma 3.5. *Let $C \in \mathbb{C}$ such that $Im(C) < 0$ and define f as*

$$f(k) := \frac{\alpha k^2 - 2\theta k - \alpha(p_0^2 + K^2/4)}{\big(k-C\big)\big(k-(ip_0+\frac{K}{2})\big)\big(k-(ip_0-\frac{K}{2})\big)\big(k-(-ip_0+\frac{K}{2})\big)\big(k-(-ip_0-\frac{K}{2})\big)}dk. \tag{23}$$

Then we get

$$\int_{\mathbb{R}} f(k)dk = 2\pi i\big[Res(f, ip_0 + \frac{K}{2}) + Res(f, ip_0 - \frac{K}{2})\big].$$

Proof. In the Figure 1 we have the contour in the counterclockwise direction as $\Gamma_R = \Gamma_R^1 \cup \Gamma_R^2$, where Γ_R^1 is half of the circle centered at $(0,0)$ with radius R and Γ_R^2 is a line from $(-R,0)$ to $(R,0)$. Observe that

$$\int_{\Gamma_R} f(z)dz = \int_{\Gamma_R^1} f(z)dz + \int_{\Gamma_R^2} f(z)dz. \tag{24}$$

Also, by the Cauchy Residue theorem in [7, Theorem on p. 235] we have that

$$\int_{\Gamma_R} f(z)dz = 2\pi i\big[Res(f, ip_0 + \frac{K}{2}) + Res(f, ip_0 - \frac{K}{2})\big].$$

In order to find desired integral from (22), we consider each integral from (24) separately. Let us consider the absolute value of the integral along the curve Γ_R^1 where $|z| = R$ and $Im(z) > 0$.

$$\Bigg|\int_{\Gamma_R^1} \frac{\alpha z^2 - 2\theta z - \alpha(p_0^2 + K^2/4)}{\big(z-C\big)\big(z-(ip_0+\frac{K}{2})\big)\big(z-(ip_0-\frac{K}{2})\big)\big(z-(-ip_0+\frac{K}{2})\big)\big(z-(-ip_0-\frac{K}{2})\big)}dz\Bigg|$$
$$\leq \int_{\Gamma_R^1} \frac{\alpha|z|^2 + 2\theta|z| + \alpha(p_0^2 + K^2/4)}{|z-C|\big|z-(ip_0+\frac{K}{2})\big|\big|z-(ip_0-\frac{K}{2})\big|\big|z-(-ip_0+\frac{K}{2})\big|\big|z-(-ip_0-\frac{K}{2})\big|}dz$$
$$\leq \int_{\Gamma_R^1} \frac{\alpha R^2 + 2\theta R + \alpha(p_0^2 + K^2/4)}{\big(R-C\big)\big(R-(ip_0+\frac{K}{2})\big)\big(R-(ip_0-\frac{K}{2})\big)\big(R-(-ip_0+\frac{K}{2})\big)\big(R-(-ip_0-\frac{K}{2})\big)}dz$$

$$= \frac{\alpha R^2 + 2\theta R + \alpha(p_0^2 + K^2/4)}{(R-C)\big(R-(ip_0+\frac{K}{2})\big)\big(R-(ip_0-\frac{K}{2})\big)\big(R-(-ip_0+\frac{K}{2})\big)\big(R-(-ip_0-\frac{K}{2})\big)} \int_{\Gamma_R^1} dz$$

$$= \frac{\pi R\big(\alpha R^2 + 2\theta R + \alpha(p_0^2 + K^2/4)\big)}{(R-|C|)\big(R-|ip_0+\frac{K}{2}|\big)\big(R-|ip_0-\frac{K}{2}|\big)\big(R-|-ip_0+\frac{K}{2}|\big)\big(R-|-ip_0-\frac{K}{2}|\big)}$$

Taking the limit of the above expression as $R \to \infty$, we find that the integral in Γ_R^1 is 0.

Now, we have that

$$\lim_{R\to\infty} \int_{\Gamma_R^2} f(z)dz = \int_{\mathbb{R}} f(k)dk.$$

Therefore, we conclude

$$\int_{\mathbb{R}} f(k)dk = 2\pi i\big[Res(f, ip_0 + \frac{K}{2}) + Res(f, ip_0 - \frac{K}{2})\big]$$

as required.

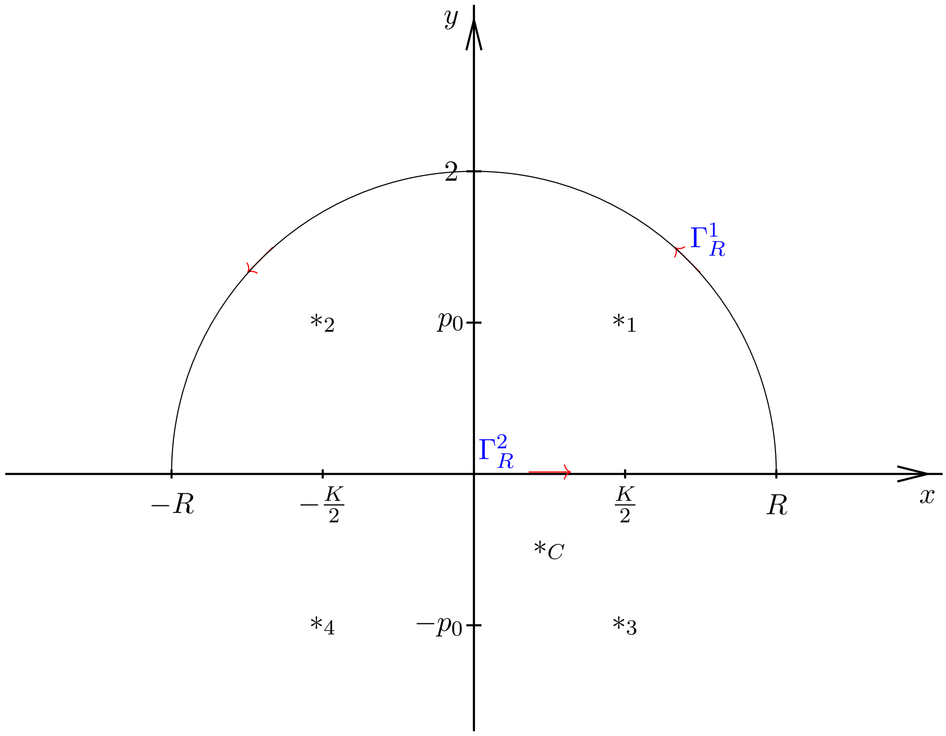

FIGURE 1. Poles of the function $f(k)$.

□

In the proposition below we find Ω for $K < 0$. So, in general, we solve the contour integral using the Cauchy Residue theorem.

Proposition 3.6. *If we insert $W_0(k) = \frac{\psi_0^2}{\pi}\frac{p_0}{k^2+p_0^2}$ into (22), then for $K < 0$ we obtain the instability growth rate:*

$$\Omega = \frac{\alpha\psi_0^2 K}{2\pi} + 2ip_0\beta K \pm i\beta K^2\sqrt{\frac{\psi_0^2}{\pi\beta K^2}(\theta - ip_0\alpha) - 1}. \tag{25}$$

Proof. As we showed in Lemma 3.5,

$$\int_{\mathbb{R}} f(k)dk = 2\pi i\big[Res(f, ip_0 + \frac{K}{2}) + Res(f, ip_0 - \frac{K}{2})\big].$$

Thus, we need to find the residues separately. So,

$$Res(f, ip_0 + \frac{K}{2}) = \frac{\alpha\big(ip_0 + \frac{K}{2}\big)^2 - 2\theta\big(ip_0 + \frac{K}{2}\big) - \alpha\big(p_0^2 + \frac{K^2}{4}\big)}{\big(ip_0 + \frac{K}{2} - C\big)K\big(2ip_0\big)\big(2ip_0 + K\big)}$$

$$= \frac{\alpha\big(2ip_0+K\big)^2 - 8\theta\big(ip_0+\frac{K}{2}\big) - 4\alpha\big(p_0^2+\frac{K^2}{4}\big)}{\big(2ip_0+K-2C\big)2K\big(2ip_0\big)\big(2ip_0+K\big)}$$
$$= \frac{\alpha\big(-4p_0^2+K^2+4ip_0K\big) - 8ip_0\theta - 4\theta K - 4\alpha p_0^2 - \alpha K^2}{\big(2ip_0+K-2C\big)2K\big(2ip_0\big)\big(2ip_0+K\big)}$$
$$= \frac{-8\alpha p_0^2 + 4ip_0\alpha K - 8ip_0\theta - 4\theta K}{\big(2ip_0+K-2C\big)2K\big(2ip_0\big)\big(2ip_0+K\big)}$$
$$= \frac{4ip_0\alpha(2ip_0+K) - 4\theta(2ip_0+K)}{(2ip_0+K-2C)4ip_0K(2ip_0+K)}$$
$$= \frac{ip_0\alpha-\theta}{(2ip_0+K-2C)ip_0K},$$

and

$$Res\big(f, ip_0-\frac{K}{2}\big) = \frac{\alpha\big(ip_0-\frac{K}{2}\big)^2 - 2\theta\big(ip_0-\frac{K}{2}\big) - \alpha\big(p_0^2+\frac{K^2}{4}\big)}{\big(ip_0-\frac{K}{2}-C\big)\big(-K\big)\big(2ip_0\big)\big(2ip_0-K\big)}$$
$$= -\frac{\alpha\big(2ip_0-K\big)^2 - 8\theta\big(ip_0-\frac{K}{2}\big) - 4\alpha\big(p_0^2+\frac{K^2}{4}\big)}{\big(2ip_0-K-2C\big)2K\big(2ip_0\big)\big(2ip_0-K\big)}$$
$$= -\frac{\alpha\big(-4p_0^2+K^2-4ip_0K\big) - 8ip_0\theta + 4\theta K - 4\alpha p_0^2 - \alpha K^2}{\big(2ip_0-K-2C\big)2K\big(2ip_0\big)\big(2ip_0-K\big)}$$
$$= -\frac{-8\alpha p_0^2 - 4ip_0\alpha K - 8ip_0\theta + 4\theta K}{\big(2ip_0-K-2C\big)2K\big(2ip_0\big)\big(2ip_0-K\big)}$$
$$= -\frac{4ip_0\alpha(2ip_0-K) - 4\theta(2ip_0-K)}{(2ip_0-K-2C)4ip_0K(2ip_0-K)}$$
$$= -\frac{ip_0\alpha-\theta}{(2ip_0-K-2C)ip_0K}.$$

Then (22) yields that

$$1 + \frac{\psi_0^2 p_0}{4\pi^2\beta}2\pi i\left(\frac{ip_0\alpha-\theta}{(2ip_0+K-2C)ip_0K} - \frac{ip_0\alpha-\theta}{(2ip_0-K-2C)ip_0K}\right)$$
$$= 1 + \frac{\psi_0^2 ip_0}{2\pi\beta}\left(\frac{ip_0\alpha-\theta}{(2ip_0+K-2C)ip_0K} - \frac{ip_0\alpha-\theta}{(2ip_0-K-2C)ip_0K}\right)$$
$$= 1 + \frac{\psi_0^2}{2\pi\beta K}\left(\frac{ip_0\alpha-\theta}{(2ip_0+K-2C)} - \frac{ip_0\alpha-\theta}{(2ip_0-K-2C)}\right)$$
$$= 1 + \frac{\psi_0^2}{2\pi\beta K}\left(\frac{2ip_0\alpha K + 2\theta K}{(2ip_0-2C+K)(2ip_0-2C-K)}\right)$$
$$= 1 - \frac{\psi_0^2}{\pi\beta}\left(\frac{ip_0\alpha-\theta}{4C^2-8ip_0C-K^2-4p_0^2}\right) = 0.$$

Thus, we arrive at a simple quadratic equation in terms of C:

$$C^2 - 2ip_0C - \frac{K^2}{4} - p_0^2 - \frac{\alpha\psi_0^2}{4\pi\beta}ip_0 + \frac{\theta\psi_0^2}{4\pi\beta} = 0.$$

Since $C = \frac{\Omega}{2\beta K} - \frac{\alpha\psi_0^2}{4\pi\beta}$, we further introduce the notations $a := \frac{\Omega}{2\beta K}$, $b := \frac{\alpha\psi_0^2}{4\pi\beta}$ and $c := \frac{\theta\psi_0^2}{4\pi\beta}$, so the above equation becomes

$$a^2 + b^2 - 2ab - 2ip_0(a-b) - K^2/4 - p_0^2 = ip_0b - c,$$
$$a^2 - a(2b+2ip_0) + b^2 + ip_0b - K^2/4 - p_0^2 + c = 0,$$

which we will solve in the variable a.

By the method of discriminant, we find the following

$$\begin{aligned} D &= (2b+2ip_0)^2 - 4b^2 - 4ip_0b + K^2 + p_0^2 - 4C \\ &= 4ip_0b + K^2 - 4C \\ &= ip_0\frac{\alpha\psi_0^2}{\pi\beta} - \frac{\theta\psi_0^2}{\pi\beta} + K^2 \\ &= K^2 - \frac{\psi_0^2}{\pi\beta}(\theta - ip_0\alpha) \\ &= (iK)^2\Big(\frac{\psi_0^2}{\pi\beta K^2}(\theta - ip_0\alpha) - 1\Big). \end{aligned}$$

Thus,

$$a = \frac{\alpha\psi_0^2}{4\pi\beta} + ip_0 \pm i\frac{K}{2}\sqrt{\frac{\psi_0^2}{\pi\beta K^2}(\theta - ip_0\alpha) - 1}.$$

Now substitute back $a = \frac{\Omega}{2\beta K}$ and obtain the following relation

$$\frac{\Omega}{\beta K} = \frac{\alpha\psi_0^2}{2\pi\beta} + 2ip_0 \pm iK\sqrt{\frac{\psi_0^2}{\pi\beta K^2}(\theta - ip_0\alpha) - 1},$$

which implies (25). □

In Proposition 3.6, we obtained an explicit expression for Ω; however, we cannot clearly see what is its imaginary part. Proposition 3.7 shows the analysis, which explains how to find explicitly the imaginary part of Ω for $K < 0$.

Proposition 3.7. *If in* (25) $Im(\Omega) > 0$, *we obtain the instability interval for* $K < 0$

$$K \in \Bigg[-\sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \Big(\frac{\psi_0^2\alpha}{4\pi\beta}\Big)^2 - 4p_0^2}\ , 0\Bigg].$$

Proof. In (25) we have complex expression under the square root

$$\sqrt{\frac{\psi_0^2}{\pi\beta K^2}(\theta - ip_0\alpha) - 1}, \tag{26}$$

which we need to rewrite in the form $A + iB$.

In order to bring the (26) to this shape, we need to use polar representation of complex numbers. Thus, define $a := \frac{\psi_0^2\theta}{\pi\beta K^2} - 1$ and $b := -\frac{\psi_0^2}{\pi\beta K^2}p_0\alpha$. Then

$$\sqrt{a+bi} = \sqrt{r}\cos\frac{\mu}{2} + i\sqrt{r}\sin\frac{\mu}{2}\ , \quad \text{where } \mu = \arctan\frac{b}{a} \text{ and } r = \sqrt{a^2+b^2}.$$

If we substitute the obtained trigonometric combination into (25) we get

$$\Omega = \frac{\alpha\psi_0^2 K}{2\pi} + 2ip_0\beta K \pm i\beta K^2\big(\sqrt{r}\cos\frac{\mu}{2} + i\sqrt{r}\sin\frac{\mu}{2}\big),$$

which can be rewritten as

$$\Omega = \frac{\alpha\psi_0^2 K}{2\pi} \pm \beta K^2\sqrt{r}\sin\frac{\mu}{2} + i\Big(2p_0\beta K \pm \beta K^2\sqrt{r}\cos\frac{\mu}{2}\Big).$$

Since

$$Im\Omega = 2p_0\beta K \pm \beta K^2\sqrt{r}\cos\frac{\mu}{2},$$

then we are interested in the expression $\sqrt{r}\cos\frac{\mu}{2}$.

Let $\mu \in [0, 2\pi]$ and

$$\frac{b}{a} = \frac{-\frac{\psi_0^2}{\pi\beta K^2}p_0\alpha}{\frac{\psi_0^2}{\pi\beta K^2}\theta - 1}.$$

We have two cases $a > 0$ and $a < 0$, in which $b < 0$ always.

Case 1: $a < 0$, then $K^2 > \frac{\psi_0^2\theta}{\pi\beta}$, and $K \in (-\infty, -\sqrt{\frac{\psi_0^2\theta}{\pi\beta}}]$.

We have that $\mu \in [\pi, 3\pi/2]$, and $\mu/2 \in [\pi/2, 3\pi/4]$. Therefore, recalling that

$$\begin{aligned} r &= \sqrt{a^2+b^2}, \\ \cos\frac{\omega}{2} &= \sqrt{\frac{1+\cos\omega}{2}}, \\ \cos(\arctan b/a) &= \pm\frac{1}{\sqrt{1+(b/a)^2}}, \end{aligned}$$

We continue with the above mentioned cos term as

$$\begin{aligned} \sqrt{r}\cos\frac{\mu}{2} &= -\sqrt{r}\sqrt{\frac{1+\cos\mu}{2}} = -\sqrt{\frac{r}{2}}\sqrt{1-\frac{1}{\sqrt{1+(b/a)^2}}} = -\sqrt{\frac{r}{2}}\sqrt{1-\frac{|a|}{r}} \\ &= -\sqrt{\frac{r}{2}}\sqrt{\frac{r-|a|}{r}} = -\frac{1}{\sqrt{2}}\sqrt{r-|a|} = -\frac{1}{\sqrt{2}}\sqrt{r+a}, \quad \text{since } a < 0. \end{aligned}$$

Thus,

$$Im(\Omega) = \pm\frac{\beta K^2}{\sqrt{2}}\sqrt{r+a} + 2p_0\beta K = \beta K(\pm\frac{K}{\sqrt{2}}\sqrt{r+a} + 2p_0) > 0.$$

Since, $K < 0$, then we must have $\frac{K}{\sqrt{2}}\sqrt{r+a} + 2p_0 < 0$

$$\begin{aligned} &\frac{K}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}+a} < -2p_0, \\ &\frac{K^2}{2}(\sqrt{a^2+b^2}+a) > 4p_0^2. \end{aligned}$$

Substituting back expressions for a and b we obtain

$$\begin{aligned} 8p_0^2 &< K^2\left(\sqrt{(\frac{\psi_0^2}{\pi\beta K^2})^2\theta^2 + 1 - 2(\frac{\psi_0^2}{\pi\beta K^2})\theta + (\frac{\psi_0^2}{\pi\beta K^2})^2 p_0^2\alpha^2} + \frac{\psi_0^2\theta}{\pi\beta K^2} - 1\right) \\ &= \sqrt{\frac{\psi_0^4\theta^2}{\pi^2\beta^2} + K^4 - \frac{2\psi_0^2\theta}{\pi\beta}K^2 + \frac{\psi_0^4}{\pi^2\beta^2}p_0^2\alpha^2} + \frac{\psi_0^2\theta}{\pi\beta} - K^2. \end{aligned}$$

Then, we have the following inequality:

$$8p_0^2 + K^2 - \frac{\psi_0^2\theta}{\pi\beta} < \sqrt{\frac{\psi_0^4\theta^2}{\pi^2\beta^2} + K^4 - \frac{2\psi_0^2\theta}{\pi\beta}K^2 + \frac{\psi_0^4}{\pi^2\beta^2}p_0^2\alpha^2},$$

$$\left(8p_0^2 + K^2 - \frac{\psi_0^2\theta}{\pi\beta}\right)^2 < \frac{\psi_0^4\theta^2}{\pi^2\beta^2} + K^4 - \frac{2\psi_0^2\theta}{\pi\beta}K^2 + \frac{\psi_0^4}{\pi^2\beta^2}p_0^2\alpha^2,$$

$$K^4 + \left(\frac{\psi_0^2\theta}{\pi\beta} - 8p_0^2\right)^2 - 2K^2\left(\frac{\psi_0^2\theta}{\pi\beta} - 8p_0^2\right) < \frac{\psi_0^4\theta^2}{\pi^2\beta^2} + K^4 - \frac{2\psi_0^2\theta}{\pi\beta}K^2 + \frac{\psi_0^4}{\pi^2\beta^2}p_0^2\alpha^2,$$

$$64p_0^4 - 16\frac{\psi_0^2\theta}{\pi\beta}p_0^2 + 16K^2p_0^2 < \frac{\psi_0^4}{\pi^2\beta^2}p_0^2\alpha^2,$$

$$64p_0^2 - 16\frac{\psi_0^2\theta}{\pi\beta} + 16K^2 < \frac{\psi_0^4\alpha^2}{\pi^2\beta^2},$$

$$16K^2 < 16\frac{\psi_0^2\theta}{\pi\beta} + \frac{\psi_0^4\alpha^2}{\pi^2\beta^2} - 64p_0^2,$$

$$K^2 < \frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2 - 4p_0^2,$$

$$K > -\sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2 - 4p_0^2}.$$

Initially, we had that $K \in (-\infty, -\sqrt{\frac{\psi_0^2\theta}{\pi\beta}}]$.

Combining this with the above result, we have an interval for $K < 0$

$$K \in \left[-\sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2 - 4p_0^2}\ , -\sqrt{\frac{\psi_0^2\theta}{\pi\beta}}\right].$$

Case 2: $a > 0$, then $K^2 < \frac{\psi_0^2\theta}{\pi\beta}$, and then $K \in [-\sqrt{\frac{\psi_0^2\theta}{\pi\beta}}, 0]$.

We have that $\mu \in [3\pi/2, 2\pi]$, and $\mu/2 \in [3\pi/4, \pi]$

$$\sqrt{r}\cos\frac{\mu}{2} = -\sqrt{r}\sqrt{\frac{1+\cos\mu}{2}} = -\sqrt{\frac{r}{2}}\sqrt{1 + \frac{1}{\sqrt{1+(b/a)^2}}} = -\sqrt{\frac{r}{2}}\sqrt{1 + \frac{|a|}{r}}$$
$$= -\sqrt{\frac{r}{2}}\sqrt{\frac{r+|a|}{r}} = -\frac{1}{\sqrt{2}}\sqrt{r+|a|} = -\frac{1}{\sqrt{2}}\sqrt{r+a}, \quad \text{since } a > 0.$$

Thus,

$$Im(\Omega) = \pm\frac{\beta K^2}{\sqrt{2}}\sqrt{r+a} + 2p_0\beta K = \beta K(\pm\frac{K}{\sqrt{2}}\sqrt{r+a} + 2p_0) > 0,$$

which is exactly the same as in Case 1, therefore, the computations are the same, which leads us to the expression

$$K > -\sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2 - 4p_0^2}.$$

However, in this case we had that $K \in [-\sqrt{\frac{\psi_0^2\theta}{\pi\beta}}, 0]$.

Combining this with the above result, we obtain another interval for $K < 0$

$$K \in \left[-\sqrt{\frac{\psi_0^2\theta}{\pi\beta}}\ , 0\right].$$

Summing up both cases $a<0$ and $a>0$, we finally obtain the full interval for $K<0$

$$K \in \left[-\sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2 - 4p_0^2}\ , 0 \right].$$

□

This lemma below is quite useful, which allows us to skip previous computation for $K>0$. It shows that instability interval is symmetric around 0.

Lemma 3.8. *Let $K>0$. If $Im(\Omega)>0$, then the instability interval for K is written as*

$$K \in \left[0,\ \sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2 - 4p_0^2}\right].$$

Proof. Remember that

$$C := \frac{\Omega}{2\beta K} - \frac{\alpha\psi_0^2}{4\beta\pi}.$$

We start with $Im(C)>0$, which is the second case of the contour integral that we solved above, and use that $Im(\overline{C})<0$ is equivalent to $Im(C)>0$.

$$\overline{C} = \frac{\Omega}{2\beta K} - \frac{\alpha\psi_0^2}{4\pi\beta} = \frac{Re(\Omega)}{2\beta K} - i\frac{Im(\Omega)}{2\beta K} - \frac{\alpha\psi_0^2}{4\beta\pi}.$$

Thus $Im(\overline{C}) = -\frac{Im(\Omega)}{2\beta K} = \frac{Im(\Omega)}{2\beta(-K)}$.

Call $L=-K$, and obtain that $Im(\overline{C}) = \frac{Im(\Omega)}{2\beta L}$.

Now, we insert L into the interval that we had for negative K

$$-\sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2 - 4p_0^2} \le L \le 0,$$

$$-\sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2 - 4p_0^2} \le -K \le 0.$$

$$\text{Then } 0 \le K \le \sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2 - 4p_0^2}.$$

□

Lemmas 3.4, 3.5 and 3.8 together with Propositions 3.6 and 3.7 imply the theorem below.

Theorem 3.9. *If we start with the solution of the Hirota equation*

$$\psi(x,t) := \frac{\psi_0}{\sqrt{2\pi}} e^{i\theta\frac{\psi_0^2}{2\pi}} + i\phi(x),$$

then, with the property that

$$\psi(x+y/2,t)\overline{\psi}(x-y/2,t) = \psi_0^2 e^{-p_0|y|},$$

we have

$$W[\psi,\psi](k) = \frac{\psi_0^2}{\pi}\frac{p_0}{k^2+p_0^2}.$$

Moreover, after inserting this into the dispersion relation (21) *we obtain the instability interval for* K

$$K \in \left[-\sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2 - 4p_0^2}\ ,\ \sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2 - 4p_0^2}\right].$$

4. CONCLUSIONS

Our main goal was to find intervals of instability for K in two different spectra:

- Dirac delta function $W_0(k) := \psi_0^2\delta(k)$, where ψ_0 is constant.
- Landau damping function $W_0(k) := \frac{\psi_0^2}{\pi}\frac{p_0}{k^2+p_0^2}$, where ψ_0 is constant and $p_0 > 0$.

All the analysis was performed for Hirota equation with exception for Landau damping function, where we let $\gamma = 0$ for simplicity.

Before we summarize the results, note that Landau damping has a more realistic shape. Both Landau damping and Dirac delta functions are solutions to Wigner-Hirota equation, and they both represent the model for a rogue wave phenomena. However, Dirac delta function is not really a wave, since it is infinity at one point (in our case at the origin) and zero everywhere else. On the other hand, Landau damping depending on p_0 shows different phenomena. For example, when p_0 is very small, then the shape of the function is similar to a rogue wave, and if p_0 tends to 0, then Landau damping function is almost the same as Dirac delta function. Though, if p_0 tends to $+\infty$, then Landau damping is just a straight line on x-axis. Thus, Landau damping function is a better model of a wave and approximation is expected to be more precise, moreover, if one seeks for a rogue wave, taking p_0 small would be enough.

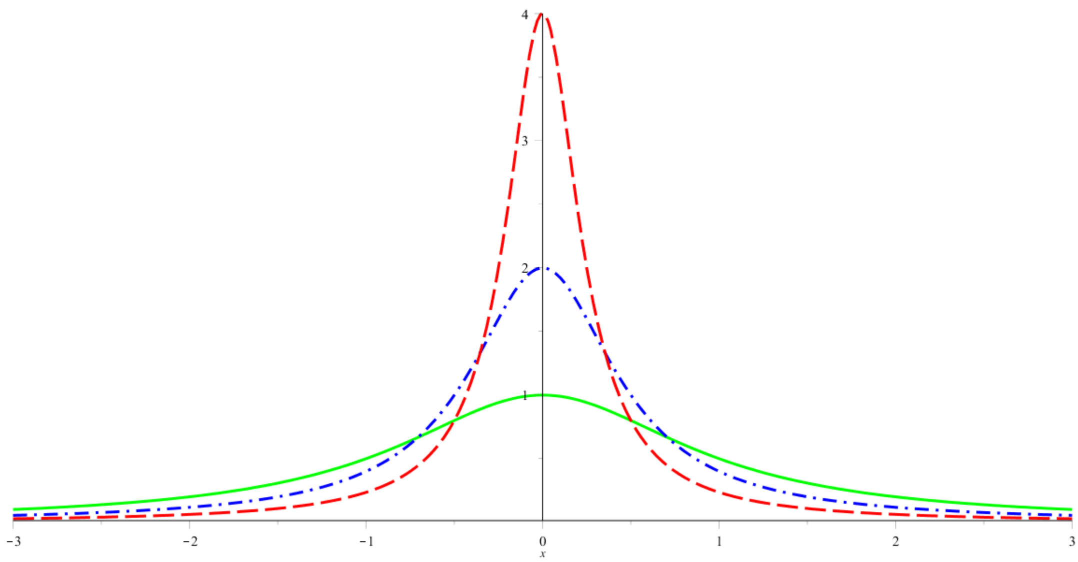

FIGURE 2. Landau damping at $p_0 = 1$ (green), $p_0 = 0.5$ (blue), $p_0 = 0.25$ (red) and $\psi_0 = \sqrt{\pi}$.

The obtained intervals for K as mentioned in Theorems 3.2 and 3.9 are:

$$I_{delta} = \left[-\sqrt{\frac{\psi_0^2\theta}{\pi\beta}}, \sqrt{\frac{\psi_0^2\theta}{\pi\beta}}\right],$$

$$I_{Landau} = \left[-\sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2 - 4p_0^2}\,,\ \sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2 - 4p_0^2}\right].$$

It is surprising that the Dirac delta case annihilates the action of α, which corresponds to a highly non-linear term in the Hirota equation. However, apriori expected that this α term should play an important tole in the instability analysis. For these reasons Dirac delta approximation is not accurate. Furthermore, we see that Landau damping brings us more information about the instability interval for K. Next, we present different cases.

(i) If $p_0 \to 0$, $I_{Landau} = \left[-\sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2}\,,\ \sqrt{\frac{\psi_0^2\theta}{\pi\beta} + \left(\frac{\psi_0^2\alpha}{4\pi\beta}\right)^2}\right].$

(ii) If $p_0 = \frac{\psi_0^2 \alpha}{8\pi\beta}$, then $I_{Landau} = \left[-\sqrt{\frac{\psi_0^2 \theta}{\pi\beta}} \ , \ \sqrt{\frac{\psi_0^2 \theta}{\pi\beta}} \right]$.

(iii) If $p_0 > \frac{\psi_0^2 \alpha}{8\pi\beta}$, then $I_{Landau} \subseteq I_{delta}$,

(iv) If $p_0 \to +\infty$, then $I_{Landau} = \varnothing$.

Thus, in (i), if p_0 is small, then we have bigger interval of instability.

Also, in case (ii) we have the same interval as in Dirac delta case, which tells us that the behavior of Dirac delta is equivalent to Landau damping with $p_0 = \frac{\psi_0 \alpha}{8\pi\beta}$.

Moreover, in (iii) for big p_0 we have smaller interval of instability.

In the last case (iv) we have the empty interval, which means that we observe stability.

In general, this shows that the Landau damping function with Wigner-Hirota equation illustrates a bigger picture about intervals of instability. Also, note that the Wigner function was useful approach to solve this problem. Remember that working directly with Hirota equation, even with the Dirac delta function is struggling (see Remark 3.3), while Landau damping situation would be much more complicated.

Finally, we state that the obtained results are new and significant for oceanography field in studying rogue wave phenomena. We developed a theory of Penrose instability analysis by performing it on Hirota equation and working with Landau damping function.

5. FUTURE PLANS

The results that we obtained in this paper are new and significant. However, we have further plans to continue these investigations. Since for simplicity we did not consider the full Hirota equation (with $\gamma \neq 0$), then this problem is still an open question.

If we undergo the same process as before, i.e. inserting the ansatz (12) with $W_0(k) = \frac{\psi_0^2}{\pi}\frac{p_0}{k^2+p_0^2}$ into the dispersion relation (13) and linearizing in ε we could deduce that

$$1 + \frac{\psi_0^2 K p_0}{2\pi^2}\int_{\mathbb{R}} \frac{-\alpha k^2 + 2\theta k + \alpha(p_0^2 + K^2/4)}{\Big(3\gamma K k^2 - 2\beta K k + \Omega + \gamma K^3/4 - \alpha K \psi_0^2/2\pi\Big)\Big(k - \Big(ip_0 + \frac{K}{2}\Big)\Big)} \times \frac{1}{\Big(k - \Big(ip_0 - \frac{K}{2}\Big)\Big)\Big(k - \Big(-ip_0 + \frac{K}{2}\Big)\Big)\Big(k - \Big(-ip_0 - \frac{K}{2}\Big)\Big)}\, dk = 0.$$

Now, this integral is very involved and if we use Cauchy Residue's theorem, then we will need to identify the poles, i.e. to say whether they have positive or negative imaginary part. However, from this expression it is very hard to find out.

Thus, we plan to solve this problem with computer software *Mathematica*. A different approach would be to use Taylor expansion in γ in order to approximate the above problematic integral.

Another plan for future is to study the emergence of rogue waves in our Hirota setting. In the paper [3] this aspect was treated for NLSE with Dirac delta function. Our goal is to continue the same approach and to investigate the rogue wave phenomena both for Dirac delta and Landau damping functions.

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