NAZARBAYEV UNIVERSITY

MATH 499 Capstone Project

Modeling dependencies between exchange rates using time-invariant and time-varying copulas

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Abstract

In this project, the bivariate dependence structures between the Japanese Yen, Chinese Yuan, and Hong Kong Dollar exchange rates against the US Dollar are studied by using time-invariant and time-varying copulas. The period from 20.03.2010- 20.03.2020 is used for numerical simulations and marginal distributions are determined by the ARMA-tGARCH approach. The optimal models are chosen based on AIC values. Then the copulas are determined by the optimal choice of marginal distributions and finally, copulas are numerically constructed and used to describe dependencies between these three exchange-rates. Changes in the linear correlation coefficient over time are studied using time-varying copulas. The R script is provided to implement this procedure. The results suggest a positive dependence and greater lower-tail dependence between pairs of Japanese Yen-Chinese Yuan and Chinese Yuan-Hong Kong Dollar exchange rates.

1. Introduction

Due to the increased globalization of financial business and the impact of financial liberalization, foreign exchange markets become more correlated and changes in one market often cause changes in another market. The same is true for the exchange rate market. The analysis of dependencies between exchange rates is essential for governments and central banks, particularly in the presence of currency interventions, to attain a certain level of appreciation/depreciation against other foreign currencies [1]. In this project, a copula method is used to model the dependence structure between three pairs of exchange rates. This method was first introduced by Sklar in 1959. The word "copula" translates from Latin as "to bond" or "to link" [2]. Copula is used to express the joint distribution of two or more random variables through marginal distributions.

Linear correlation is one of the most popular tools to model dependence structures because of its simplicity. However, it provides adequate results only when variables are Normally distributed. For variables with asymmetric distribution and distributions with fat tails, another measure of dependence is needed.

Following Albulescu et. al. (2017) [3], this paper presents an analysis of the dependence structures associated with a bivariate distribution of Japanese Yen, Chinese Yuan, and Hong Kong Dollar. Here two measures of exchange rate dependence are considered: correlation and tail dependence.

Gaussian copula is used to model correlation, while Frank, BB1 and BB7 copulas are used to model upper and lower tail dependencies. The data spans the period of 20.03.2010- 20.03.2020. The paper proceeds as follows. Firstly, a general theory on the concepts of copula is given in the next section. Then, the description of the data set is provided in section 3. The application of copula and discussion of the results are in section 4. Conclusion and further suggestions are given in the end.

2. Theory

2.1. Concept and properties

Copula function is a powerful tool when there is a need to represent a joint distribution of random variables using their marginal distributions. Given marginal distributions, one can find a copula function that best fits the joint distribution that approximates the sample. This section discusses some basic concepts about copula. Since this work focuses on dependence between 2 exchange rates, all concepts are provided for bivariate cases.

Definition: For a map *C*: $[0,1]^2 \rightarrow [0,1]$ if the following conditions are satisfied:

1. For $\forall\, u, v \in [0,1], C\,(u, v) \geq 0$
2. For $\forall\, u, v \in [0,1], C(u, 1) = u$ and $C\,(1, v) = v$
3. For $\forall\, u_1, u_2, v_1, v_2 \in [0,1], \sum_{i=j} C\,(u_i, v_j) - \sum_{i \neq j} C\,(u_i, v_j) \geq 0$, where $u_1 \leq u_2,\ v_1 \leq v_2$

, then *C* is called a bivariate copula [4]

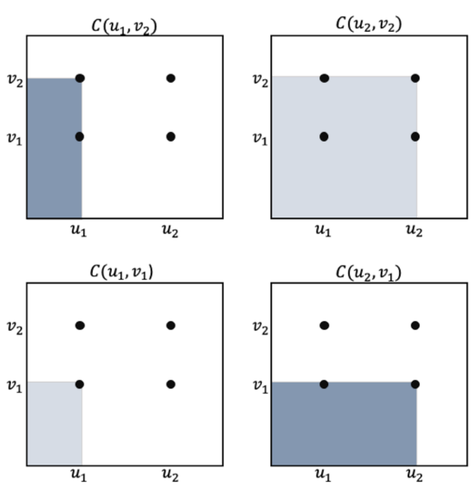

Figure 1: Rectangle inequality

C is non-decreasing in both variables; one variable can be obtained from the function by setting another variable to 1; *C* can be thought as a function that assigns a number from the unit interval to the rectangle $[0, u] \times [0, v]$, and third property states that that the number assigned to each rectangle $[u_1, u_2] \times [v_1, v_2]$ must be non-negative [8].

Every copula is bounded by its upper and lower bounds, called Frechet-Hoeffding bounds.

Theorem: For every copula C and every $(u, v) \in [0,1]^2$,

$$\max(u + v - 1, 0) \leq C(u, v) \leq \min(u, v) \tag{1}$$

The probability integral transform states that for a random variable X with continuous CDF F_X and a random variable $U = F_X(X)$, $U \sim U\,[0,1]$. Using this result, it can be seen that the joint distribution of $F_X(X)$ and $F_Y(Y)$ is a copula function.

$$C(u, v) = P(F_X(X) < u, F_Y(Y) < v) = P\left(X < F_X^{-1}(u), Y < F_Y^{-1}(v)\right) = F_{XY}\left(F_X^{-1}(u), F_Y^{-1}(v)\right) \tag{2}$$

According to Sklar's theorem, for a set of random variables (X, Y) with joint distribution $F_{XY}(x, y)$, their joint distribution can be written in terms of copulas and marginal distributions of X and Y. This result can be obtained by letting $u = F_X(x), v = F_Y(y)$ in equation (2).

Sklar's theorem: Suppose that $F_{XY}(x, y)$ is a joint distribution on R^2 with one-dimensional distributions F_X, F_Y. Then there exists a copula C such that,

$$F_{XY}(x, y) = C\left(F_X(x), F_Y(y)\right) \tag{3}$$

If $F_{XY}(x, y)$ is continuous, then C is unique. Otherwise, C is uniquely determined on $Ran(F_X) \times Ran(F_Y)$ where $Ran(F_i)$ denotes the range of F_i [4].

Copula splits the joint distribution into the marginal distributions and copula function, where the copula represents the "linkage" between marginal.

Definition: Density of the copula function $C\,(u, v)$ is given by

$$c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v} \tag{4}$$

where $u = F_X(x)$ and $v = F_Y(y)$

For continuous random variables, density of the joint distribution function can be obtained from the density of the copula using the chain rule as follows:

$$f_{XY}(x, y) = \frac{\partial^2 C(u,v)}{\partial u \partial v} \cdot \frac{dF_X(x)}{dx} \cdot \frac{dF_Y(y)}{dy} = c(u, v) \cdot f_X(x) \cdot f_Y(y) \tag{5}$$

Then, it can be seen that the density of the copula function is equal to the ratio of the density of the joint density to the product of the density of the marginal densities

$$c\left(F_X(x), F_Y(y)\right) = \frac{f_{XY}(x, y)}{f_X(x) f_Y(y)} \tag{6}$$

where $f_X(x), f_Y(y)$ are densities of X and Y variables.

Another important property of copulas is that they are invariant under increasing and continuous transformations [4].

Proposition: If (X, Y) has copula C and T_1, T_2 are increasing continuous functions, then $(T_1(X_1), T_2(X_2))$ also has copula C.

Proof [8]:

$F_{T(X)}(x) = P(T(X) < x) = P(X < T^{-1}(x)) = F_X(T^{-1}(x))$

$C_{T(X)T(Y)}\left(F_{T(X)}(x), F_{T(Y)}(y)\right) = F_{T(X)T(Y)}(x, y) = P(T(X) < x, T(Y) < y) =$

$= P\left(X < T^{-1}(x), Y < T^{-1}(y)\right) = F_{XY}\left(T^{-1}(x), T^{-1}(y)\right) = C_{XY}\left(F_X(T^{-1}(x)), F_Y\left(T^{-1}(y)\right)\right)$

$= C_{XY}\left(F_{T(X)}(x), F_{T(Y)}(y)\right)$

The later proposition is important because instead of working with exchange rates directly, their log-returns were used.

2.2. Dependence structures

2.2.1 Independence

Two random variables are independent if and only if their joint distribution can be represented as the product of its marginal distributions. In case of copulas,

Corollary: Two random variables are independent if and only if they have the product copula given by $C(u, v) = uv$

2.2.2 Co-monotonicity and Counter-monotonicity

Two random variables X and Y are said to be associated if they are not independent. X and Y are co-monotone or perfectly positively dependent if their copula is the upper Frechet-Hoeffding bound and are counter-monotone or perfectly negatively dependent if their copula is the lower Frechet-Hoeffding bound [4].

2.2.3 Linear correlation

Definition: For random variables X and Y with non-zero variances, the linear correlation coefficient is given as:

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X) * Var(Y)}} \tag{7}$$

where $Var(X) = E((X - EX)^2)$ and $Cov\,(X, Y) = E(XY) - E(X)E(Y)$

2.2.4 Tail dependence

Tail dependence is used to show the amount of dependence in the lower-left and upper-right corners of the bivariate distribution. It describes the co-movement of two random variables.

Definition: Let X, Y be random variables with CDF F_X, F_Y. Then

$$\lambda_u = \lim_{u\to 1^-} P(\mathrm{X} > F_X^{-1}(u) \mid \mathrm{Y} > F_Y^{-1}(u)) = \lim_{u\to 1^-} P(\mathrm{Y} > F_Y^{-1}(u) \mid \mathrm{X} > F_X^{-1}(u)) = \lim_{u\to 1^-} \frac{1-2u+C(u,u)}{1-u} \tag{8}$$

define upper tail dependence if and only if $\lambda_u \in [0,1]$, and

$$\lambda_l = \lim_{u\to 0^+} P(\mathrm{X} \le F_X^{-1}(u) \mid \mathrm{Y} \le u) = \lim_{u\to 0^+} P(\mathrm{Y} \le F_Y^{-1}(u) \mid \mathrm{X} \le F_X^{-1}(u)) = \lim_{u\to 0^+} \frac{C(u,u)}{u} \tag{9}$$

define lower tail dependence if and only if $\lambda_l \in [0,1]$

Equations (8) and (9) are derived as follows:

$$\lambda_u = \lim_{u\to 1^-} P(\mathrm{X} > F_X^{-1}(u) \mid \mathrm{Y} > F_Y^{-1}(u)) = \lim_{u\to 1^-} \frac{P\left(\mathrm{X} > F_X^{-1}(u), \mathrm{Y} > F_Y^{-1}(u)\right)}{P\left(\mathrm{Y} > F_Y^{-1}(u)\right)}$$

$$= \lim_{u\to 1^-} \frac{1 - P\left(\mathrm{X} \le F_X^{-1}(u)\right) - P\left(\mathrm{Y} \le F_Y^{-1}(u)\right) + P(\mathrm{X} \le F_X^{-1}(u), \mathrm{Y} \le F_Y^{-1}(u))}{1 - P\left(\mathrm{Y} \le F_Y^{-1}(u)\right)}$$

$$= \lim_{u\to 1^-} \frac{1-2u+C(u,u)}{1-u}$$

$$\lambda_l = \lim_{u\to 0^+} P(\mathrm{X} \le F_X^{-1}(u) \mid \mathrm{Y} \le u) = \lim_{u\to 0^+} \frac{P\left(\mathrm{X} \le F_X^{-1}(u), \mathrm{Y} \le F_Y^{-1}(u)\right)}{P\left(\mathrm{Y} \le F_Y^{-1}(u)\right)} = \lim_{u\to 0^+} \frac{C(u,u)}{u}$$

Two random variables are asymptotically independent in upper tail if $\lambda_u = 0$ and are asymptotically independent in lower tail if $\lambda_l = 0$ [4].

2.3. Models for the copula

This paper focuses on Gaussian, BB1, BB7 and Frank copulas with time-invariant and Gaussian copula with time-varying parameter.

Bivariate Gaussian copula with Time-invariant parameter

The bivariate Gaussian copula is defined as:

$$C_{Ga}(u, v|\rho_{XY}) = \int_0^u \int_0^v \frac{1}{\sqrt{1-\rho_{XY}^2}} exp\left\{\frac{2\rho_{XY}xy - x^2\rho_{XY}^2 - y^2\rho_{XY}^2}{2(1-\rho_{XY}^2)}\right\} \tag{10}$$

Bivariate Gaussian copula can be obtained by the following procedure:

Assume $(X, Y) \sim BiNorm\ (0, \Sigma)$, where Σ is a correlation matrix defined as $\Sigma = \begin{bmatrix} 1 & \rho_{XY} \\ \rho_{XY} & 1 \end{bmatrix}$ [10].

Then PDF of (X, Y) is given by:

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{1-\rho_{XY}^2}} exp\left\{\frac{x^2 - 2\rho_{XY}xy + y^2}{2(1-\rho_{XY}^2)}\right\} \tag{11}$$

Theorem: Let (X, Y) have bivariate normal distribution. Then the marginal distributions of X and Y are $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$ [10].

From the theorem above, it can be seen that X and Y have standard univariate normal distributions. Then applying equation (2), copula function can be written as [4]:

$$C(u, v) = \Phi_{XY}\left(\Phi_X^{-1}(x), \Phi_Y^{-1}(y)\right) \tag{12}$$

which is another way of expressing (10).

Probability density function of Gaussian copula can be obtained directly from (6) and is given by:

$$c_{Ga}(u, v|\, \rho_{XY}) = \frac{1}{\sqrt{1-\rho_{XY}{}^2}} \exp\left\{-\frac{x^2\rho_{XY}^2 + 2\rho_{XY}xy + y^2\rho_{XY}^2)}{2(1-\rho_{XY}^2)}\right\} \tag{13}$$

where $x = \Phi_X^{-1}(u)$, $y = \Phi_Y^{-1}(v)$, Φ_i is standard normal distribution function and ρ_{XY} is a correlation coefficient between X and Y.

Property: For Gaussian Copula, $\rho_{XY} \in [-1,1]$, where $\rho_{XY} = -1$ corresponds to counter-monotonicity and $\rho_{XY} = 1$ corresponds to co-monotonicity. $\rho_{XY} = 0$ implies independence. [4]

The Gaussian copula does not generate a standard normal distribution function for any other marginal distribution except standard normal and expresses no tail dependence unless $\rho_{XY} = \pm 1$. It generates both positively quadrant dependence and negatively quadrant dependence [4].

Archimedean copula

Definition: A continuous, decreasing, convex function $\varphi: I \longrightarrow R$ with $\varphi(1) = 0$ is called a generator function. φ is a strict generator if $\varphi(0) = +\infty$ [4].

Definition: Given a function $\varphi$and its pseudo-inverse, an Archimedean copula is defined as follows

$$C(u, v) = \varphi^{-1}\left(\varphi(u) + \varphi(v)\right) \tag{14}$$

Archimedean copulas that were useful for this project are provided in table 1 [11].

Table 1: Archimedean copulas with specified parameter ranges and generator functions

Name	α	$\varphi(t)$	Copula function
Gumbel	$[1,\infty)$	$exp\left\{-t^{\frac{1}{\alpha}}\right\}$	$\exp\left\{-((-\ln u)^{\alpha}+(-\ln v)^{\alpha})^{\frac{1}{\alpha}}\right\}$
Clayton	$(0,\infty)$	$(1+t)^{-\frac{1}{\alpha}}$	$(u^{-\alpha}+v^{-\alpha}-1)^{-\frac{1}{\alpha}}$
Frank	$(0,\infty)$	$-\frac{[\log(e^{-t}(e^{-\alpha}-1)+1)]}{\alpha}$	$-\frac{1}{\alpha}\ln\frac{(1-e^{-\alpha})-(1-e^{-u\alpha})(1-e^{-v\alpha})}{2}$
Joe	$[1,\infty)$	$1-(1-e^{-t})^{\frac{1}{\alpha}}$	$1-(u^{-\alpha}+v^{-\alpha}-u^{-\alpha}v^{-\alpha})^{\frac{1}{\alpha}}$

Bivariate BB1, BB7 and Frank copulas with Time-invariant parameters

The following two copulas were obtained by mixing two other copulas from Archimedean family mentioned above. BB1 is a mix of Clayton and Gumbel copulas.

For the Gumbel copula, the degree of dependence in the upper tail is higher than in the lower, while the opposite is true for the Clayton copula. BB1 copula is useful since it combines both extreme cases of those two copulas. BB1 copula is defined as:

$$C_{BB1}(u,v) = \left(1+\left[\left(u^{-\theta}-1\right)^{\delta}+\left(v^{-\theta}-1\right)^{\delta}\right]^{\frac{1}{\delta}}\right)^{-\frac{1}{\theta}} \tag{15}$$

where $\delta \geq 1$ is the parameter corresponding to Clayton copula and $\theta > 1$is the parameter corresponding to Gumbel copula [5].

The upper and lower tail dependencies of BB1 copula are given by [6]:

$$\lambda_l = 2^{-\frac{1}{\theta\delta}}, \lambda_u = 2- 2^{\frac{1}{\delta}} \tag{16}$$

BB7 is another mixture of two copulas: Joe and Clayton, which is given by:

$$C_{BB7}(u,v) = 1- [1-((1-u^{\theta})^{-\delta}+(1-v)^{-\delta}-1)^{-1/\delta}]^{1/\theta} \tag{17}$$

where $\theta \geq 1, \delta > 0$ [5].

This copula expresses upper and lower tail with the following coefficients [6]:

$$\lambda_l = 2^{-\frac{1}{\delta}}, \lambda_u = 2- 2^{\frac{1}{\theta}} \tag{18}$$

Frank copula is another member of Archimedean class given by:

$$C_F(u,v) = -\frac{1}{\theta}\ln\left[\frac{\left(1-e^{-\theta}\right)-\left(1-e^{-\theta u}\right)\left(1-e^{-\theta v}\right)}{2}\right] \qquad -\infty < \theta < \infty \tag{19}$$

Frank copula does not express any tail dependence [4].

Bivariate Gaussian copula with Time-varying parameter

For bivariate cases, Patton (2001) extended the concept of standard copula to the conditional copula to model time-varying dependencies. The Sklar's theorem for the conditional bivariate copula is given as follows:

$$F_t(x_t, y_t | \mathfrak{I}_t) = C_t\left(F_{X_t}(x_t|\mathfrak{I}_t), F_Y(y_t|\mathfrak{I}_t)|\mathfrak{I}_t\right) \tag{20}$$

$\mathfrak{I}_t = \sigma\{x_{t-1}, y_{t-1}, x_{t-2}, y_{t-2}, \dots\}$ is a sigma algebra generated by all past observations. Each marginal and copula must have the same conditional set.

The joint distribution of (X_t, Y_t) is assumed to be the same for $t = 1, \dots T$, while conditional means evolve according to autoregressive process and conditional variances evolve according to tGARCH model [5]. The form of conditional copula is also fixed, but parameters evolve. For Gaussian copula, the only parameter is linear correlation. Patton proposed that the linear correlation evolves in time according to autoregressive (AR) moving average (MA) process, ARMA (1, q) as:

$$\rho_t = \Lambda\left(\varphi_0 + \varphi_1\rho_{t-1} + \varphi_3\frac{1}{q}\sum_{i=1}^{q}\Phi^{-1}(u_{t-i})\,\Phi^{-1}(v_{t-i})\right) \tag{21}$$

where the transformation $\Lambda(x) = \frac{1-e^{-x}}{1+e^{-x}}$ is necessary to keep $\rho_t \in (-1, 1)$.

The regression term ρ_t captures changes in the dependence parameter and average sum $\frac{1}{q}\sum_{i=1}^{q}\Phi^{-1}(u_{t-i})\,\Phi^{-1}(v_{t-i})$ captures variation in the dependence in the last q days. In this project, q = 10 [4,12].

2.4 Models of marginal distribution

Marginal distributions F_X and F_Y are modeled using ARMA (p, q) – tGARCH (n, m) model with skewed t-distribution. Before moving to ARMA (p, q) model, the revision of AR (p) and MA(q) models was done.

Autoregressive term AR (1) is given by

$$r_t = \alpha + \lambda r_{t-1} + \varepsilon_t \tag{22}$$

where r_t- depends only on the first lag, $\alpha, \lambda$are constants [7].

The model can produce a stable time-series for $|\lambda| < 1$

AR(p) is given by

$$r_t = \alpha + \sum_{i=1}^{p} \lambda_i r_{t-i} + \varepsilon_t \tag{23}$$

where λ_i are constants with $|\lambda_i|$< 1 for a stable time-series, ε_t is the disturbance term at time t.

Moving average, MA (q), is another important model in time-series. It is given by:

$$r_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_n \varepsilon_{t-q} \tag{24}$$

In combination with AR (p), ARMA (p, q) model is formed:

$$r_t = \alpha + \sum_{i=1}^{p} \lambda_i r_{t-i} + \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} \tag{25}$$

Let $x_t$represent an observation generated from $F_X$at time t. Then the mean equation of x_t is given by:

$$x_t = E(x_t|\Im_t) + \varepsilon_t \tag{26}$$

, where $E(x_t|\Im_t)$ is a conditional mean of x_t given $\Im_t$. In order to model conditional mean, ARMA (p, q) model was used. However, ARMA model is limited because it assumes that the disturbance term has a constant variance. To model conditional variance generalized autoregressive conditional heteroscedasticity (GARCH) model was used [13].

The disturbance terms are taken from ARMA equation and defined such that

$$\varepsilon_t = \sigma_t \eta_t \tag{27}$$

where η_t is a sequence of independent identically distributed skewed student-t random variables with ϑ degrees of freedom, σ_t is conditional variance of error that changes over time according to:

$$\sigma_t^2 = \omega + \sum_{i=1}^{n} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{m} \beta_i \sigma_{t-i}^2 \tag{28}$$

where $\alpha_i$and $\beta_i$are constants, $\varepsilon_{t-i}$is responsible for the lags of disturbance, $\sigma_{t-i}^2$is responsible for the lags of variance itself [13].

Then the sequence ε_t is a sequence of IID random variables

$$\sqrt{\frac{\vartheta}{\sigma_t^2(\vartheta-2)}}\varepsilon_t \sim iid\ t \tag{29}$$

Overall, the model is called ARMA-tGARCH model, where ARMA is responsible for modeling conditional mean and tGARCH is responsible for modeling conditional variance of the disturbance term.

Numbers of lags p, q for ARMA(p,q) and n, m for GARCH (n, m) models are selected according to Akaike Information Criteria (AIC). The preferred model is the one that has the lowest AIC value [3]. The equation used to calculate AIC value is provided in the next section.

2.5 Parameters Estimation

Statistical modeling problem for copula can be decomposed in two steps:

1. Estimation of the marginal distributions
2. Choosing appropriate copula function

From the density of the joint distribution function specified in (5), log-likelihood function is obtained [5]:

$$l(\hat{\theta}) = \sum_{t=1}^{N} \ln c\big(F_X(x_t), F_Y(y_t); \hat{\theta}\big) + \sum_{t=1}^{N} \ln\,(f_X(x_t;\ \hat{\theta}) + \sum_{t=1}^{N} \ln\,(f_Y\big(y_t;\ \hat{\theta}\big) \qquad (30)$$

where $\hat{\theta}$ is set of all parameters of both the marginal distributions and copula function and N is the total number of observations. Since it is hard to use direct maximum likelihood approach when there are too many unknown parameters (parameters of marginal distributions from ARMA-tGARCH, parameters of copula), the procedure which separates the estimation of parameters of marginals and estimation of copula's parameters were used.

The procedure which separates the estimation of the marginal and estimation of the copula was proposed by Joe and Xu (1996) [5].

1. At first step, parameters of the marginal distributions are estimated according to:

$$\hat{\theta}_1 = Argmax\left(\sum_{t=1}^{T} \ln\,(f_X(x_t; \hat{\theta}_1)\ + \sum_{t=1}^{T} \ln\,(f_Y(y_t;\ \hat{\theta}_1)\right) \qquad (31)$$

   where $\hat{\theta}_1$is the set of all parameters of the marginal distribution

2. As the second step, given $\hat{\theta}_1$ from the previous step, parameters of the copula are estimated as:

$$\hat{\theta}_2 = Argmax\left(\sum_{t=1}^{T} \ln c\,\big(F_X\big(x_t;\ \hat{\theta}_1\big), F_Y\big(y_t;\ \hat{\theta}_1\big);\ \hat{\theta}_2\big)\right) \qquad (32)$$

The method is called Inference for the margins (IFM). For the goodness-of-fit, use AIC criteria for large sample was used.

$$AIC =\ 2k - 2\log(L) \qquad (33)$$

where k- number of parameters estimated by the model and L- maximized value of the log-likelihood function. The AIC value shows the amount of information lost by the estimated model and the less information is lost, the higher is the quality of the model.

3. Data

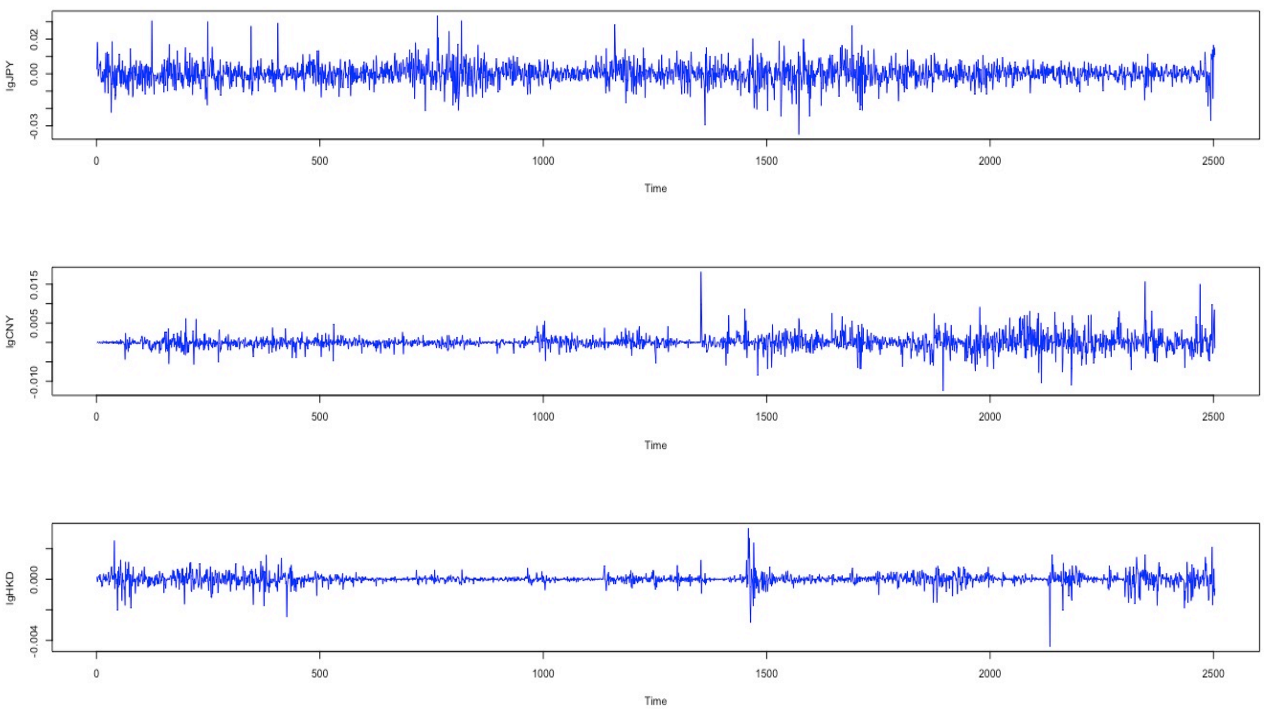

Figure 2: Exchange rate returns for JPY, CNY and HKD against USD

Table 2: Summary statistics for exchange rate returns

	JPY	CNY	HKD
Mean	$8.478 * 10^{-5}$	$1.541 * 10^{-5}$	$2.830 * 10^{-7}$
Median	$8.878 * 10^{-5}$	0.0000	$1.283 * 10^{-5}$
Maximum	$3.343 * 10^{-2}$	$1.816 * 10^{-2}$	$3.345 * 10^{-3}$
Minimum	$-3.498 * 10^{-2}$	$-1.242 * 10^{-2}$	$-4.422 * 10^{-3}$
St. Dev.	0.0058	0.0019	0.0004
Skewness	0.1122	0.8427	-0.8098
Kurtosis	3.9859	11.2923	18.8399
Jarque-Bera	1666.9*	13622*	37360*
p-value	0.0000	0.0000	0.0000

Notes to Table2: The table provides summary statistics of daily returns for the period 20.03.2010-20.03.2020. An asterisk (*) indicates rejection of null-hypothesis (data is normally distributed) at the 1% significance level.

Table 3: Linear pairwise correlations

	JPY	**CNY**	**HKD**
JPY	1	0.0922	-0.0050
CNY		1	0.1772
HKD			1

Data set consists of daily returns of three exchange rates against U.S. Dollar: Japanese Yen (JPY), Chinese Yuan (CNY), Hong Kong Dollar (HKD). Data for exchange rates was extracted from the FRED database (Federal Reserve Bank of St. Louis). Observations span period March 10, 2014 to March 20, 2020. Days when at least one of foreign exchange markets did not provide data were not considered. Daily exchange returns are calculated as the difference between log-values of two consecutive exchange rates.

Figure 2 shows plots of exchange rate return series. Table 2 and Table 3 provide summary statistics for log-returns of exchange rates. Median and mean are approximately zero for all three cases. Greater volatility is observed in JPY. JPY and CNY exchange rate returns exhibit positive skewness meaning that the distribution has longer right tails, while HKD has longer left tails and negative skewness. Positive value of skewness reflects the probability of large increases in exchange rate returns. High value of kurtosis statistics indicates heavy tails and means that probabilities of extreme returns are high. The highest kurtosis among 3 exchange rate returns was observed for HKD. Jarque-Bera test for normality is a goodness-of-fit test, which indicates if returns are normally distributed. For all series, the Jarque-Bera test of normality rejects unconditional normality at 1% significance level. Linear correlation matrices show positive correlation between all pairs of exchange rate returns except JPY and HKD. CNY-HKD pairs exhibit the highest degree of correlation.

4. Application

Results for models of marginal distribution

Marginal distributions of exchange rate returns are generated numerically by selecting the ARMA (p, q)-GARCH (n, m) model that fits the data best according to the AIC values. p and q ranged from 1 to 4 and 0 to 3 respectively; n and m ranged from 1 to 4 and 0 to 2 respectively. For each of exchange rate returns, a model which minimizes AIC value was chosen.

Table 4: Parameters estimates for the marginal distributions model

	JPY	**CNY**	**HKD**
Mean model			
α	$7.9327 * 10^{-5}$	$-7.0065 * 10^{-6}$	$-2.8218 * 10^{-7}$
λ_1	$-4.3303 * 10^{-5}$	$-2.3721 * 10^{-2}$	$5.4843 * 10^{-2}$
λ_2	$-4.4818 * 10^{-3}$	$3.7665 * 10^{-4}$	$-1.5868 * 10^{-2}$
λ_3	$1.9299 * 10^{-2}$	$2.8556 * 10^{-2}$	$-4.3329 * 10^{-2}$
Variance model			
ω	$3.7806 * 10^{-7}$	$2.7552 * 10^{-6}$	$1.4060 * 10^{-10}$
α_1	$5.4429 * 10^{-2}$	$9.9810 * 10^{-1}$	$5.0000 * 10^{-2}$
β_1	$9.3647 * 10^{-1}$		$4.5000 * 10^{-1}$
β_2			$4.5000 * 10^{-1}$
Ljung-Box test	28.967 [0.0884]	35.875 [0.0159]	33.419 [0.0303]
JB	1674.3 [0.00]	13937 [0.00]	35963 [0.00]

Notes: the p-value for the parameters of ARMA-GARCH are given in parentheses. Ljung-box test for autocorrelations was computed with 20 lags.

The optimal models are ARMA (3,0)-GARCH (1,1) for JPY, ARMA (3,0)-GARCH (1,0) for CNY, ARMA (3,0)-GARCH (1,2) for HKD. Parameters for the models are provided in Table 4 above.

Obtained marginal models show evidences of fat tails. The goodness-of-fit analysis was conducted on the basis of Ljung-Box test. The null-hypothesis states that residuals are independent and was failed to reject for all cases at the 1% level, which concludes that the autocorrelations in residuals of the marginal model are zero. The Jarque-Bera test for normality indicates non-normal distribution of residuals.

Discussion of the Stationary copula results

Parametric estimates of Gaussian, BB1, BB7 and Frank copulas are presented in Table 5. Parameters of the bivariate copulas were estimated by using maximum likelihood approach.

Table 5: Parameter estimates for the stationary copula models

	JPY-CNY	**JPY-HKD**	**CNY-HKD**
Gaussian copula			
ρ_{XY}	0.097* (<0.01)	-0.0047 (0.0765)	0.185* (<0.01)
AIC	-20.719	1.947	-76.512
	BB1 copula	**Frank copula**	**BB7 copula**
θ	0.0330* (<0.01)	0.2046 (0.0765)	1.0959* (<0.01)
δ	1.0467* (<0.01)		0.1106* (<0.01)
AIC	-27.691	-0.830	-87.917
λ_u	$1.9699 * 10^{-9}$	0	0.0019
λ_l	0.061	0	0.1177

Note: * indicates significance at 5% level

Before estimating parameters of the Gaussian copula, marginal distributions were transferred so that they will follow normal distribution. The estimated values of ρ_{XY} are close to those presented in the linear pairwise correlations table. Japanese Yen-Chinese Yuan and Chinese Yuan- Hong Kong Dollar pairs present evidences of positive dependence, with a stronger dependence for CNY-HKD pair. Meanwhile, Japanese Yen- Hong Kong Dollar shows negative dependence.

The values of upper and lower tails give us more information about dependencies in extreme cases. Since margins of the Copula must be uniform, marginal distributions were transferred to follow uniform distribution. Copula model for each pair of exchange rates was chosen using “BiCopSelect” function of “VineCopula” package in R using maximum likelihood estimation. The best copula was chosen based on AIC value. For JPY-CNY, JPY-HKD and CNY-HKD the following models were obtained: BB1 copula, Frank copula and BB7 copula. For both JPY-CNY and CNY-HKD pairs, the value of $\lambda_l$is greater than the value of λ_u. This means that those pairs of foreign currencies are more likely to appreciate together against US Dollar, than to depreciate [1]. Frank copula exhibits no tail dependence.

Discussion of the Time-varying Gaussian copula results

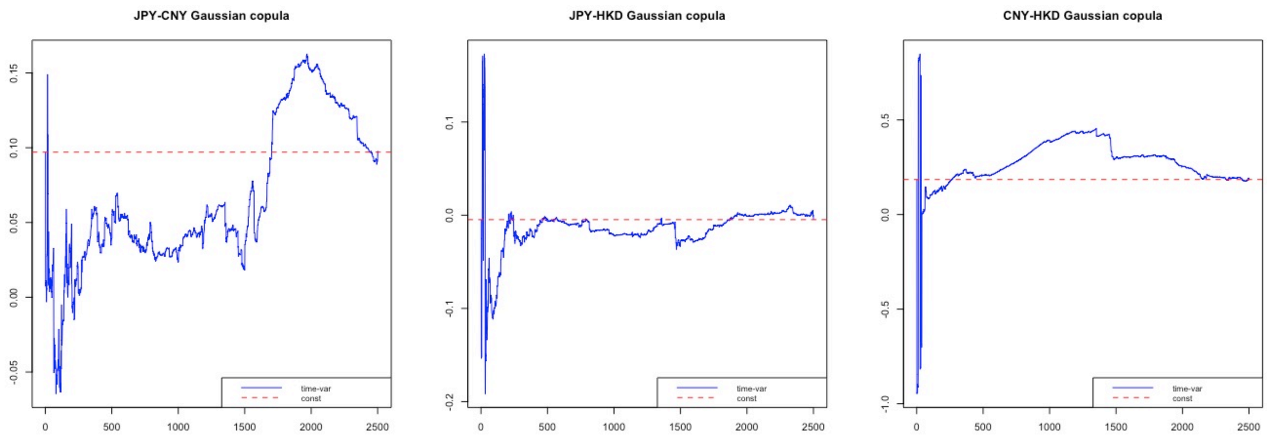

Figure 3: Time-varying correlation for Gaussian copula. X-axis represents the time period from 20.03.2010 to 20.03.2020 (x = 1000 corresponds to 14.03.14, x = 1500 corresponds to 14.03.16, x = 2000 corresponds to 13.03.2018).

Figure 3 shows the time-dynamics of the linear correlation between exchange rates during the sampling period. X-axis represents time period, where each number corresponds to a date when data was taken. Evolution of the correlation can be affected by several reasons such as trade between two countries, capital flow or investments. For instance, increase in correlation between JPY-CNY pair might have happened due to the improvement in China-Japan relationships during China-United State trade war. The same reason might be responsible for the decrease in CNY-HKD correlation in 2019.

5. Conclusion

In this project, dependence between Japanese Yen, Chinese Yuan, Hong Kong Dollar exchange rates against US dollar were analyzed using time-invariant and time-varying copulas. Gaussian copula was used to model linear correlation, while BB1, BB7, and Frank copulas were used to model upper and lower tail dependence. Copula function allowed to construct joint distribution of two exchange rates using their marginal distributions that were obtained using ARMA-tGARCH model. The results indicate positive dependence between pairs of JPY-CNY and CNY-HKD, and negative dependence between JPY-HKD pair. Tail dependence coefficient showed that JPY-CNY, CNY-HKD pairs are more correlated in the lower tail of their joint distribution. Finally, changes in linear correlation coefficient can be associated with several factors such as trade and cash-flow between countries.

6. Suggestions for further research

This work can be extended in many ways. One of options is to increase the number of implemented copulas and use another measure of dependency. Also, since Frank copula does not exhibit tail dependence, another copula can be used to model tail dependence between JPY-HKD pair of exchange rates. Lastly, one may consider other copulas with time-varying coefficients besides Gaussian copula.

R script

```
library(readxl)
DataExc <- read_excel("~/Desktop/DataExc.xls")
mean(DataExc$JPY)
mean(DataExc$CNY)
mean(DataExc$HKD)
n<-length(DataExc$JPY)

#statistics summary
lgJPY<-diff(log(DataExc$JPY),lag = 1)
summary(lgJPY)
lgCNY<-diff(log(exchrates$RU),lag = 1)
summary(lgCNY)
lgHKD<-diff(log(exchrates$CN),lag=1)
summary(lgHKD)

sd(lgJPY)
sd(lgCNY)
sd(lgHKD)

library(e1071)
skewness(lgJPY)
skewness(lgCNY)
skewness(lgHKD)
kurtosis(lgJPY)
kurtosis(lgCNY)
kurtosis(lgHKD)

cor(lgJPY,lgCNY,method="pearson")
cor(lgJPY,lgHKD,method="pearson")
cor(lgCNY,lgHKD,method="pearson")

par(mfrow=c(2,2))
plot.ts(lgJPY, col='blue')
plot.ts(lgCNY, col='blue')
plot.ts(lgHKD, col='blue')

install.packages("normtest")
library(normtest)
```

```
jb.norm.test(lgJPY,nrepl=2000)
jb.norm.test(lgCNY,nrepl = 2000)
jb.norm.test(lgHKD,nrepl = 2000)

#marginal models
#ARMA-GARCH for JPY
library(aTSA)
library(forecast)
library(rugarch)

JPY<-as.numeric(lgJPY)
JPY<-JPY[!is.na(JPY)]
JPYarma.aicfin<-Inf
JPYarma.orderfin<-c(0,0)
for (p in 1:4) {
    for (q in 0:3) {
        JPYarma.aiccur<-AIC(arima(JPY,order=c(p,q,0), optim.control=list(maxit=1000)))
        if (JPYarma.aiccur<JPYarma.aicfin) {
            JPYarma.aicfin<-JPYarma.aiccur
            JPYarma.orderfin<-c(p,q)
            JPYarma<-arima(JPY,order = c(p,q,0))
        }
    }
}
JPYarma.orderfin

JPYgarch.order<-c(0,0)
JPYgarch.aicfin<-Inf

for (n in 1:4) {
    for (m in 0:2) {
        JPYgarch.model<-ugarchspec(variance.model=list(model="sGARCH",
garchOrder=c(n,m)), mean.model = list(armaOrder=c(3,0), distribution.model = “sstd”)
        JPYgarch.fit<-ugarchfit(JPYgarch.model, data = JPY, out.sample = 0)
        JPYgarch.aiccur<-infocriteria(JPYgarch.fit)[1]
        if (JPYgarch.aiccur<JPYgarch.aicfin) {
            JPYgarch.aicfin<-JPYgarch.aiccur
            JPYgarch.order<-c(n,m)
        }
    }
```

```
}

JPYgarch.m<-ugarchspec(variance.model = list(model="sGARCH", garchOrder=c(1,1)),
mean.model = list(armaOrder=c(3,0)), distribution.model = "sstd")
JPYgarch<-ugarchfit(JPYgarch.m, data=JPY, out.sample = 0)
JPYgarch.resid<-residuals(JPYgarch)
Box.test(JPYgarch.resid, lag=20, type = "Ljung-Box")
JPYgarch@fit[["coef"]]
jb.norm.test(JPYgarch.resid, nrepl = 2000)

#ARMA-GARCH for CNY
CNY<-as.numeric(lgCNY)
CNY<-CNY[!is.na(CNY)]
CNYarma.aicfin<-Inf
CNYarma.orderfin<-c(0,0)
for (p in 1:4) {
  for (q in 0:3) {
    CNYarma.aiccur<-AIC(arima(CNY,order=c(p,q,0), optim.control=list(maxit=1000)))
    if (CNYarma.aiccur<CNYarma.aicfin) {
      CNYarma.aicfin<-CNYarma.aiccur
      CNYarma.orderfin<-c(p,q)
      CNYarma<-arima(CNY,order = c(p,q,0))
    }
  }
}
CNYarma.orderfin

CNYgarch.order<-c(0,0)
CNYgarch.aicfin<-Inf

for (n in 1:4) {
  for (m in 0:2) {
    CNYgarch.model<-ugarchspec(variance.model=list(model="sGARCH",
garchOrder=c(n,m)), mean.model = list(armaOrder=c(3,0), distribution.model = “sstd”)
    CNYgarch.fit<-ugarchfit(CNYgarch.model, data = CNY, out.sample = 0)
    CNYgarch.aiccur<-infocriteria(CNYgarch.fit)[1]
    if (CNYgarch.aiccur<CNYgarch.aicfin) {
      CNYgarch.aicfin<-CNYgarch.aiccur
      CNYgarch.order<-c(n,m)
    }
```

```
    }
}

CNYgarch.m<-ugarchspec(variance.model = list(model="sGARCH", garchOrder=c(1,0)),
mean.model = list(armaOrder=c(3,0)), distribution.model = "sstd")
CNYgarch<-ugarchfit(CNYgarch.m, data=CNY, out.sample = 0)
CNYgarch.resid<-residuals(CNYgarch)
Box.test(CNYgarch.resid, lag=20, type = "Ljung-Box")
CNYgarch@fit[["coef"]]
jb.norm.test(CNYgarch.resid, nrepl = 2000)

#ARMA-GARCH for HKD
HKD<-as.numeric(lgHKD)
HKD<-HKD[!is.na(HKD)]
HKDarma.aicfin<-Inf
HKDarma.orderfin<-c(0,0)
for (p in 1:4) {
    for (q in 0:3) {
        HKDarma.aiccur<-AIC(arima(HKD,order=c(p,q,0), optim.control=list(maxit=1000)))
        if (HKDarma.aiccur<HKDarma.aicfin) {
            HKDarma.aicfin<-HKDarma.aiccur
            HKDarma.orderfin<-c(p,q)
            HKDarma<-arima(HKD,order = c(p,q,0))
        }
    }
 }
HKDarma.orderfin

HKDgarch.order<-c(0,0)
HKDgarch.aicfin<-Inf

for (n in 1:4) {
    for (m in 0:2) {
        HKDgarch.model<-ugarchspec(variance.model=list(model="sGARCH",
garchOrder=c(n,m)), mean.model = list(armaOrder=c(3,0), distribution.model = “sstd”)
        HKDgarch.fit<-ugarchfit(HKDgarch.model, data = HKD, out.sample = 0)
        HKDgarch.aiccur<-infocriteria(HKDgarch.fit)[1]
        if (HKDgarch.aiccur<HKDgarch.aicfin) {
            HKDgarch.aicfin<-HKDgarch.aiccur
            HKDgarch.order<-c(n,m)
```

```
        }
    }
}

HKDgarch.m<-ugarchspec(variance.model = list(model="sGARCH", garchOrder=c(1,2)),
mean.model = list(armaOrder=c(3,0)), distribution.model = "sstd")
HKDgarch<-ugarchfit(HKDgarch.m, data=HKD, out.sample = 0)
HKDgarch.resid<-residuals(HKDgarch)
Box.test(HKDgarch.resid, lag=20, type = "Ljung-Box")
HKDgarch@fit[["coef"]]
jb.norm.test(HKDgarch.resid, nrepl = 2000)

#Fitting time-invariant Gaussian copula
> library(copula)
> library(VineCopula)
jp<-rstandard(lm(JPY~JPYgarch@fit$fitted.values))
ch<-rstandard(lm(CNY~CNYgarch@fit$fitted.values))
hk<-rstandard(lm(HKD~HKDgarch@fit$fitted.values))
jp<-pnorm(jp)
ch<-pnorm(ch)
hk<-pnorm(hk)
gauss1<-BiCopEst(jp,ch,family = 1, method = "mle")
summary(gauss1)
gauss2<-BiCopEst(jp,hk,family = 1, method = "mle")
summary(gauss2)
gauss3<-BiCopEst(ch,hk,family = 1, method = "mle")
summary(gauss3)

#Fitting time-invariant BB1, BB7 and Frank copulas
hk1<-rstandard(lm(HKD~HKDgarch@fit$fitted.values))
hk1<-to.uniform(hk1)
library(tiger)
jp1<-rstandard(lm(JPY~JPYgarch@fit$fitted.values))
jp1<-to.uniform(jp1)
ch1<-rstandard(lm(CNY~CNYgarch@fit$fitted.values))
ch1<-to.uniform(ch1)
sel1<-BiCopSelect(jp1,ch1, familyset = NA, selectioncrit = "AIC")
sel1
sel2<-BiCopSelect(jp1,hk1, familyset = NA, selectioncrit = "AIC")
sel2
```

```
sel3<-BiCopSelect(ch1,hk1, familyset = NA, selectioncrit = "AIC")
sel3

#Time-varying Gaussian copula
n<-length(jp)
jccor<-vector('numeric', n)
for (i in 3:n) {
    b<-vector('numeric',i)
    c<-vector('numeric',i)
    b[1]= jp[1]
    b[2]= jp[2]
    c[1]=ch[1]
    c[2]=ch[2]
    for (j in 3:i) {
        b[j]=jp[j]
        c[j]=ch[j]
    }
    g<-BiCopEst(b,c,family=1,method="mle")
    d<-g$par
    jccor[i]<-d
 }
jc.timevar.cor<-vector('numeric', n-1)
for (i in 2:n-1) {
    jc.timevar.cor[i]=jccor[i+1]
 }
 jc.timevar.cor[1]=gauss1$par

library(tseries)
jpch<-arma(jc.timevar.cor, order = c(1,10))

chcor<-vector('numeric', n)
rm(b,c)
for (i in 3:n) {
    b<-vector('numeric',i)
    c<-vector('numeric',i)
    b[1]= ch[1]
    b[2]= ch[2]
    c[1]=hk[1]
    c[2]=hk[2]
    for (j in 3:i) {
```

```
        b[j]=ch[j]
        c[j]=hk[j]
    }
    g<-BiCopEst(b,c,family=1,method="mle")
    d<-g$par
    chcor[i]<-d
 }
 ch.timevar.cor<-vector('numeric', n-1)
 for (i in 2:n-1) {
    ch.timevar.cor[i]=chcor[i+1]
 }
ch.timevar.cor[1]=gauss3$par
chhk<-arma(ch.timevar.cor, order = c(1,10))

jhcor<-vector('numeric', n)
 rm(b,c)
 for (i in 4:n) {
    b<-vector('numeric',i)
    c<-vector('numeric',i)
    b[1]= jp[1]
    b[2]= jp[2]
    b[3]=jp[3]
    c[1]=hk[1]
    c[2]=hk[2]
    c[3]=hk[3]
    for (j in 4:i) {
        b[j]=jp[j]
        c[j]=hk[j]
    }
    g<-BiCopEst(b,c,family=1,method="mle")
    d<-g$par
    jhcor[i]<-d
 }
jh.timevar.cor<-vector('numeric', n-2)
for (i in 2:n-2) {
    jh.timevar.cor[i]=jhcor[i+2]
 }
 jh.timevar.cor[1]=gauss2$par
jphk<-arma(jh.timevar.cor, order = c(1,10))
```

```
par(mfrow=c(2,2))
 plot(seq.int(1,2502), jc.timevar.cor, main="JPY-CNY Gaussian copula", xlab="", ylab="", type="l", col="blue")
 abline(h=gauss1$par,lty=2,col="red")
 legend("bottomright", legend = c("time-var", "const"), col = c("blue", "red"), lty = 1:2, cex = 0.8)
 plot(seq.int(1,2501), jh.timevar.cor, main="JPY-HKD Gaussian copula", xlab="", ylab="", type="l", col="blue")
 abline(h=gauss2$par,lty=2,col="red")
 legend("bottomright", legend = c("time-var", "const"), col = c("blue", "red"), lty = 1:2, cex = 0.8)
 plot(seq.int(1,2502), ch.timevar.cor, main="CNY-HKD Gaussian copula", xlab="", ylab="", type="l", col="blue")
 abline(h=gauss3$par,lty=2,col="red")
 legend("bottomright", legend = c("time-var", "const"), col = c("blue", "red"), lty = 1:2, cex = 0.8)
```

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