(P,Q)-SUB-LAPLACIANS ON THE HEISENBERG GROUP

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Abstract. The main purpose of this capstone project is to do some analysis related to (p,q)-sub-Laplacians on the Heisenberg group. In the first part of the project, Green's identities for (p,q)-sub-Laplacians are given on the Heisenberg group and used further in proof of the uniqueness of a weak solution of a nonlinear Dirichlet boundary value problem for the (p,q)-sub-Laplacian. Moreover, concepts of CC and Kaplan balls are discussed to illustrate the smoothness of the considered domain for the BVP.

1. Introduction

Group, G is a set given with the binary operation, $(G, \circ)$ which satisfies some so called *Group axioms*:

- **Closure** : $\forall a, b \in G,$ the result of operation $a \circ b \in G$;
- **Associativity** : $\forall a, b, c \in G, (a \circ b) \circ c = a \circ (b \circ c)$;
- **Identity element** : $\exists e \in G,$ such that for every element in $G, e \circ a = a \circ e = a$ holds;
- **Inverse element** : For each $a \in G, \exists b \in G,$ denoted as a^{-1}, such that $a \circ b = c \circ a = e,$ where e is the identity element.

The Heisenberg group is $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ)$. We use the notation

$$\xi := (z, t) = (z_1, z_2, ..., z_n, t) = (x_1, y_1, ..., x_n, y_n, t)$$

for the points of $\mathbb{H}^n$ (see, e.g. [2]). In simpler case when $n = 1$ group law:

$$(x_1, y_1, t_1) \circ (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(y_1x_2 - x_1y_2)),$$

and dilation rule:

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t), \quad (x, y, t) \in \mathbb{R}^3$$

are defined as given above. The Laplace operator in classical (commutative) analysis is defined as:

$$\Delta := \nabla \cdot \nabla = \sum_{j=1}^{n} \frac{\partial^2}{\partial^2 x_j}, \quad (x_1, ..., x_n) \in \mathbb{R}^n$$

where ∇ is known as gradient:

$$\nabla := \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \right). \tag{1.1}$$

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Whereas on the Heisenberg group the sub-Laplacian operator $\mathcal{L}$ on $\mathbb{H}^n$ is defined as follows:

$$\mathcal{L} := \sum_{j=1}^{n} \left(X_j^2 + Y_j^2\right),$$

where X_j and Y_j are left invariant (with respect to the group law) vector fields:

$$X_j = \partial_{x_j} + 2y_j\partial_t, \quad Y_j = \partial_{y_j} - 2x_j\partial_t.$$

The gradient (horizontal) on $\mathbb{H}^n$ is given by

$$\nabla_H := (X_1, \ldots, X_n, Y_1, \ldots, Y_n).$$

For further discussions on the analysis on the Heisenberg group, we refer to [7] and [9].

The p-Laplacian in the classical analysis is a quasilinear elliptic partial differential operator of 2nd order:

$$\Delta_p u := \nabla \cdot (|\nabla u|^{p-2}\nabla u), \;\; 1 < p < \infty$$

where ∇ is defined in (1.1). It is a nonlinear generalization of the above-mentioned Laplace operator for any u in $\mathbb{R}^n$, where $|\nabla u|^{p-2}$ is defined as

$$|\nabla u|^{p-2} := \left(\left(\frac{\partial u}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial u}{\partial x_n}\right)^2\right)^{\frac{p-2}{2}}, \;\; (x_1, \ldots x_n) \in \mathbb{R}^n.$$

Again, the p-Laplacian can be generalized to the so-called (p, q)-Laplacian. The case when $p > q$ (p, q)-Laplacian is used to model steady state solutions of reaction-diffusion equations that arise in many areas such as biophysics, plasma physics, models of elementary particles, etc (see, e.g. [1]).

Analogously, the (p, q)-Laplacian on $\mathbb{H}^n$, that is, on the Heisenberg group is known as the (p, q)-sub-Laplacian and is defined as follows

$$\mathcal{L}_{p,q} := -\nabla_H \cdot (|\nabla_H|^{p-2}\nabla_H) - \nabla_H \cdot (|\nabla_H|^{q-2}\nabla_H), \; 1 < q < p. \tag{1.2}$$

The aim of this paper is to extend Green's identities for the (p, q)-sub-Laplacians and use it in the process of proving the uniqueness for the solution of BVP. Green's identities defined for more general stratified groups were established in [6]. Then they were obtained for p-sub-Laplacians [8] (see also [5] and [4]). In this paper, we extend them to the (p, q)-sub-Laplacian on the Heisenberg group and consider some of their applications.

This Capstone project report has the following structure: extension of Green's first identity on the Heisenberg group for (p, q)-sub-Laplacians is discussed in Section 2; in Section 3 a weak formulation of the Dirichlet Boundary Value Problem involving (p, q)-sub-Laplacian is given and uniqueness of its solution is proven using some mathematical tools, such as Young inequality, comparison principle, etc; Section 4 is dedicated to the discussion of Carnot-Caratheodory Ball and Kaplan Ball to give better understanding of the smoothness of domain in the Heisenberg group, balls were printed out using the 3D printer.

2. (p,q)-sub-Laplacian Green's Identities

We say that $d\nu$ is the the volume element on $\mathbb{H}^n$. Note that the Lebesque measure on $\mathbb{R}^{2n+1}$ is the (left) Haar measure for $\mathbb{H}^n$ (see, e.g. [2, Proposition 1.3.21]). Throughout this paper we assume that a domain $\Omega \subset \mathbb{H}^n$ is an admissible domain.

Theorem 2.1. [6] *Let $f_k \in C^1(\Omega) \bigcap C(\overline{\Omega})$ and $k = 1, ..., n$ (where n is the topological dimension of $\mathbb{H}^n$), we then have*

$$\int_\Omega (X_k + Y_k) f_k d\nu = \int_{\partial\Omega} f_k \langle (X_k + Y_k), d\nu \rangle. \tag{2.1}$$

Therefore,

$$\int_\Omega \sum_{k=1}^n (X_k + Y_k) f_k d\nu = \int_{\partial\Omega} \sum_{k=1}^n f_k \langle (X_k + Y_k), d\nu \rangle, \tag{2.2}$$

where $d\nu$ is the volume element on $\mathbb{H}^n$.

$f \in C^1(\Omega)$ means $\nabla_H f \in C(\Omega)$. Green's first identity for the (p,q)-sub-Laplacian is obtained from the above-mentioned divergence formula:

Theorem 2.2. *[Green's first identity]. For $1 < p < \infty$ (as well as $1 < q < \infty$) let $v \in C^1(\Omega) \bigcap C(\overline{\Omega})$ and $u \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$. So*

$$\begin{aligned} &\int_\Omega \Big((|\nabla_H u|^{p-2} \widetilde{\nabla} v) u + (|\nabla_H u|^{q-2} \widetilde{\nabla} v) u + v \mathcal{L}_{p,q} u \Big) d\nu \\ &= \int_{\partial\Omega} (|\nabla_H u|^{p-2} + |\nabla_H u|^{q-2}) v \langle \widetilde{\nabla} u, d\nu \rangle, \end{aligned} \tag{2.3}$$

$$\widetilde{\nabla} u = \sum_{k=1}^n \Big((X_k u) X_k + (Y_k u) Y_k \Big). \tag{2.4}$$

$\mathcal{L}_{p,q}$ is the (p,q)-sub-Laplacian operator defined on $\mathbb{H}^n$.

Proof of Theorem 2.2. Let us define f as follows: $f_k = v|\nabla_H u|^{p-2} X_k u + v|\nabla_H u|^{q-2} Y_k u$, then we have

$$\sum_{k=1}^n (X_k + Y_k) f_k = (|\nabla_H u|^{p-2} \widetilde{\nabla} v) u + (|\nabla_H u|^{q-2} \widetilde{\nabla} v) u + v \mathcal{L}_{p,q} u.$$

By taking integrals of both sides over Ω and considering Theorem 2.1 we arrive at

$$\begin{aligned}
&\int_\Omega \Big((|\nabla_H u|^{p-2}\widetilde{\nabla} v)u + (|\nabla_H u|^{q-2}\widetilde{\nabla} v)u + v\mathcal{L}_{p,q}u\Big)\,d\nu \\
&= \int_\Omega \sum_{k=1}^{n}(X_k+Y_k)f_k d\nu \\
&= \int_{\partial\Omega} \sum_{k=1}^{n}\langle f_k(X_k+Y_k), d\nu\rangle \\
&= \int_{\partial\Omega} \sum_{k=1}^{n}\langle v|\nabla_H u|^{p-2}X_k u X_k + v|\nabla_H u|^{q-2}Y_k u Y_k, d\nu\rangle \\
&= \int_{\partial\Omega} (|\nabla_H u|^{p-2} + |\nabla_H u|^{q-2}) v\langle \widetilde{\nabla} u, d\nu\rangle.
\end{aligned}$$

□

3. Uniqueness of a positive weak solution

This section is devoted to the proof of the weak formulation of the Dirichlet boundary value problem (3.1). Math tools involved in the process are comparison principle, green's identities proven in the section (2), Young's inequality. The method was based on the paper [3].

We start by considering the Dirichlet boundary value problem containing $(p,q)-$sub-Laplacian

$$\mathcal{L}_{p,q}u = f(\xi)h(u), u > 0 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega, \tag{3.1}$$

where $1 < q < p$, $f(\xi)$ is a non-negative bounded function, Ω is the smooth domain (its importance is explained in Section 4)and h satisfies the following conditions:

- $h : (0,\infty) \to (0,\infty)$ is a non-decreasing function
- $h(s)s^{1-\beta}$ is non-increasing for some β such that $1 \le \beta < q$.

We expand the $(p,q)-$sub-Laplacian in the equation (3.1) using the fact that $\mathcal{L}_{p,q}u = -\nabla_H \cdot (|\nabla_H u|^{p-2}\nabla_H u) - \nabla_H \cdot (|\nabla_H u|^{q-2}\nabla_H u)$ and get the following formulation of the equation:

$$f(\xi)h(u) = -\nabla_H \cdot (|\nabla_H u|^{p-2}\nabla_H u) - \nabla_H \cdot (|\nabla_H u|^{q-2}\nabla_H u).$$

In order to get weak formulation of the equation we start by multiplying the it by the some function from the same class as u, which is ϕ, $\forall\phi \in S_0^{1,p}(\Omega)\,, \phi \ge 0$, we get:

$$f(\xi)h(u)\phi = -\nabla_H \cdot (|\nabla_H u|^{p-2}\nabla_H u)\phi - \nabla_H \cdot (|\nabla_H u|^{q-2}\nabla_H u)\phi.$$

By integrating it over Ω and using the Theorem 2.2 about Green's first identity the equation transforms to

$$\int_\Omega f(\xi)h(u)\phi d\nu = \int_\Omega (|\nabla_H u|^{p-2} + |\nabla_H u|^{q-2})\nabla_H u \cdot \nabla_H \phi d\nu.$$

Definition 3.1. A function $u(x) \in S_0^{1,p}(\Omega) \cap C(\bar{\Omega})$ is called a weak solution to (3.1) if we have

$$\int_\Omega (|\nabla_H u|^{p-2} + |\nabla_H u|^{q-2})\nabla_H u \cdot \nabla_H \phi d\nu = \int_\Omega f(\xi)h(u)\phi d\nu. \tag{3.2}$$

Theorem 3.2. *The equation* (3.1) *has at most one positive weak solution.*

We apply comparison principle in order to prove the uniqueness result for the given equation.

Theorem 3.3. *Let $u(x) \in S_0^{1,p}(\Omega) \cap C(\bar{\Omega})$ be a positive solution to*

$$\begin{aligned}&\int_\Omega (|\nabla_H u|^{p-2} + |\nabla_H u|^{q-2})\nabla_H u \cdot \nabla_H \phi d\nu \leq \\ &\int_\Omega f(\xi)h(u)\phi d\nu, \ \ \forall \phi \in S_0^{1,p}(\Omega), \ \ \phi \geq 0,\end{aligned} \tag{3.3}$$

and let $v(x) \in S_0^{1,p}(\Omega) \cap C(\bar{\Omega})$ be a positive solution to

$$\begin{aligned}&\int_\Omega (|\nabla_H v|^{p-2} + |\nabla_H v|^{q-2})\nabla_H v \cdot \nabla_H \phi d\nu \geq \\ &\int_\Omega f(\xi)h(v)\phi d\nu, \ \ \forall \phi \in S_0^{1,p}(\Omega), \ \ \phi \geq 0.\end{aligned} \tag{3.4}$$

Then, $u \leq v$ in Ω.

Proof of Theorem 3.3. First of all, let us define some set K elements of which satisfy the condition: $K = \{\xi \in \Omega \ : \ u(\xi) > v(\xi)\}$. The theorem is proved by using contradiction argument, which means by claiming that K is non-empty set. Assume $u_\epsilon = u + \epsilon$ and $v_\epsilon = v + \epsilon$ for $\epsilon > 0$. By introducing test function

$$\psi_1(\xi) = \max\left[\frac{u_\epsilon^\beta(\xi) - v_\epsilon^\beta(\xi)}{u_\epsilon^{\beta-1}(\xi)}, 0\right],$$

the inequality (3.3) can be rewritten as

$$\begin{aligned}&\int_K (|\nabla_H u|^{p-2} + |\nabla_H u|^{q-2})\nabla_H u \cdot \nabla_H \left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{u_\epsilon^{\beta-1}}\right) d\nu \leq \\ &\int_K f(\xi)h(u)\left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{u^{\beta-1}}\right)\left(\frac{u}{u_\epsilon}\right)^{\beta-1} d\nu.\end{aligned} \tag{3.5}$$

Using another test function

$$\psi_2(\xi) = \max\left[\frac{u_\epsilon^\beta(\xi) - v_\epsilon^\beta(\xi)}{v_\epsilon^{\beta-1}(\xi)}, 0\right],$$

the inequality (3.4) can be transformed similarly

$$\begin{aligned}&\int_K (|\nabla_H v|^{p-2} + |\nabla_H v|^{q-2})\nabla_H v \cdot \nabla_H \left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{v_\epsilon^{\beta-1}}\right) d\nu \leq \\ &\int_K f(\xi)h(v)\left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{v^{\beta-1}}\right)\left(\frac{v}{v_\epsilon}\right)^{\beta-1} d\nu.\end{aligned} \tag{3.6}$$

Subtracting (3.6) from (3.5) we obtain the following inequality

$$\begin{aligned}
&\int_K |\nabla_H u|^{p-2}\nabla_H u \cdot \nabla_H \left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{u_\epsilon^{\beta-1}}\right) d\nu \\
&+ \int_K |\nabla_H u|^{q-2}\nabla_H u \cdot \nabla_H \left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{u_\epsilon^{\beta-1}}\right) d\nu \\
&+ \int_K |\nabla_H v|^{p-2}\nabla_H v \cdot \nabla_H \left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{v_\epsilon^{\beta-1}}\right) d\nu \\
&+ \int_K |\nabla_H v|^{q-2}\nabla_H v \cdot \nabla_H \left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{v_\epsilon^{\beta-1}}\right) d\nu \\
&\leq \int_K f(\xi)h(u)\left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{u^{\beta-1}}\right)\left(\frac{u}{u_\epsilon}\right)^{\beta-1} d\nu \\
&+ \int_K f(\xi)h(v)\left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{v^{\beta-1}}\right)\left(\frac{v}{v_\epsilon}\right)^{\beta-1} d\nu.
\end{aligned} \tag{3.7}$$

On the other hand, we have

$$\nabla_H\left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{u_\epsilon^{\beta-1}}\right) = \nabla_H u + (\beta-1)\left(\frac{v_\epsilon}{u_\epsilon}\right)^\beta \nabla_H u - \beta\left(\frac{v_\epsilon}{u_\epsilon}\right)^{\beta-1}\nabla_H v,$$

and

$$\nabla_H\left(\frac{v_\epsilon^\beta - u_\epsilon^\beta}{v_\epsilon^{\beta-1}}\right) = \nabla_H v + (\beta-1)\left(\frac{u_\epsilon}{v_\epsilon}\right)^\beta \nabla_H v - \beta\left(\frac{u_\epsilon}{v_\epsilon}\right)^{\beta-1}\nabla_H u.$$

Thus, the sum of second and fourth integrals in (3.7) on the left hand side can be expanded as follows:

$$\begin{aligned}
&\int_K |\nabla_H u|^{q-2}\nabla_H u \cdot \nabla_H \left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{u_\epsilon^{\beta-1}}\right) d\nu \\
&+ \int_K |\nabla_H v|^{q-2}\nabla_H v \cdot \nabla_H \left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{v_\epsilon^{\beta-1}}\right) d\nu \\
&= \int_K \left\{|\nabla_H u|^q\left(1 + (\beta-1)\left(\frac{v_\epsilon}{u_\epsilon}\right)^\beta\right) - \beta|\nabla_H u|^{q-2}\left(\frac{v_\epsilon}{u_\epsilon}\right)^{\beta-1}\nabla_H u \cdot \nabla_H v\right\} d\nu \\
&+ \int_K \left\{|\nabla_H V|^q\left(1 + (\beta-1)\left(\frac{u_\epsilon}{v_\epsilon}\right)^\beta\right) - \beta|\nabla_H v|^{q-2}\left(\frac{u_\epsilon}{v_\epsilon}\right)^{\beta-1}\nabla_H v \cdot \nabla_H u\right\} d\nu.
\end{aligned} \tag{3.8}$$

Let us recall the Young inequality which states the following

$$A \cdot B \leq \frac{1}{s}|A|^s + \frac{1}{q}|B|^q, \;\; \frac{1}{s} + \frac{1}{q} = 1 \;\text{ for } A, B \in \mathbb{R}^n. \tag{3.9}$$

Plugging $|A|^{q-2}A$ instead of A we get

$$|A|^{q-2}A \cdot B \leq \frac{1}{s}|A|^q + \frac{1}{q}|B|^q.$$

Also replacing A by λA (λ >0) we have

$$|A|^{q-2}\lambda^{q-1}A\cdot B\leq \frac{1}{s}\lambda^q|A|^q+\frac{1}{q}|B|^q.$$

We set $\lambda=\left(\frac{v_\epsilon}{u_\epsilon}\right)^{\frac{\beta-1}{q-1}}$, $A=\nabla_H u$ and $B=\nabla_H v$, so

$$|\nabla_H u|^{q-2}\left(\frac{v_\epsilon}{u_\epsilon}\right)^{\beta-1}\nabla_H u\cdot\nabla_H v\leq \frac{1}{s}\left(\frac{v_\epsilon}{u_\epsilon}\right)^{s(\beta-1)}|\nabla_H u|^q+\frac{1}{q}|\nabla_H v|^q.$$

Similarly, we set $\lambda=\left(\frac{u_\epsilon}{v_\epsilon}\right)^{\frac{\beta-1}{q-1}}$, $A=\nabla_H v$ and $B=\nabla_H u$, that is, we have

$$|\nabla_H v|^{q-2}\left(\frac{u_\epsilon}{v_\epsilon}\right)^{\beta-1}\nabla_H v\cdot\nabla_H u\leq \frac{1}{s}\left(\frac{u_\epsilon}{v_\epsilon}\right)^{s(\beta-1)}|\nabla_H v|^q+\frac{1}{q}|\nabla_H u|^q.$$

Finally, equation (3.8) can be rewritten as

$$\begin{aligned}&\int_K|\nabla_H u|^{q-2}\nabla_H u\cdot\nabla_H\left(\frac{u_\epsilon^\beta-v_\epsilon^\beta}{u_\epsilon^{\beta-1}}\right)d\nu\\&+\int_K|\nabla_H v|^{q-2}\nabla_H v\cdot\nabla_H\left(\frac{u_\epsilon^\beta-v_\epsilon^\beta}{v_\epsilon^{\beta-1}}\right)d\nu\\&\geq\int_K\left\{|\nabla_H u|^q\left[1+(\beta-1)\left(\frac{v_\epsilon}{u_\epsilon}\right)^\beta-\frac{\beta}{s}\left(\frac{v_\epsilon}{u_\epsilon}\right)^{s(\beta-1)}-\frac{\beta}{q}\right]\right\}\\&+\left\{|\nabla_H v|^q\left[1+(\beta-1)\left(\frac{u_\epsilon}{v_\epsilon}\right)^\beta-\frac{\beta}{s}\left(\frac{u_\epsilon}{v_\epsilon}\right)^{s(\beta-1)}-\frac{\beta}{q}\right]\right\}d\nu.\end{aligned}$$

By introducing

$$\phi(t)=1+(\beta-1)t^\beta-\frac{\beta}{s}t^{s(\beta-1)}-\frac{\beta}{q}, \tag{3.10}$$

the latter inequality takes the form

$$\begin{aligned}&\int_K|\nabla_H u|^{q-2}\nabla_H u\cdot\nabla_H\left(\frac{u_\epsilon^\beta-v_\epsilon^\beta}{u_\epsilon^{\beta-1}}\right)d\nu\\&+\int_K|\nabla_H v|^{q-2}\nabla_H v\cdot\nabla_H\left(\frac{u_\epsilon^\beta-v_\epsilon^\beta}{v_\epsilon^{\beta-1}}\right)d\nu\\&\geq\int_K\left\{|\nabla_H u|^q\phi\left(\frac{v_\epsilon}{u_\epsilon}\right)+|\nabla_H v|^q\phi\left(\frac{u_\epsilon}{v_\epsilon}\right)\right\}.\end{aligned}$$

Since $\phi(1)=0$ and $\phi'(t)=\beta(\beta-1)t^{\beta-1}(1-t^{\frac{\beta-q}{q-1}})$, so $\phi(t)\geq 0$ for $t>0$, and we have

$$\begin{aligned}&\int_K|\nabla_H u|^{q-2}\nabla_H u\cdot\nabla_H\left(\frac{u_\epsilon^\beta-v_\epsilon^\beta}{u_\epsilon^{\beta-1}}\right)d\nu\\&+\int_K|\nabla_H v|^{q-2}\nabla_H v\cdot\nabla_H\left(\frac{u_\epsilon^\beta-v_\epsilon^\beta}{v_\epsilon^{\beta-1}}\right)d\nu\geq 0.\end{aligned}$$

Thus, (3.7) implies

$$\int_K |\nabla_H u|^{p-2}\nabla_H u\cdot\nabla_H\left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{u_\epsilon^{\beta-1}}\right)d\nu + \int_K |\nabla_H v|^{p-2}\nabla_H v\cdot\nabla_H\left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{v_\epsilon^{\beta-1}}\right)d\nu \le \int_K f(\xi)h(u)\left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{u^{\beta-1}}\right)\left(\frac{u}{u_\epsilon}\right)^{\beta-1}d\nu + \int_K f(\xi)h(v)\left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{v^{\beta-1}}\right)\left(\frac{v}{v_\epsilon}\right)^{\beta-1}d\nu. \tag{3.11}$$

Left part of the latter inequality can be rewritten the same way as equation (3.8) with q replaced by p:

$$\begin{aligned}&\int_K |\nabla_H u|^{p-2}\nabla_H u\cdot\nabla_H\left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{u_\epsilon^{\beta-1}}\right)d\nu \\ &+ \int_K |\nabla_H v|^{p-2}\nabla_H v\cdot\nabla_H\left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{v_\epsilon^{\beta-1}}\right)d\nu \\ &= \int_K |\nabla_H u|^p\left(1+(\beta-1)\left(\frac{v_\epsilon}{u_\epsilon}\right)^\beta\right)d\nu \\ &- \int_K \beta|\nabla_H u|^{p-2}\left(\frac{v_\epsilon}{u_\epsilon}\right)^{\beta-1}\nabla_H u\cdot\nabla_H v d\nu \\ &+ \int_K |\nabla_H v|^p\left(1+(\beta-1)\left(\frac{u_\epsilon}{v_\epsilon}\right)^\beta\right)d\nu \\ &- \int_K \beta|\nabla_H v|^{p-2}\left(\frac{u_\epsilon}{v_\epsilon}\right)^{\beta-1}\nabla_H v\cdot\nabla_H u d\nu.\end{aligned} \tag{3.12}$$

From the inequality

$$|A|^{p-2}\lambda^{p-1}A\cdot B \le \frac{1}{r}\beta^p|A|^p + \frac{1}{p}|B|^p,\ \ \frac{1}{r}+\frac{1}{p}=1,$$

it follows that

$$|\nabla_H u|^{p-2}\left(\frac{v_\epsilon}{u_\epsilon}\right)^{\beta-1}\nabla_H u\cdot\nabla_H v \le \frac{1}{r}\left(\frac{v_\epsilon}{u_\epsilon}\right)^{r(\beta-1)}|\nabla_H u|^p + \frac{1}{p}|\nabla_H v|^p,$$

as well as

$$|\nabla_H v|^{p-2}\left(\frac{u_\epsilon}{v_\epsilon}\right)^{\beta-1}\nabla_H v\cdot\nabla_H u \le \frac{1}{r}\left(\frac{u_\epsilon}{v_\epsilon}\right)^{r(\beta-1)}|\nabla_H v|^p + \frac{1}{p}|\nabla_H u|^p.$$

Thus, combining (3.11) and (3.12) we obtain

$$
\begin{aligned}
\int_K \Bigg\{ |\nabla_H u|^p \Bigg(1 + (\beta-1)\left(\frac{v_\epsilon}{u_\epsilon}\right)^{\beta} - \frac{\beta}{r}\left(\frac{v_\epsilon}{u_\epsilon}\right)^{r(\beta-1)} - \frac{\beta}{p} \Bigg) \\
+ |\nabla_H v|^p \Bigg(1 + (\beta-1)\left(\frac{u_\epsilon}{v_\epsilon}\right)^{\beta} - \frac{\beta}{r}\left(\frac{u_\epsilon}{v_\epsilon}\right)^{r(\beta-1)} - \frac{\beta}{p} \Bigg) \Bigg\} d\nu \\
\leq \int_K f(\xi) h(u) \left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{u^{\beta-1}} \right) \left(\frac{u}{u_\epsilon} \right)^{\beta-1} d\nu \\
+ \int_K f(\xi) h(v) \left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{v^{\beta-1}} \right) \left(\frac{v}{v_\epsilon} \right)^{\beta-1} d\nu.
\end{aligned}
$$

By inserting

$$
\phi(t) = 1 + (\beta-1)t^\beta - \frac{\beta}{r} t^{r(\beta-1)} - \frac{\beta}{p},
$$

we get the following

$$
\begin{aligned}
&\int_K \left\{ |\nabla_H u|^p \phi\left(\frac{v_\epsilon}{u_\epsilon}\right) + |\nabla_H v|^p \phi\left(\frac{u_\epsilon}{v_\epsilon}\right) \right\} d\nu \\
&\leq \int_K f(\xi) h(u) \left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{u^{\beta-1}} \right) \left(\frac{u}{u_\epsilon} \right)^{\beta-1} d\nu + \int_K f(\xi) h(v) \left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{v^{\beta-1}} \right) \left(\frac{v}{v_\epsilon} \right)^{\beta-1} d\nu.
\end{aligned}
$$

By the definitions for h, we obtain

$$
\begin{aligned}
\lim_{\epsilon\to 0} \int_K f(\xi) h(u) \left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{u^{\beta-1}} \right) \left(\frac{u}{u_\epsilon} \right)^{\beta-1} d\nu \\
+ \lim_{\epsilon\to 0} \int_K f(\xi) h(v) \left(\frac{u_\epsilon^\beta - v_\epsilon^\beta}{v^{\beta-1}} \right) \left(\frac{v}{v_\epsilon} \right)^{\beta-1} d\nu \\
= \int_K f(\xi) \left(\frac{h(u)}{u^{\beta-1}} - \frac{h(v)}{v^{\beta-1}} \right) (u^\beta - v^\beta) d\nu \leq 0.
\end{aligned}
$$

As $\epsilon \to 0$

$$
\int_K \left\{ |\nabla_H u|^p \phi\left(\frac{v_\epsilon}{u_\epsilon}\right) + |\nabla_H v|^p \phi\left(\frac{u_\epsilon}{v_\epsilon}\right) \right\} d\nu \leq 0. \tag{3.13}
$$

$\phi(1) = 0$ and $\phi'(t) = \beta(\beta-1)t^{\beta-1}(1 - t^{\frac{\beta-p}{p-1}})$, hence $\phi(t) > 0$ for $t \neq 1$. Since $\frac{u}{v} > 1$ in K, by (3.13) we must have $|\nabla_H u| = |\nabla_H v| = 0$ in K. Therefore, $\nabla_H(u-v) = 0$ in K and $u - v = 0$ on ∂K. So, $u(x) = v(x)$, which contradicts the definition of K. $\square$

Proof of Theorem 3.2. Any weak solution must satisfy (3.2). That is, at the same time a (weak) solution can be considered both sub and sup-solution. Let us assume that the equation has two solutions, say, u_1 and u_2. Thus, u_1 can be considered a sub-solution and u_2 is a sup-solution. Therefore, by Theorem 3.3 we get $u_1 \leq u_2$. Now by exchanging the roles of u_1 and u_2 , we get $u_1 \geq u_2$. This yields $u_1 = u_2$ which means we can't have two different solutions, that is, we have uniqueness. $\square$

4. Kaplan and Carnot-Carathéodory balls in the Heisenberg group

This section is dedicated to explanation of the distance in non-commutative analysis. Kaplan and Carnot-Carathéodory balls were printed out using 3D printer and their 3D model is given as well. We start by explaining what is distance in classical (commutative) analysis by recalling the fundamental solution for the p-Laplacian equation. The fundamental solution for the following equation

$$-\Delta\varepsilon_c(x) = \delta(x), x \in \mathbb{R}^n$$

is given by

$$\varepsilon_c(x) := \frac{1}{\omega_n |x|^{n-2}}, \quad |x| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}.$$

Let Q be the homogeneous dimension (it's the same as topological dimension in commutative analysis) of $\mathbb{H}^n$ and let ε be the fundamental solution for the sub-Laplacian.

$$d(x) := \varepsilon(x)^{\frac{1}{2-Q}}$$

is known as distance. Let $\delta_\lambda(x)$ be dilation, then:

$$d(\delta_\lambda(x)) = \lambda d(x).$$

Using the same logic it was found by Folland that the fundamental solution on the Heisenberg group $\mathbb{H}^1$ is given by:

$$\varepsilon(x) := \left(C\ d(x)^{-1}\right)^{\frac{1}{2-Q}}, \tag{4.1}$$

where Q is 4 and $d(x)$ is

$$d(x) := \left(\left(\sqrt{x_1^2 + x_2^2}\right)^4 + 16x_3^2\right)^{\frac{1}{2}}. \tag{4.2}$$

This distance is called the Kaplan distance.

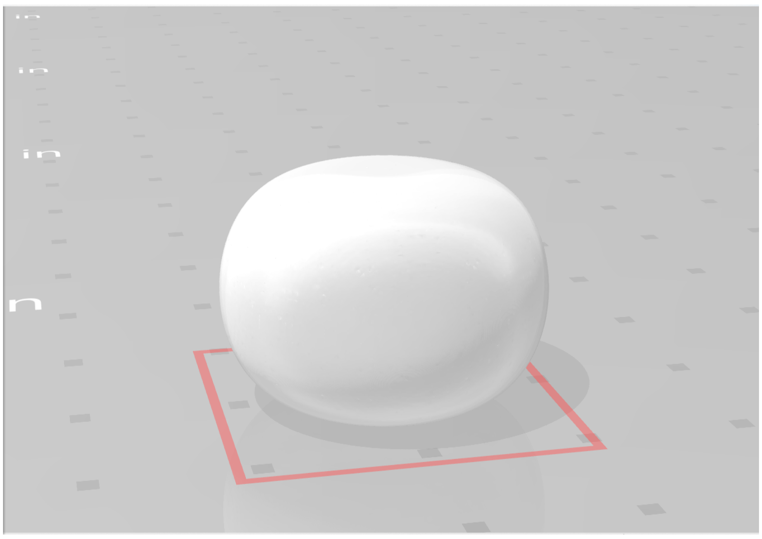

Figure 4.1. Kaplan Ball.

The 3D model of the Kaplan Ball is illustrated in Fig. 1.

The Carnot-Carathéodory distance between two points is known as minimum time needed to connect these points by curves, whose derivatives are spanned by the vector fields X_j, Y_j of the Heisenberg group [10]. The parametric equation for the unit CC-ball is given by:

$$x(\theta,\phi) = \frac{cos\theta(1 - cos\phi) + sin\theta sin\phi}{\phi};$$

$$y(\theta,\phi) = \frac{-sin\theta(1 - cos\phi) + cos\theta sin\phi}{\phi};$$

$$t(\theta,\phi) = \frac{2(\phi - sin\phi)}{\phi^2}; 0 \leq \theta \leq 2\pi, -2\pi \leq \phi \leq 2\pi.$$

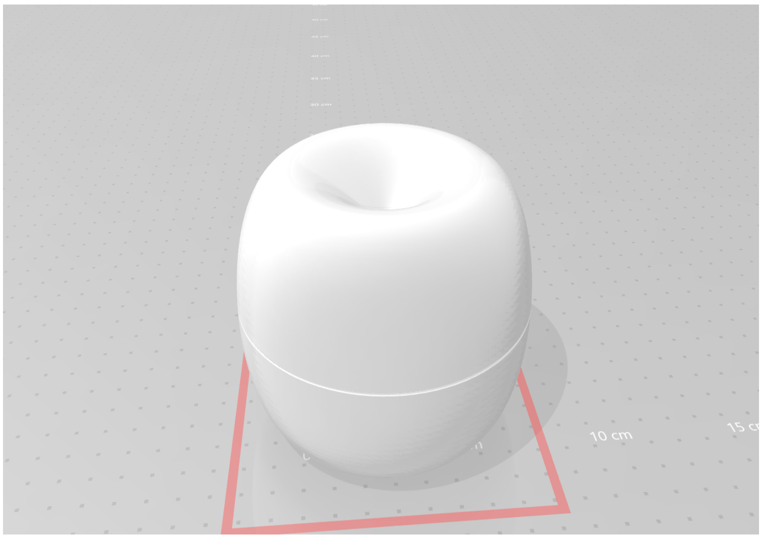

Figure 4.2. Carnot-Carathéodory Ball.

References

[1] V. Benci, P. D'Avenia, D. Fortunato, L. Pisani. Solitons in several space dimensions: Derricks Problem and infinitely many solutions. *Arch. Ration. Mech. Anal.*, 41:297–324, 2000.

[2] A. Bonfiglioli, E. Lanconelli, and F.Uguzzoni. Stratified Lie groups and potential theory for their sub-Laplacians. Springer-Verlag, Berlin, Heidelberg, 2007.

[3] B. Emamizadeh, Y. Liu, G. Porru. Extensions of a result by G. Talenti to (p,q)-Laplace equations. *Nonlinear Analysis*, 180:41-51, 2019.

[4] A. Kassymov and D. Suragan. Existence of solutions for p-sub-Laplacians with nonlinear sources on the Heisenberg group. *Complex Variables and Elliptic Equations*, DOI: 10.1080/17476933.2020.1731737, 2020.

[5] M. Mustafa and D. Suragan. Green-type identities for Rockland operators on graded Lie groups. *Complex Analysis and Operator Theory*, 13 (3):959-966, 2019.

[6] M. Ruzhansky and D. Suragan. Layer potentials, Kac's problem, and refined Hardy inequality on homogeneous Carnot groups. *Adv. Math.*, 308:483–528, 2017.

[7] M. Ruzhansky, D. Suragan. Hardy inequalities on homogeneous groups: 100 years of Hardy inequalities, *Progress in Math.* Vol. 327, Birkhäuser, 588 pp, (2019) (open access book).

[8] M. Ruzhansky and D. Suragan. Green's identities, comparison principle, and uniqueness of positive solutions for nonlinear p-sub-Laplacian equations on stratified Lie groups. *Potential Analysis*, 2019.

[9] S. Thangavelu. Harmonic analysis on the Heisenberg group, *Progress in Math.* Vol. 159, Birkhäuser, 192 pp, (1998).

[10] CC-ball Parametric equation Roberto Monti Some properties of Carnot-Carathéodory balls in the Heisenberg group *Rendiconti Lincei Matematica E Applicazioni*, 2000.

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