NAZARBAYEV UNIVERSITY

BACHELOR THESIS

Brick-By-Brick: A Construction of ’t Hooft’s Brick Wall

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Declaration of Authorship

I, Narynbek Gilman, declare that this thesis titled "Brick-By-Brick: A Construction of 't Hooft's Brick Wall" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- I have acknowledged all main sources of help.

Signed:

Date:

"Everyone can dream. Who is ready to build it step-by-step, brick-by-brick?"

Rasheed Ogunlaru

NAZARBAYEV UNIVERSITY

Abstract

Department of Physics
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Brick-By-Brick:
A Construction of 't Hooft's Brick Wall

by Narynbek Gilman

A pedagogical construction and review, aimed at the undergraduate level, are given of the brick wall model. The brick wall is a finite region just outside the event horizon of a black hole. It is introduced in order to calculate the entropy of the collapsed star. The statistical origin of the entropy is still unknown, but the brick wall model offers an elementary exercise in counting quantum field solutions to arrive at the famous entropy-area result. The brick wall is a cut-off, effectively regularizing an otherwise divergent result. The model encompasses many fields of physics, including general relativity, quantum theory and statistical mechanics. Its multidisciplinary approach hints at what a more sophisticated solution will look like to the problem of a statistical explanation of the entropy of black holes.

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Contents

Dedicated to Bricklayers

Chapter 1

Introduction

"I like talking to a brick wall - it's the only thing in the world that never contradicts me! "

Oscar Wilde

The brick wall model was introduced [1] as an elementary exercise by 't Hooft to arrive at the entropy [2] of a black hole from an underlying statistical mechanical perspective. The model offers a way to think about the random black body Hawking radiation [3] of particles leading to information loss at the horizon of a black hole. The quantum field at the horizon must be complicated by gravitational interactions, and so, as a first step, it proves instructive to consider a simple cut-off (a brick wall). The field is then zero inside the brick wall (brick thickness b) as well as inside the black hole including the horizon:

$$\phi(r) = 0 \quad \text{if} \quad r \leq r_s + b. \tag{1.1}$$

where b is the small distance very near the horizon (UV cut-off) and $r_s = 2GM/c^2$ is the Schwarzschild radius. See Figure 1.1 for a picture of a spherical brick wall in the form of a hot pizza oven.

FIGURE 1.1: A spherical brick wall.

1.0.1 Mirrors and bricks

One way to look at this situation is to consider the brick wall as a moving mirror to an observer who is freely falling into the black hole. The moving mirror model is well-known to have the very nice advantage of mathematical simplicity. This is both in general, and in the context of the recent one-to-one correspondence with a black hole [4–7], which found that, for the specific example of trajectory motion, the particle production is exactly the same as in the black hole collapse case in (1+1)-dimensions.

It is good to emphasize that, of course, a moving mirror is unlikely to fully resolve the statistical origin of the entropy of black hole or the related and famous information paradox of an astrophysical black hole (e.g., a rotating Kerr black hole with temperature $2\pi T = g - k$ [8], not to mention an extremal black hole [9]), but understanding the subtleties of quantum field theory in a moving mirror model is a promising first baby step toward the more complicated physics of black holes and their related information mysteries.

To further drive home this point, there are interesting subtleties that are overlooked in a moving mirror model, notably, any calculation of entanglement entropy, for instance, necessitates regularizing ultraviolet divergences (the kind that do not affect the spin-statistics connection [10]). One notices that imposing a cutoff is sometimes a tricky procedure, since modes, that have sufficiently high energy at some point in the spacetime, can be red-shifted at some other point due to spacetime curvature.

An asymptotically inertial mirror [11, 12], which is a particular moving mirror that has a one-to-one correspondence to the exactly solvable black hole case [4], may correspond to the original brick wall case. See, for instance, potential candidates like the first known asymptotically inertial solution found by Walker-Davies [13], the asymptotically static case in Good-Anderson-Evans [14] and a drifting case in Good-Ong [15–17]. There are other recent mirror extensions [18–22], including interesting uniformly accelerated trajectories which may be explored in (3+1)-dimensions [23, 24]. One particular trajectory stands out as well [25], which has finite particle emission, long periods of thermal radiation, no information loss and a solvable spectrum.

Another somewhat surprising issue, that is currently being studied, is the difference in fall-off of the particle creation in time, that is known to be much slower than what would be expected [26]. This is in addition to the non-trivial curvature outside the black hole that the accelerated boundary model does not describe [27].

Recent studies have analyzed the Schwarzschild [4], Schwarzschild with Planck length [28], Reissner-Nordström (RN) [29], extreme RN [30] and Kerr [31] cases through a transformation of the (3+1)-dimensional metric to a (1+1)-dimensional accelerated boundary trajectory in flat spacetime. This moving mirror model approach has also been surprisingly used in a cosmological context with respect to the cosmological horizon of de Sitter/anti-de Sitter space [32], where an exact eternal-for-all-times thermal Planck distribution was derived explicitly via Bogoliubov coefficients. In addition, in terms of entanglement harvesting, one study has explored differences between horizonless mirrors and mirrors with strict horizons [33].

Regardless of these interesting directions, the basic object of both models is the quantum field, ϕ, which can be the massless scalar of the Klein-Gordon equation (for a nonlinear investigation of the KG equation in the more general context of quantum field theory under external conditions, see [34, 35]), $\Box\phi = 0$, and whose value is zero when evaluated at the position of the moving mirror, $z(t)$, i.e. $\phi|_z = 0$. This will be treated in explicit details in the next sections, but, for this thesis, we make it clear, the mirror or brick wall will be completely stationary, sitting right outside the event horizon. It will act as a cut-off only, and its own particle and energy production will not be calculated.

1.0.2 Works on the wall

Chronologically, there have been many published works on the brick wall model since 1985. A few of those works stand out and we highlight them here, so that the reader can get a sense of directions that are of interest to the community.

In 1990, Mann et al. [36] investigated brick walls in different dimensions. Looking at N dimensions, they found for any $N > 3$ that the cutoff occurs as a consequence of the causal structure of spacetime, independently of the strength of the source (mass, charge), in agreement with the four-dimensional case. They separately discuss the special case $N = 2$, showing why in this case the cutoff depends on the strength of the source, demonstrating the special nature of the $N = 2$ case.

In 1995, Demers-Lafrance-Myers [37] found that, in the Pauli-Villars regulated theory, the 't Hooft's brick wall can be removed by introducing five regulator fields. With 't Hooft's model and Pauli-Villars regulation, they found that the statistical-mechanical entropy, arising from the minimally coupled quantum scalar field in a general nonextreme static black hole, has a first part that matches the usual Bekenstein-Hawking entropy after renormalization of the gravitational constant.

In 1998, Mukohyama and Israel [38] have shown that the brick wall model, having seemingly doubtful properties, is actually self-consistent. More exactly, thermal excitations near the wall are problematic, but they can be counteracted by correctly defining the ground state, which is the Boulware state due to the absence of horizons above the wall. Specifically, negative energy of the state neutralizes positive energy of the excitations.

In 2000, Winstanley [39] investigated the entropy of a quantum scalar field outside the event horizon of a black hole via the brick wall method. However, in this case, the background of the field was a spherically symmetric black hole geometry in anti-de Sitter space. In this space, there was no need to introduce an infrared cut-off. In addition, all ultraviolet divergences can be included in a renormalization of the coupling constants in the one-loop effective gravitational Lagrangian, which gives a finite entropy.

In 2000, Liu [40] proposed a thin layer near the horizon, which appears to avoid some drawbacks in the original method. Using two parameters, to describe the film, Liu focused on streamlining the little mass approximation, and then neglected logarithmic terms by means of the thin film. Moreover, he deals with the contribution of the vacuum surrounding a black hole by use of the thin film.

In 2004, Jing [41] found that the brick wall model is quite robust even for different coordinates. The entropies for the quantum scalar field are computed for specific coordinates different than the usual Schwarzschild. This seems at odds with the original work of 't Hooft because general covariance (physics being the same under change of coordinates), is usually violated with a cut-off on integration since the integration is done in a particular coordinate system and therefore the cut-off is considered coordinate-dependent. Moreover, the particular coordinate systems, Painlevé and Lemaître coordinates, don't have coordinate singularities at the event horizon, which would seemingly affect the final results for the entropies. These issues are worked around by utilizing a special scheme called Pauli-Villars regularization.

In 2007, Sarkar, Shankaranarayanan and Sriramkumar [42] illustrated that the usual considered zeroth-order term in the WKB approximation leads to corrections for the Bekenstein-Hawking entropy, provided that the metric functions are expanded beyond the linear order near the horizon. They also showed that all the higher-order terms in the WKB approximation have the same form as the zeroth-order term, and found that higher-order WKB terms actually contribute more to the entropy than the lower order terms.

In 2012, Kim and Kulkarni [43] extended and treated the brick wall entropy to higher order via the WKB approximation and generalized it for arbitrary spins. This is with respect to the program of investigation into different types of fields other than the usual scalar ones.

In 2019, Arzano et al. [44] considered the brick wall model of the Schwarzschild geometry in Eddington-Finkelstein coordinates by replacing the by-hand brick wall with an inherent quantum ergosphere 'wall'. The authors are keen on the basic ideas surrounding the notion that backreaction of Hawking radiation can excite the quasinormal modes of the black hole, effectively creating a "wall" of oscillations in the geometry close to the horizon [45]. This is one of pictures of the Bekenstein-Hawking entropy emerging from an interplay between the degrees of freedom of the geometry and those of the field. To realize this idea, they incorporate a small luminosity which creates a "quantum ergosphere" region between the apparent horizon and the event horizon, which effectively acts as a brick wall providing a finite horizon contribution to the entropy.

Chapter 2

Quantum: Field Equation

"May I create plain fields by collecting clouds and bedeck them with arching rainbows."

Suman Pokhrel

2.0.1 Synopsis

The field will be described by the general covariant wave equation,

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\,g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)\phi = 0. \tag{2.1}$$

That is, the field can be expressed in an arbitrary coordinate system and curved spacetime geometry of our choices. Moreover, it is completely free (no potential), massless (zero mass), and scalar (spin-0 or spinless).

2.0.2 Origin of the field equation

It ultimately comes from the action, which defines our theory, governing the scalar field in a curved spacetime, namely,

$$S = \int d^4x\,\mathcal{L}, \tag{2.2}$$

where our Lagrangian density (Lagrangian can be without $\sqrt{-g}$) is chosen as, using $(-,+,+,+)$ signature metric convention,

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{m^2c^2}{\hbar^2}\phi^2\right). \tag{2.3}$$

Notice the $\hbar$ and c which ostensibly signal that we are working in a quantum relativistic regime. Our action then reads:

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{m^2c^2}{\hbar^2}\phi^2\right). \tag{2.4}$$

The equation of motion is found by extremizing the action, which gives the simple Lagrangian equation of motion:

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right) = \frac{\delta\mathcal{L}}{\delta\phi}. \tag{2.5}$$

The derivative in parenthesis is taken carefully like so:

$$\begin{aligned}\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} &= \frac{\delta}{\delta(\partial_\mu \phi)}\left(-\frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{m^2c^2}{\hbar^2}\phi^2\right)\right) \\ &= \frac{\delta}{\delta(\partial_\mu \phi)}\left(-\frac{1}{2}\sqrt{-g}\left(g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi\right)\right).\end{aligned} \tag{2.6}$$

Since index derivatives work like so:

$$\frac{\delta(\delta_i\phi)}{\delta(\delta_j\phi)} \equiv \delta^i_j,$$

we get, after a product rule,

$$\begin{aligned}\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} &= -\frac{1}{2}\sqrt{-g}\,g^{\rho\sigma}\left(\delta^\rho_\mu(\partial_\sigma\phi) + (\partial_\rho\phi)\delta^\sigma_\mu\right) \\ &= -\frac{1}{2}\sqrt{-g}\left(g^{\mu\sigma}\,\partial_\sigma\phi + g^{\rho\mu}\,\partial_\rho\phi\right) \\ &= -\frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\,\partial_\nu\phi + g^{\nu\mu}\,\partial_\nu\phi\right),\end{aligned} \tag{2.7}$$

where we have substituted dummy indices. As the metric is symmetric, $g^{\mu\nu} = g^{\nu\mu}$,

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} = -\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\phi. \tag{2.8}$$

Then, the left-hand side of Eq. (2.5) is

$$\partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\right) = \frac{\partial}{\partial x^\mu}\left(-\sqrt{-g}\,g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)\phi. \tag{2.9}$$

Now, we deal with the right-hand side of Eq. (2.5):

$$\begin{aligned}\frac{\delta \mathcal{L}}{\delta \phi} &= \frac{\delta}{\delta \phi}\left(-\frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{m^2c^2}{\hbar^2}\phi^2\right)\right) \\ &= \frac{\delta}{\delta \phi}\left(-\frac{1}{2}\sqrt{-g}\left(\frac{m^2c^2}{\hbar^2}\phi^2\right)\right) \\ &= -\sqrt{-g}\,\frac{m^2c^2}{\hbar^2}\phi.\end{aligned} \tag{2.10}$$

So, Eq. (2.5) becomes (cancelling out negative signs on both sides),

$$\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\,g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)\phi = \sqrt{-g}\,\frac{m^2c^2}{\hbar^2}\phi. \tag{2.11}$$

Moving $\sqrt{-g}$ to the left,

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\,g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)\phi = \frac{m^2c^2}{\hbar^2}\phi. \tag{2.12}$$

and, defining the left-hand side with the operator $D_\mu D^\mu$,

$$D_\mu D^\mu \phi = \frac{m^2c^2}{\hbar^2}\phi. \tag{2.13}$$

In this work, for simplicity, we treat the massless field and take $m = 0$, so we will have

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g}\, g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right) \phi = 0, \tag{2.14}$$

which is exactly Eq. (2.1). This is the curved spacetime version of the flat spacetime equation of motion, the usual KG equation:

$$\partial_\mu \partial^\mu \phi = 0 \tag{2.15}$$

or

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \phi = \partial_t^2 \phi - \vec{\nabla}^2 \phi = 0. \tag{2.16}$$

In general relativity, the inclusion of the effects of gravity is done by replacing partial with covariant derivatives, so that the result is the Klein–Gordon equation in curved spacetime:

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \to g^{\mu\nu} \nabla_\mu \nabla_\nu = D^\mu D_\mu. \tag{2.17}$$

Notice that $d^4x\sqrt{-g}$ is the invariant spacetime volume interval, i.e. it remains the same under coordinate transformations.

2.0.3 Appearance of Planck constant

Even though $\hbar$ disappears in the wave equation by the use of a massless field, it will reappear in the brick wall itself. The brick wall thickness of 't Hooft is arguably a quantum object:

$$b = \frac{\hbar}{720\pi c M} = \frac{G k_B}{c^4} \frac{T_H}{90} = \frac{k_B T_H}{360 F_m} = \frac{\hbar G}{360\pi c^3 r_s} = \frac{\ell_P^2}{360\pi r_s}. \tag{2.18}$$

Notice the first equality has no dependence on the gravitational constant G. Here the temperature of the black hole is Hawking's temperature, $T_H = \hbar c^3/8\pi G M k_B$. We have gone ahead and introduced the radius of the black hole, $r_s = 2GM/c^2$, and the squared Planck length, $\ell_P^2 = \hbar G/c^3$, for convenience.

A fun way to think about $\hbar$ is to push it into Hawking's temperature, keeping its introduction limited to T_H. This is done by the use of the maximum force [8] of general relativity which has been introduced as a curiosity, $F_m = c^4/(4G)$. One can think of the thermal work divided by the maximum force, up to some constant, 360, as the distance the force acts. This distance is the brick thickness.

2.0.4 Discrete solutions as quantum input

To count the number of field-wave solutions of frequency below ω, giving us the total number of microscopic states, $N(\omega)$, one sums over quantum numbers m and l. Summing over the magnetic number m yields $2l + 1$:

$$N(\omega) = \sum_{l,m} n_r(\omega, l, m) = \sum_l (2l+1) n_r(\omega, l). \tag{2.19}$$

This quantum sum is a little less quantum by assuming a near continuous distribution of states via semi-classical quantization, where the sum becomes an integral:

$$N(\omega) = \int_l dl\ (2l+1) \int_r dr \frac{k_r(\omega, l)}{\pi}. \tag{2.20}$$

Here we can see that a solution exists for discrete values of the wave number k_r:

$$k_r(\omega, l) = \pi n_r(\omega, l). \tag{2.21}$$

It is this quantity that we count, not degrees of freedom or Planck areas on the surface of the black hole.

2.0.5 Recap

We introduced the central object of the model, the quantum field. The equation of motion is in the form of a wave equation in an arbitrary curved spacetime background expressed via any use of coordinates. The type of field was chosen to be as simple as possible by ignoring spin, mass and interactions (except curved spacetime interactions!).

The quantum nature of the field is not particularly manifest because there is no $\hbar$ associated with the dynamics, but we are assured that the model will be quantum because $\hbar$ shows up in Hawking's temperature, which will be a key assumption for the system.

Moreover, we find that ultimately we are counting the number of state solutions of the field outside the brick wall. Using the semi-classical quantization approach, the quantum nature of the solution is further blurred by integrating rather than summing over the quantum numbers.

Chapter 3

Gravity: Black Hole Background

"The ships hung in the air the way that bricks don't."

Douglas Adams

3.0.1 Synopsis

The metric is taken as Schwarzschild background geometry,

$$ds^2 = -fc^2dt^2 + f^{-1}dr^2 + r^2d\Omega^2, \tag{3.1}$$

where $f \equiv 1 - \frac{r_s}{r}$ and $r_s = 2GM/c^2$ is the Schwarzschild radius. In an especially explicit form, the line element of the Schwarzschild geometry is

$$ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)(cdt)^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.2}$$

where $\{t, r, \theta, \phi\}$ are called Schwarzschild coordinates and the respective metric $g_{\mu\nu}$ is called the Schwarzschild metric.

As it can be seen, the metric components do not depend on time, i.e. on the t coordinate. The metric has spherical symmetry, which is seen by the last term that is the spherical line element,

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2. \tag{3.3}$$

The Schwarzschild solution is the unique vacuum solution to Einstein's field equations for a static, spherically symmetric spacetime. This is the most simple yet non-trivial background we might choose, as any other choice (like charged black holes or spinning black holes) would prove to unnecessarily complicate what is meant to be an elementary model and, ultimately, an approximate framework. See Figure 3.1 for a picture of the predicted appearance of a non-rotating black hole.

3.0.2 Explicit components

The explicit metric components needed for the quantum field equation, Eq. (2.1),

$$D_\mu D^\mu = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right), \tag{3.4}$$

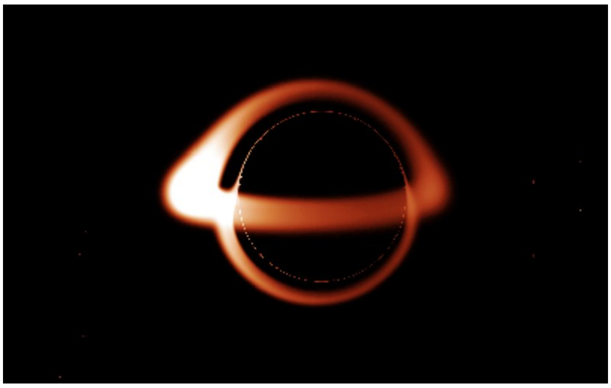

FIGURE 3.1: A non-rotating Schwarzschild black hole with a nice looking ring of ionised matter [46]. The aesthetic asymmetry is due to the Doppler effect resulting from a large orbital speed of the ring.

are as follows. First, we have the metric in matrix form:

$$g_{\mu\nu} = \begin{pmatrix} -fc^2 & 0 & 0 & 0 \\ 0 & f^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{pmatrix}, \tag{3.5}$$

and its inverse:

$$g^{\mu\nu} = \begin{pmatrix} -f^{-1}c^{-2} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\csc^2(\theta)}{r^2} \end{pmatrix}. \tag{3.6}$$

The Jacobian is

$$\sqrt{-g} = cr^2 \sin\theta \tag{3.7}$$

and the pieces of the symmetric interval are

$$g^{tt} = -\frac{1}{fc^2}, \tag{3.8}$$

$$g^{rr} = f, \tag{3.9}$$

$$g^{\theta\theta} = \frac{1}{r^2}, \tag{3.10}$$

$$g^{\phi\phi} = \frac{1}{r^2\sin^2\theta}, \tag{3.11}$$

where all other components are zero.

3.0.3 The field equation in the background

We can now write out the field equation operator in its full explicit glory:

$$\begin{aligned} D_\mu D^\mu &= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^t}\left(\sqrt{-g}g^{tt}\frac{\partial}{\partial x^t}\right) \\ &+ \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^r}\left(\sqrt{-g}g^{rr}\frac{\partial}{\partial x^r}\right) \\ &+ \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\theta}\left(\sqrt{-g}g^{\theta\theta}\frac{\partial}{\partial x^\theta}\right) \\ &+ \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\phi}\left(\sqrt{-g}g^{\phi\phi}\frac{\partial}{\partial x^\phi}\right), \end{aligned} \tag{3.12}$$

where the zero off-diagonal terms leave us with just four main pieces. We write this more simply as, of course:

$$\begin{aligned} D_\mu D^\mu &= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial t}\left(\sqrt{-g}g^{tt}\frac{\partial}{\partial t}\right) \\ &+ \frac{1}{\sqrt{-g}}\frac{\partial}{\partial r}\left(\sqrt{-g}g^{rr}\frac{\partial}{\partial r}\right) \\ &+ \frac{1}{\sqrt{-g}}\frac{\partial}{\partial \theta}\left(\sqrt{-g}g^{\theta\theta}\frac{\partial}{\partial \theta}\right) \\ &+ \frac{1}{\sqrt{-g}}\frac{\partial}{\partial \phi}\left(\sqrt{-g}g^{\phi\phi}\frac{\partial}{\partial \phi}\right). \end{aligned} \tag{3.13}$$

Further explicit substitution gives:

$$\begin{aligned} D_\mu D^\mu &= \frac{1}{cr^2\sin\theta}\frac{\partial}{\partial t}\left(cr^2\sin\theta\frac{-1}{fc^2}\frac{\partial}{\partial t}\right) \\ &+ \frac{1}{cr^2\sin\theta}\frac{\partial}{\partial r}\left(cr^2\sin\theta f\frac{\partial}{\partial r}\right) \\ &+ \frac{1}{cr^2\sin\theta}\frac{\partial}{\partial \theta}\left(cr^2\sin\theta\frac{1}{r^2}\frac{\partial}{\partial \theta}\right) \\ &+ \frac{1}{cr^2\sin\theta}\frac{\partial}{\partial \phi}\left(cr^2\sin\theta\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\right). \end{aligned} \tag{3.14}$$

Simplification via canceling and utilizing partials gives:

$$\begin{aligned} D_\mu D^\mu &= -\frac{1}{fc^2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\right) \\ &+ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f\frac{\partial}{\partial r}\right) \\ &+ \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) \\ &+ \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(\frac{\partial}{\partial \phi}\right). \end{aligned} \tag{3.15}$$

And last but not least, let's clean up the partial styles for t and ϕ:

$$D_\mu D^\mu = -\frac{1}{fc^2}\frac{\partial^2}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}. \quad (3.16)$$

We now have the explicit curved spacetime operator that will act on the quantum field. We are in a good position to introduce statistical considerations next.

3.0.4 Recap

The curved spacetime of the Schwarzschild background, will be fixed. The quantum field will live on this background and will not affect the geometry. This is what is referred to as semi-classical gravity. That is, the curvature and geometry affects the field, but the field does not affect the geometry. There is no backreaction of the field back on the geometry.

The Schwarzschild geometry is parametrized by a single scale, M, which characterizes the entire black hole. The black hole has no charge and does not rotate. The solution is a vacuum solution outside the radius, meaning that the metric $g_{\mu\nu}$ is a solution to Einstien's field equations, where $T_{\mu\nu} = 0$.

Chapter 4

Statistical: Thermal Equilibrium

"As a young man I tried to read thermodynamics, but I always came up against entropy as a brick wall that stopped my further progress."

James Swinburne

4.0.1 Synopsis: Stefan-Boltzmann derivation in natural units

Using natural units in this subsection, one can see that for thermal equilibrium the free energy of a black body, as known from the usual Stefan-Boltzman law, is

$$F = -N_0 \frac{\pi^4 T^4}{15}, \tag{4.1}$$

which results from the famous Bose–Einstein integral, often called the polylogarithm, or just simply the Riemann zeta function, $4\zeta(4) = \pi^4/15$. An easy derivative gives the entropy, $S = -\partial_T F$,

$$S = N_0 \frac{4}{15} \pi^4 T^3. \tag{4.2}$$

A substitution of the Hawking temperature, $T = (4\pi r_s)^{-1}$, gives

$$S = N_0 \frac{4}{15} \pi^4 \left(\frac{1}{4\pi r_s} \right)^3 = N_0 \frac{\pi}{15 \cdot 16} \frac{1}{r_s^3} = \frac{N_0 \pi}{240 r_s^3}. \tag{4.3}$$

The scaling for the particle count is found from the brick wall thickness, $b = 1/(360\pi r_s)$ (which we will choose to make the answer work out right!) and the total solution count:

$$N = \frac{2 r_s^4 \omega^3}{3\pi b} = N_0 \omega^3, \tag{4.4}$$

which will be found later. The count gives $N_0 = 240 r_s^5$, which yields the area of a hole with radius r_s:

$$S = \pi r_s^2. \tag{4.5}$$

The answer is the area of a hole (a black circle, if you will, with radius r_s).

4.0.2 From thermodynamics in natural units

The first law of thermodynamics in closed, reversible system with no change in volume is

$$dE = T dS. \tag{4.6}$$

In natural units, we can apply this to a black hole using Hawking temperature, $T = 1/8\pi M$ to immediately obtain the entropy:

$$dM = \frac{1}{8\pi M} dS. \tag{4.7}$$

We have set the energy of the spacetime as the mass of the black hole. Grouping gives,

$$8\pi M dM = dS, \tag{4.8}$$

integrating gives,

$$4\pi M^2 = S, \tag{4.9}$$

which is, using $r_s = 2M$,

$$S = \pi r_s^2. \tag{4.10}$$

This derivation is deemed insufficient because we know thermodynamics is based on statistical mechanics. What is it that we shall count?

4.0.3 Equipartition theorem and a black hole

One hint at what could be counted comes from the equal parts theorem. The non-rotating, uncharged black hole obeys equipartition if the degrees of freedom are considered to be the number of Planck areas covering the surface area:

$$D = \frac{A}{\ell_P^2}, \tag{4.11}$$

where D is the degrees of freedom. The equipartition theorem says:

$$E = \frac{D}{2} k_B T. \tag{4.12}$$

One can see, in natural units, that

$$E = \frac{A}{2} T, \tag{4.13}$$

and plugging in $T = 1/8\pi M$, and $A = 4\pi r_s^2 = 16\pi M^2$, we get

$$E = \frac{(16\pi M^2)}{2} \left(\frac{1}{8\pi M} \right) = M. \tag{4.14}$$

Since the total energy is the total mass, this result seems to suggest we count Planck areas. However, the equal parts theorem is well-known for breaking down when describing quantum systems, so we must be at least a little wary that this could be one those nice coincidences where the answer is right but the approach is wrong.

In the brick wall model, we don't exactly count degrees of freedom, per say, but instead, field solutions right outside the horizon.

4.0.4 Partition function to free energy

We can start with the canonical ensemble which assigns a probability P to each distinct microstate q given by the following exponential

$$P = e^{\beta(F-E)}. \tag{4.15}$$

Here E is the total energy of the system in the microstate and F is the constant free energy that provides the normalization via the partion function $Z = e^{-\beta F}$ which describes a system of fixed composition that is in thermal equilibrium with a heat bath of a precise temperature, T.

We can start by looking at massless scalar particles with definite frequency ω. A state with N such spinless particles has energy $E = N\hbar\omega$. Summing over all N gives us the partition function for scalars at fixed frequency:

$$Z_\omega = 1 + e^{-\beta\hbar\omega} + e^{-2\beta\hbar\omega} + ... = \frac{1}{1 - e^{-\beta\hbar\omega}}. \tag{4.16}$$

where we have used the sum of convergent geometric series,

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad \text{for} \quad |r| < 1. \tag{4.17}$$

which is true because $e^{-\beta\hbar\omega} < 1$ since the frequency and temperature are both real and positive. We now need to sum over all possible frequencies. Independent partition functions multiply, which means that any logs we apply will add. Our full partition function is then:

$$Z = \left(\frac{1}{1 - e^{-\beta\hbar\omega_1}}\right)\left(\frac{1}{1 - e^{-\beta\hbar\omega_2}}\right) ... \left(\frac{1}{1 - e^{-\beta\hbar\omega_q}}\right) \tag{4.18}$$

or

$$Z = \prod_q \frac{1}{1 - e^{-\beta\hbar\omega_q}}. \tag{4.19}$$

So therefore, using Z, we have the freedom to use log addition:

$$\ln Z = \ln \prod_q \frac{1}{1 - e^{-\beta\hbar\omega_q}} = \sum_q \ln\left(\frac{1}{1 - e^{-\beta\hbar\omega_q}}\right). \tag{4.20}$$

That is, written a little bit more explicitly,

$$\ln Z = \ln\left[\left(\frac{1}{1 - e^{-\beta\hbar\omega_1}}\right)\left(\frac{1}{1 - e^{-\beta\hbar\omega_2}}\right)\left(\frac{1}{1 - e^{-\beta\hbar\omega_3}}\right) ...\right] \tag{4.21}$$

is then

$$\ln Z = \sum_q \left[\ln 1 - \ln(1 - e^{-\beta\hbar\omega_q})\right]. \tag{4.22}$$

or, pulling out a negative in front,

$$\ln Z = -\sum_q \ln(1 - e^{-\beta\hbar\omega_q}). \tag{4.23}$$

This is effectively writing,

$$\ln Z = \sum_q \ln Z_{\omega_q}. \tag{4.24}$$

The particular problem we have now though is that we need to know how many scalar states there are with some particular frequency ω. This is what is referred to as the density of states: $g(\omega)$. That is, this $g(\omega)d\omega$ is the number of states available to a single scalar particle with frequency between ω and $\omega + d\omega$. We can convert the sum to an integral with the density of states, going directly from Z_ω:

$$\ln Z = \int_0^\infty d\omega g(\omega) \ln Z_\omega. \tag{4.25}$$

Using $\ln Z_\omega = -\ln(1 - e^{-\beta\hbar\omega})$, we have

$$\ln Z = -\int_0^\infty d\omega g(\omega) \ln(1 - e^{-\beta\hbar\omega}). \tag{4.26}$$

The free energy is defined in terms of the partition function, $Z = e^{-\beta F}$, and is the closest object related to it, so using

$$F = -k_B T \ln Z, \tag{4.27}$$

we find that the free energy is

$$F = +k_B T \int_0^\infty d\omega g(\omega) \ln(1 - e^{-\beta\hbar\omega}). \tag{4.28}$$

This sum is replaced by a integration because the energy is effectively continuous, where $g(\omega) = dN(\omega)/d\omega$ is commonly known as the density of the states, and $N(\omega)$ is the total number of microstates; i.e. what we want to count. Let's express this integral in terms of the microstates:

$$F = k_B T \int_0^{+\infty} dN(\omega) \ln(1 - e^{-\beta\hbar\omega}), \tag{4.29}$$

Let's move the $\beta = 1/k_B T$ to other side for an integration by parts, which gives:

$$\beta F = N(\omega) \ln(1 - e^{-\beta\hbar\omega})\Big|_0^{+\infty} - \int_0^{+\infty} d\omega \frac{N(\omega)(-e^{-\beta\hbar\omega})(-\beta\hbar)}{1 - e^{-\beta\hbar\omega}}. \tag{4.30}$$

It is easy to see that the first term disappears because $\ln 1 = 0$, and $N(0) = 0$. However, the second term remains and can be recognized as that of thermal equilibrium where the particle spectrum is a Bose-Einstein distribution. The free energy as a function of temperature is the sum of all energies $d\omega$ in the available states,

$$F(T) = -\int_0^\infty d\omega \, \frac{\hbar N(\omega)}{e^{\hbar\omega/k_B T} - 1}. \tag{4.31}$$

By definition, a Planck distribution, even for scalar particles, has microstates which scale as $N(\omega) \sim \omega^3$. This is also derived explicitly in the next section, but we can go ahead and use this fact as a given in $3+1$ dimensions for thermal equilibrium. Writing the microstates as

$$N(\omega) \equiv N_0 \omega^3, \tag{4.32}$$

will allow us to go ahead and extend our thermal treatment as far as possible and leave the numerical coefficients, N_0, for later. The result is

$$F(T) = -\int_0^\infty d\omega \, \frac{\hbar N_0 \omega^3}{e^{\hbar\omega/kT} - 1}, \tag{4.33}$$

which is the well known Bose-Einstein integral,

$$\int dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}. \tag{4.34}$$

So we have for the free energy:

$$F = -\hbar N_0 \frac{k_B^4 T^4}{\hbar^4} \left(\frac{\pi^4}{15} \right) = -N_0 \frac{k_B^4}{\hbar^3} \frac{\pi^4}{15} T^4. \tag{4.35}$$

4.0.5 Free energy to entropy and use of Hawking temperature

The entropy can be found with a single derivative, $S = -\partial_T F$, which gives:

$$S = +N_0 \frac{k_B^4}{\hbar^3} \frac{4\pi^4}{15} T^3 \tag{4.36}$$

Now we use the Hawking temperature for our black hole,

$$T = \frac{\hbar c^3}{8\pi G M k_B} = \frac{c\hbar}{4\pi k_B r_s}, \tag{4.37}$$

and obtain the entropy,

$$S = k_B \pi \frac{N_0 c^3}{240 r_s^3}, \tag{4.38}$$

as a function of our unknown N_0 quantity, which has important numerical and dimensional information, obtained by use of the brick wall thickness, b.

4.0.6 Numerical factor and scaling of the radius

We will see in the next section on Building the Brick Wall that using the brick wall amounts to choosing

$$N_0 = \frac{240 r_s^5}{\hbar G}, \tag{4.39}$$

so that our final entropy result will be:

$$S = k_B \frac{c^3}{\hbar G} \pi r_s^2 = \frac{k_B}{\ell_P^2} \frac{A}{4}. \tag{4.40}$$

We remark here that from our equipartition result, where the degrees of freedom were the number of Planck areas on the surface, $D = A/\ell_P^2$, then the entropy is just the degrees of freedom divided by four:

$$\frac{S}{k_B} = \frac{D}{4}. \tag{4.41}$$

4.0.7 Recap

We start with the partition function and can extend the statistical mechanical treatment all the way to a final expression for entropy, dependent on the N_0 numerical quantity (which is dependent on the size of the brick wall). It is clear that the statistical framework of the canonical ensemble takes us quite far.

Moreover, if we are willing to start from the $F \sim T^4$ scaling of the Stefan-Boltzmann law, for the free energy, we can go straight to the entropy with a single derivative and the use of Hawking's temperature.

Along the way we encountered an unexpected agreement with the equipartition theorem for the Schwarzschild black hole. Here the degrees of freedom are counted by the number of Planck areas on the surface of the black hole.

We also derived the thermodynamics, (non-counting) result of the entropy as a convenience, and reminder to the reader that macroscopically, the entropy result is almost trivial to obtain from Hawking's temperature alone (no brick walls).

Chapter 5

Building the Brick Wall

"Boundaries are just made of brick."

Nikita Dudani

5.0.1 Spherical symmetry separability ansatz

A good assumption is spherical symmetry of the field, since the background is spherically symmetric. In this case, the field takes the simplified and seperable form:

$$\phi(r,\theta,\phi,t) = R(r)Y_{lm}(\theta,\phi)e^{-i\omega t}. \tag{5.1}$$

This symmetry will take us quite far. Most notably, the angular dimensions are seen to considerly simplify via harmonics. It is interesting how the effect of the angular dimensions are felt into the radial solution with the appearance of an $l(l+1)/r^2$ term operator. A simplification could be done by looking at $l = 0$ case exclusively, a regime called the 's-wave' sector.

5.0.2 Operator

The field equation operator is therefore interesting only radially, and, substituting the Jacobian and all terms of the metric explicitly, we get Eq. (3.16):

$$D_\mu D^\mu = -\frac{1}{fc^2}\frac{\partial^2}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{5.2}$$

The two time derivatives bring down a frequency ω from the wave ansatz, while the two angular pieces is well-known from Laplace's spherical harmonics and separability, i.e. $r^2\nabla^2 Y_{lm} = -l(l+1)Y_{lm}$, where $l = 0, 1, 2...$ and $m = -l, -l+1..., +l$. The operator on the massless scalar field in the Schwarzschild geometry becomes, after splitting the operator into space and time pieces,

$$0 = (D_t^2 + D_x^2)R(r)Y(\theta,\phi)e^{-i\omega t}, \tag{5.3}$$
$$\tag{5.4}$$
$$0 = R(r)Y(\theta,\phi)D_t^2 e^{-i\omega t} + e^{-i\omega t}D_x^2\left[R(r)Y(\theta,\phi)\right], \tag{5.5}$$
$$\tag{5.6}$$
$$0 = R(r)Y(\theta,\phi)\left(-(-i)(-i)\frac{\omega^2}{c^2 f}\right)e^{-i\omega t} + e^{-i\omega t}D_x^2\left[R(r)Y(\theta,\phi)\right], \tag{5.7}$$
$$\tag{5.8}$$
$$0 = +R(r)Y(\theta,\phi)\frac{\omega^2}{c^2 f} + D_x^2\left[R(r)Y(\theta,\phi)\right], \tag{5.9}$$
$$\tag{5.10}$$
$$0 = R(r)Y(\theta,\phi)\frac{\omega^2}{c^2 f} + Y(\theta,\phi)D_x^2 R(r) + R(r)D_x^2 Y(\theta,\phi), \tag{5.11}$$
$$\tag{5.12}$$
$$0 = R(r)Y(\theta,\phi)\frac{\omega^2}{c^2 f} + Y(\theta,\phi)D_x^2 R(r) + R(r)\left(-\frac{l(l+1)}{r^2}Y(\theta,\phi)\right), \tag{5.13}$$
$$\tag{5.14}$$
$$0 = R(r)\frac{\omega^2}{c^2 f} + D_x^2 R(r) - R(r)\frac{l(l+1)}{r^2}, \tag{5.15}$$

So that one has, putting the zero on the right,

$$\frac{\omega^2}{c^2 f}R(r) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f\frac{\partial}{\partial r}R(r)\right) - \frac{l(l+1)}{r^2}R(r) = 0, \tag{5.16}$$

So only the radial part is interesting, when using separability. Recall that $f \equiv 1 - r_s/r$ is dimensionless. The equation can be written more suitably putting the remaining operator pieces on the right-hand side,

$$\left(\frac{\omega^2}{c^2 f} - \frac{l(l+1)}{r^2}\right)R(r) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f\frac{\partial}{\partial r}R(r)\right). \tag{5.17}$$

5.0.3 WKB

Furthermore, we express the radial part by phase, $R(r) = e^{iS(r)}$, and obtain the radial wave number, k_r, by acting the field operator on $S(r)$.

$$\left(\frac{\omega^2}{c^2 f} - \frac{l(l+1)}{r^2}\right)e^{iS(r)} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f\frac{\partial}{\partial r}e^{iS(r)}\right). \tag{5.18}$$

After one, then two derivatives on the right-hand side,

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f\frac{\partial}{\partial r}e^{iS(r)}\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f iS'(r)e^{iS(r)}\right), \tag{5.19}$$
$$= -\frac{1}{r^2}\left[r^2 f iS'(r) iS'(r)e^{iS(r)} + ie^{iS(r)}\frac{\partial}{\partial r}\left(r^2 f S'(r)\right)\right] \tag{5.20}$$
$$= fS'(r)^2 e^{iS(r)} - i\frac{e^{iS(r)}}{r^2}\frac{\partial}{\partial r}\left(r^2 f S'(r)\right). \tag{5.21}$$

Canceling out the radial wave $e^{iS(r)}$, on both sides, we obtain:

$$\left(\frac{\omega^2}{c^2 f} - \frac{l(l+1)}{r^2}\right) = fS'(r)^2 - i\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f S'(r)\right), \tag{5.22}$$

Before we neglect the imaginary piece, as is usual in the WKB approximation, let us write it out more explicitly, using $f = f(r)$,

$$\begin{aligned} -i\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f(r) S'(r)\right) &= -i\left[S''(r)f(r) + S'(r)f'(r) + \frac{2f(r)S'(r)}{r}\right], && (5.23)\\ &= -i\left[S''(r)f(r) + S'(r)\left(f'(r) + \frac{2f(r)}{r}\right)\right]. && (5.24)\end{aligned}$$

The full expression is,

$$\left(\frac{\omega^2}{c^2 f} - \frac{l(l+1)}{r^2}\right)\frac{1}{f} = S'(r)^2 - i\left[S''(r) + S'(r)\left(\frac{f'}{f} + \frac{2}{r}\right)\right], \tag{5.25}$$

Our approximation only holds if the imaginary piece is small compared to $S'(r)^2$ term. Notice the curvature term, f'/f, proportional to $S'(r)$ which grows very large as $r \to r_s$ where $f \to 0$. The first approximation where the entire imaginary term remains small (where our phase has absorbed factors of $\hbar$ and $S'(r)^2$ is leading order [47]), gives

$$S'(r)^2 = \left(\frac{\omega^2}{c^2 f} - \frac{l(l+1)}{r^2}\right)\frac{1}{f}, \tag{5.26}$$

or expressed as a result for k_r in terms of the energy levels of the field, $\omega(l, m, n)$:

$$k_r^2 = \left(\frac{\partial S(r)}{\partial r}\right)^2 = \left(\frac{\omega^2}{c^2 f} - \frac{l(l+1)}{r^2}\right)\frac{1}{f}. \tag{5.27}$$

This will be our essential wave number, related to the state, n, by

$$n(r, \omega, l, m)\pi = k(r, \omega, l, m). \tag{5.28}$$

We want to count all the states for all the relevant quantum numbers, and determine the scaling of ω by integrating over r.

5.0.4 Counting states

This wave number can be used to count the number of wave solutions of frequency below ω, giving us the total number of microscopic states, $N(\omega)$, which is easy to sum over the magnetic number m; $2l + 1$ for every l:

$$N(\omega) = \sum_{l,m} n_r(\omega, l, m) = \sum_l (2l+1) n_r(\omega, l). \tag{5.29}$$

Assuming near continuous distribution of states via semi-classical quantization,

$$N(\omega) = \int_l dl\ (2l+1) \int_r dr \frac{k_r(\omega, l)}{\pi}. \tag{5.30}$$

This scheme counts the states, and now all we need to do is two integrals. Once we have the bounds on the angular integral, we will find it to be large but straightforward; while the radial integral will require an approximation to give tractable results.

5.0.5 Lower and upper limits of angular momentum integral

The angular momentum integral has a lower limit of zero by definition. The upper limit is chosen so that k_r^2 is non-negative:

$$k_r = \pm\sqrt{\left(\frac{\omega^2}{c^2 f} - \frac{l(l+1)}{r^2}\right)\frac{1}{f}} > 0. \tag{5.31}$$

Here we choose the $+$ sign so $k_r > 0$ when the argument is also positive. Simply setting to zero and solving for l gives,

$$l_{\text{upper}} = \frac{1}{2}\left(\pm\sqrt{\frac{4r^2\omega^2}{c^2 f} + 1} - 1\right), \tag{5.32}$$

where we will choose the $+$ sign again in the quadratic formula in order to have the upper limit be positive.

5.0.6 Angular momentum and radial integral combined

Thus we have, putting it all together, with the upper limit and the value of k_r:

$$N(\omega) = \int_r dr \int_0^{l_{\text{upper}}} dl\ (2l+1)\frac{1}{\pi}\sqrt{\left(\frac{\omega^2}{c^2 f} - \frac{l(l+1)}{r^2}\right)\frac{1}{f}}. \tag{5.33}$$

Despite appearances, this integral, over l, is straightforward to compute.

5.0.7 Angular integration

The result of the angular integration is in brackets:

$$N(\omega) = \int_r dr \left[\frac{2}{3\pi}\left(\frac{\omega}{c}\right)^3 \frac{r^2}{f^2}\right]. \tag{5.34}$$

The terms independent of r can be moved outside the integrad:

$$N(\omega) = \frac{2}{3\pi}\left(\frac{\omega}{c}\right)^3 \int_r dr\ \frac{r^2}{f^2}. \tag{5.35}$$

It is quite pleasing to see such a simple result emerge, but moreover, one sees the curvature effect directly by the influence of f for the integration over r.

5.0.8 Laurent series

Since the infinite contribution happens near the horizon, a Laurent series around $r = r_s$ gives the zeroth order terms,

$$\frac{r^2}{f^2} = \frac{r_s^4}{(r-r_s)^2} + \frac{4r_s^3}{(r-r_s)} + 6r_s^2 + \mathcal{O}(r-r_s). \tag{5.36}$$

We will only use the leading order term since most of the effect on the quantum field happens near the event horizon:

$$\frac{r^2}{f^2} = \frac{r_s^4}{(r - r_s)^2}. \tag{5.37}$$

This approximation is a nice convenient way to getting a non-divergent result, as integrating r^2/f^2 directly using our bounds, does not converge because of vacuum terms. An indefinite integral is possible, and then subsequent series near $r = r_s$, also results in an equivalent approximation as Eq. (5.37).

5.0.9 Radial integration

Now we integrate the main contribution near the horizon,

$$N(\omega) = \frac{2}{3\pi}\left(\frac{\omega}{c}\right)^3 \int_{r_s+b}^{\infty} dr\, \frac{r_s^4}{(r - r_s)^2} = -\frac{2}{3\pi}\left(\frac{\omega}{c}\right)^3 \frac{r_s^4}{(r - r_s)}\bigg|_{r_s+b}^{\infty}. \tag{5.38}$$

The contribution at the upper infrared limit is zero:

$$N(\omega) = \frac{2}{3\pi}\left(\frac{\omega}{c}\right)^3 \left(\frac{r_s^4}{b}\right). \tag{5.39}$$

The modes at the upper limit do not contribute to the counting as the field at large distances is undisturbed there by gravitation, i.e. we have ignored the vacuum contributions (they never show up!) and used only the leading order term for r near r_s by the previous use of the Laurent series of the integrand near the horizon. This avoids the need to even write down any vacuum contributions as they are zeroed out by the Laurent series.

5.0.10 Utilizing the brick wall thickness

Using the brick wall thickness of 't Hooft,

$$b = \frac{\ell_P^2}{360\pi r_s}, \tag{5.40}$$

gives the number of modes,

$$N(\omega) = \frac{2}{3\pi}\left(\frac{\omega}{c}\right)^3 \left(\frac{360\pi r_s^5}{\ell_P^2}\right) = \frac{2}{3\pi} \cdot 360\pi \left(\frac{r_s\omega}{c}\right)^3 \left(\frac{r_s^2}{\ell_P^2}\right). \tag{5.41}$$

In th expression on the right, we have arranged it so you can see it is a dimensionless quantity, as it should be, because the number of mode solutions is just that, a number with no units. Simply expressed:

$$N(\omega) = \frac{240\omega^3 r_s^5}{\hbar G} \equiv N_0\omega^3, \tag{5.42}$$

where $N_0 \equiv 240r_s^5/\hbar G$, a dimensionful object, used to sweep up all the scaling, numerical factors, and units; leaving only the dependence on frequency visible in $N(\omega)$.

5.0.11 Entropy using mode count

One sees that from equilibrium considerations, already iterated in the statistical section, that the entropy is,

$$S = k_B c^3 \frac{N_0 \pi}{240 r_s^3}, \tag{5.43}$$

So that, using $N_0 \equiv 240 r_s^5/\hbar G$ and $\ell_P^2 = \hbar G/c^3$:

$$S = \frac{k_B}{\ell_P^2} \pi r_s^2. \tag{5.44}$$

We remark here that the area of a circle, πr_s^2, or 'hole', fits the nomenclature of the name black hole, quite well (a rarity sometimes in physics), especially since we are learning more and more that information content and entropy as its measure, is an increasingly more accurate and useful description of fundamental physical law. Using the surface area of a sphere $A = 4\pi r^2$, with the black hole radius as $r = r_s$, we obtain

$$S = \frac{k_B}{\ell_P^2} \frac{A}{4}. \tag{5.45}$$

As we see again, entropy is no longer an extensive object in our system. Extensive means 'proportional to the amount of quantity present'. That is, one would expect $S \sim V$, but instead we see $S \sim A$! This result is shocking, to say the least; no matter how many times we see it.

Chapter 6

Conclusion

"I found Rome a city of bricks and left it a city of marble."

Augustus

Generally, the quest for a statistical understanding of black hole entropy may lead us to a more fundamental theory of gravity and quantum matter. While this model is enlightening as an elementary exercise, it does have obvious limitations as presented. In particular, the principle that physics is invariant under coordinate transformations has been abandoned at the horizon due to the introduction of the brick wall cut-off while using Schwarzschild coordinates. A model that keeps general covariance is worth addressing. The upside is that this model has no loss of information, and abides by the laws of quantum mechanics. That is, quantum coherence is maintained. 't Hooft places a strong emphasis on the question of finding a model that maintains quantum coherence and general covariance.

6.0.1 How much entropy does a black hole have anyways?

Since we are using SI units, we might as well actually plug in real-life values to get a handle on how much entropy is contained in a run-of-the-mill black hole. Let's consider a black hole sun ($M \sim 10^{30}$ kg):

$$S \sim 10^{54} \text{ J/K}. \tag{6.1}$$

This should be compared to the entropy of the sun:

$$S \sim 10^{35} \text{ J/K}. \tag{6.2}$$

The black hole sun entropy is larger by 19 orders of magnitude! Consider one bit of information is $S = k_B \ln 2$ and one nat of information is $S/\ln 2 = k_B \sim 10^{-23}$ J/K, while the information content of the universe is $S_U = 10^{81}$ J/K. The biggest black hole, M87, with a billion solar masses has $S \sim 10^{73}$ J/K.

6.0.2 Why so much entropy?

The reason black holes have so much entropy effectively comes down to the fact that the scaling is

$$S_{BH} \sim M^2, \tag{6.3}$$

while a thermal ball of radiation scales as

$$S_R \sim M^{3/2}. \tag{6.4}$$

This is because for the ball of thermal radiation which could be the source of formation, the Stefann-Boltzmann law gives volume times fourth power of temperature,

$$M \sim T^4 R^3, \tag{6.5}$$

We then know that $R \sim M$, is the size of the ball to form a black hole, so that

$$M \sim T^4 M^3 \to M^{-2} \sim T^4, \tag{6.6}$$

or just, rearranging for temperature,

$$T \sim M^{-1/2}, \tag{6.7}$$

so when one takes the derivative of the Stefan-Boltzmann law, one sees entropy scaling as

$$S \sim T^3 R^3, \tag{6.8}$$

which gives, plugging in Eq. (6.7), and $R \sim M$ gives

$$S \sim M^{3/2}, \tag{6.9}$$

which is Eq. (6.4), the entropy of a thermal ball of radiation.

6.0.3 How thick is the brick?

It was originally argued that the brick wall thickness is invariant with respect to the size of the black hole, despite the suggestive form:

$$b = \frac{\ell_P^2}{360\pi r_s}, \tag{6.10}$$

showing inverse proportional scaling to the radius of the black hole. This was done by finding the proper length of the brick wall, using

$$L_{\text{proper}} = \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu}. \tag{6.11}$$

This is written, defining $L_{\text{proper}} \equiv b_P$ as

$$b_P = \int_{r_s}^{r_s+b} \sqrt{g_{rr} dr^2} = \int_{r_s}^{r_s+b} dr \sqrt{g_{rr}}, \tag{6.12}$$

where we use the Schwarzschild metric in spherical Schwarzschild coordinates, so that $g_{rr} = f^{-1}$, where we write,

$$b_P = \int_{r_s}^{r_s+b} \frac{dr}{f^{1/2}}, \tag{6.13}$$

with $f = 1 - r_s/r$, so that

$$b_P = \int_{r_s}^{r_s+b} \frac{dr}{(1 - \frac{r_s}{r})^{1/2}}. \tag{6.14}$$

This is not as harmless of an integral as it may seem, since the answer is,

$$b_P(r_s) = \sqrt{\frac{1}{(360\pi r_s)^2} + \frac{1}{360\pi}} + \frac{r_s}{2}\ln\left(1 + \frac{\sqrt{360\pi r_s^2+1}+1}{180\pi r_s^2}\right), \tag{6.15}$$

which is clearly dependent on the size of the black hole! A series expansion for a large black hole, around $r_s = \infty$, gives:

$$b_P = \frac{1}{\sqrt{90\pi}} + \frac{1}{6480\sqrt{10}\pi^{3/2} r_s^2} + \mathcal{O}(r_s^{-4}). \tag{6.16}$$

At this point, we see one can just as easily make a series expansion in the small brick limit, relative to the radius of the black hole (bricks are usually smaller than black holes anyways), $b \ll r_s$, on the evaluated integral in terms of b, and we obtain,

$$b_P = 2\sqrt{br_s} = 2\sqrt{2bM} = \frac{1}{\sqrt{90\pi}}. \tag{6.17}$$

Notice the typo in the original manuscript where the square root sign does not extend as far as it should, but the final answer is correct regardless. So indeed, the brick wall thickness is independent of the size of the black hole, when viewed after calculating the proper thickness (and utilizing the appropriate regime where $r_s \gg \ell_P$).

6.0.4 What could be smaller than the Planck length?

A natural question arises though about whether this number has any significance, i.e., why is it smaller than the Planck length?

$$b_P = \frac{1}{\sqrt{90\pi}} = 0.059\ell_P. \tag{6.18}$$

One may view such a cut-off as a signal of a serious liability on the brick wall model. It suggests that there is limited utility for the model. One should not look too deeply at the brick wall as much more than a useful but ultimately elementary exercise in establishing the basics of a statistical foundation for understanding the entropy of black holes.

6.0.5 Where is the mass of the black hole located?

Interestingly, with the free energy, F, we can also compute quite easily, the total energy:

$$U = \frac{\partial}{\partial\beta}(\beta F) = F + TS. \tag{6.19}$$

Using our free energy value, with $N_0 = 240 r_s^5$, and Hawking temperature $T = 1/(4\pi r_s)$,

$$F = -N_0\frac{\pi^4}{15}T^4 = -\frac{r_s}{16}, \tag{6.20}$$

we find from $U = F + TS$,

$$U = \frac{-r_s}{16} + \frac{1}{4\pi r_s}(\pi r_s^2) = \frac{3}{16}r_s = \frac{3}{8}M. \tag{6.21}$$

This should raise an eyebrow, since it is a sizable fraction of the total mass, M, of the spacetime! This result is suggesting, technically, that $3/8 = 37.5\%$ of the mass of the black hole is outside the black hole radius, in the form of thermal radiated energy. Immediately, one is suspect that this outside energy should significantly affect the curvature of the assumed geometry, i.e. backreaction.

6.0.6 What about a massive scalar field?

Does a massive scalar change anything? We can see that it only serves to complicate the model. The entire calculation proceeds the same way, with the only difference occurring when the mass appears in the angular integration and upper limit. The integral:

$$N(\omega) = \int_r dr \int_0^{l_{\text{upper}}} dl\ (2l+1)\frac{1}{\pi}\sqrt{\left(\frac{\omega^2}{c^2 f} - \frac{l(l+1)}{r^2} - m^2\right)\frac{1}{f}}, \tag{6.22}$$

now includes a small particle mass m as well as in the upper limit, l_{upper},

$$l_{\text{upper}} = \frac{1}{2}\left(-1 \pm \sqrt{\frac{4r^2\omega^2}{c^2 f} - 4m^2r^2 + 1}\right), \tag{6.23}$$

The scary looking integral is easy to do, and gives

$$N(\omega) = \frac{2}{3\pi}\int_r dr \frac{r^2}{f^2}\left(w^2 - fm^2\right)^{3/2}. \tag{6.24}$$

While this looks quite different with respect to the radial integration, the Laurent series will in fact negate the particle mass. The leading order term near the horizon is still mass-free:

$$\frac{r^2}{f^2}\left(w^2 - fm^2\right)^{3/2} = \frac{\omega^3 r_s^4}{(r - r_s)^2}. \tag{6.25}$$

This gives the usual particle count as before:

$$N(\omega) = -\frac{2}{3\pi}\omega^3 \left.\frac{r_s^4}{(r - r_s)}\right|_{r_s+b}^{\infty} = \frac{2}{3\pi}\omega^3\left(\frac{r_s^4}{b}\right). \tag{6.26}$$

This also clear from the fact that near the horizon, $f \sim 0$, so in a sense, its the curvature f that negates the mass term in the integrand, Eq. (6.24), if all we care about is a halo of hot particles (massless scalars!) close to the black hole .

6.0.7 What about a charged black hole?

What if we change metrics and add charge to the mix?

$$f_q = 1 - \frac{r_s}{r} + \frac{r_q^2}{r^2}. \tag{6.27}$$

This means that the horizon is now different than $r = r_s$. The outer horizon, we will subscript with r_p for positive sign for the square root, is at

$$r_p = \frac{1}{2}(r_s + \sqrt{r_s^2 - 4r_q^2}). \tag{6.28}$$

We can do a series approximation of the radial integrand,

$$\frac{r^2}{f_q^2} = \frac{r_p^6}{(r-r_p)^2(2r_p-r_s)^2}, \tag{6.29}$$

where we have kept only the leading order term around the horizon $r = r_p$ and expressed r_q in terms of r_p. Integrating this from the brick wall position outward gives

$$\int_{r_p+b_q}^{\infty} dr \frac{r_p^6}{(r-r_p)^2(r_s-2r_p)^2} = \frac{r_p^6}{b_q\,(2r_p-r_s)^{\,2}}. \tag{6.30}$$

Since the rest of the calculation is the same, we can identify a new N_0, from:

$$N_q(\omega) \equiv N_0\omega^3 = \frac{2}{3\pi}\omega^3\left(\frac{r_p^6}{b_q\,(2r_p-r_s)^{\,2}}\right). \tag{6.31}$$

The entropy calculation is as before, where

$$S = N_0\frac{4\pi^4}{15}T^3. \tag{6.32}$$

But now we plug in the colder Hawking temperature of the charged black hole,

$$T_q = \frac{1}{4\pi r_s} - \frac{r_s}{4\pi}\left(\frac{1}{r_p}-\frac{1}{r_s}\right)^2 = \frac{2r_p-r_s}{4\pi r_p^2}. \tag{6.33}$$

This gives our $S_q(N_0)$,

$$S_q = \frac{\pi N_0\,(2r_p-r_s)^{\,3}}{240 r_p^6}, \tag{6.34}$$

which we know must equal the known answer of $S = \pi r_p^2$. Solving for N_0 gives,

$$N_0 = \frac{240 r_p^8}{(2r_p-r_s)^{\,3}}. \tag{6.35}$$

Plugging this into our $N(\omega)$ value, Eq. (6.31), and solving for b_q gives:

$$b_q = \frac{2r_p-r_s}{360\pi r_p^2}. \tag{6.36}$$

We can see that as $r_p \to r_s$, then the usual non-rotating results are obtained. It is easy to see that the new brick wall is still the usual ratio associated with the new temperature, that is,

$$b_q = \frac{T_q}{90}, \tag{6.37}$$

which is as it was in the neutral charge case. This electric black hole works so well because of spherical symmetry that is left uncorrupted by the addition of charge.

The proper length can be calculated in the same way as the uncharged case. Writing the integral as

$$b_P = \int_{r_p}^{r_p+b} \frac{dr}{f^{1/2}} = \int_{r_p}^{r_p+b} \frac{r dr}{\sqrt{(r-r_p)(r+r_p-r_s)}}. \tag{6.38}$$

This gives, remembering the conditions, $2r_p > r_s > r_p > 0$ and $b > 0$,

$$b_P = \sqrt{b(b + 2r_p - r_s)} + r_s \sinh^{-1}\left(\sqrt{\frac{b}{2r_p - r_s}}\right). \tag{6.39}$$

For small bricks, $b \ll r_p$, to leading order,

$$b_P = \frac{2r_p\sqrt{b}}{\sqrt{2r_p - r_s}}. \tag{6.40}$$

Plugging in our brick wall, we find

$$b_P = \frac{1}{\sqrt{90\pi}}\frac{r_p}{r_s}. \tag{6.41}$$

A dimensionless quantity, that decreases as more charge is added.

6.0.8 What about a cosmological horizon?

What if we change the metric to one where every observer is surrounded by a cosmic event horizon?

$$f_{dS} = 1 - \frac{r^2}{\ell^2}. \tag{6.42}$$

Here ℓ is the curvature radius. This is called de Sitter spacetime. The horizon is at $r = \pm\ell$ where we will take the positive root. A series of r^2/f^2 near the horizon, ℓ, gives to leading order,

$$\frac{r^2}{f^2} = \frac{\ell^4}{4(r-\ell)^2}. \tag{6.43}$$

Integrating from $r = 0$ to $r = \ell - b$, and keeping only the highest order term because $b \ll \ell$ (bricks are much smaller than the universe), gives $\ell^4/(4b)$. The count is therefore,

$$N(\omega) \equiv N_0\omega^3 = \frac{2}{3\pi}\left(\frac{\ell^4}{4b}\right)\omega^3. \tag{6.44}$$

The entropy is therefore,

$$S = N_0\frac{4\pi^4}{15}T^3 = \left(\frac{\ell^4}{6\pi b}\right)\frac{4\pi^4}{15}\left(\frac{1}{2\pi\ell}\right)^3 = \frac{\ell}{180b}, \tag{6.45}$$

where we have plugged in the Hawking temperature of the de Sitter horizon, $T = 1/(2\pi\ell)$. If the brick wall has thickness $b = T/90$, then $b = 1/(180\pi\ell)$, and we see that the entropy is

$$S = \pi\ell^2, \tag{6.46}$$

the surface area of the universe, divided by 4.

6.0.9 Brick wall in spherically symmetric spacetimes

It is seen that spherically symmetric spacetimes with black holes will have

$$b = \frac{T}{90}. \tag{6.47}$$

This is possible because the integrand term r^2/f^2 can be written to leading order via series expansion as

$$\frac{r^2}{f^2} = \frac{1}{(r-R)^2}\frac{R^2}{f'(r)|_R^2}, \tag{6.48}$$

where R is the location of the horizon where $f(r)$ is zero. The derivative piece has the curvature information in it and amounts to

$$g'_{tt}(r)\big|_{r=R} = f'(r)\big|_{r=R} = 4\pi T. \tag{6.49}$$

The integral near the horizon of the r^2/f^2 term then is always, to leading order,

$$\int dr\frac{r^2}{f^2} = \frac{R^2}{(4\pi T)^2}\int_{R+b} dr\frac{1}{(r-R)^2} = \frac{R^2}{(4\pi T)^2}\frac{1}{b}. \tag{6.50}$$

Combined with the angular integration constant $2/(3\pi)$, the Stefan-Boltzman factor $\pi^4/15$, the factor of 4 from the T^4 free energy derivative gives the entropy,

$$S = \frac{2}{3\pi}\left(\frac{R^2}{(4\pi T)^2}\frac{1}{b}\right)\frac{\pi^4}{15}(4)T^3 = \pi R^2\frac{T}{90b} = \pi R^2. \tag{6.51}$$

It is easy to see that in last step, we choose to know from thermodynamics, we must have $S = \pi R^2$, so then our result gives, Eq. (6.47), $b = T/90$, where T is the Hawking temperature of the horizon R.

6.0.10 List of add-ons and upgrades in construction from 1985

- We have used SI units.
- We have shown the model works fine for a massless field.
- We have used a Laurent series to avoid vacuum terms.
- We have revealed an exact dependence of brick wall thickness on the size of the black hole.
- We have provided a dimensional argument for ω^3 dependence from equilibrium assumption.
- We have used a shortcut via Stefan-Boltzmann T^4 dependence.
- The maximum force association grouping the introduction of $\hbar$ entirely into Hawking's temperature T_H is illuminating.
- We have demonstrated that N_0 is not explicitly relativistic (at least not ostensibly), as it contains only $\hbar$ and G and the scale of the system, r_s absorbs all factors of c.
- We have simplified by not including species Z treatment, nor any λ controversy.
- We have found that the WKB extra term may more strongly restrict the physical regime of applicability.
- We have included a brick wall outside a charged black hole and generally spherically symmetric spacetimes, including a cosmological horizon of de Sitter.

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