Some applications of potential theory for a degenerate type diffusion equation

by

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Abstract

This thesis is aimed to study boundary conditions for heat potentials for degenerate-type diffusion equations with initial condition and conductivity coefficient given by a time variable. Equations can be one-dimensional or multi-dimensional. The latter gives a new look to the problem. It is worth to mention that the coefficient is not always positive and therefore, it causes a degeneracy for the equation. The found boundary conditions make the solution, which is in the form of potential, unique. These boundary conditions are commonly called transparent boundary conditions or Kac's boundary conditions. This kind of results first appeared in M. Kac's work in the middle of last century. He developed potential theory and established further applications. Since then, many researchers have been studying potential theory related to the field. Some problems can be solved by simple integration, whereas others need more efforts to put in. Even though the works of past decades play an important role in potential theory, there exist problems which are still hard to solve. Hence, potential theory needs further developments and modern approaches.

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Chapter 1

Introduction

The history of potential theory for parabolic equations such as heat equations (diffusion equations) is long enough. It has been used to handle the initial-boundary value problems of parabolic equations with different levels of difficulty for several years. This strategy is implemented by using volume potentials, Poisson integrals, single and double layer potentials which are the elements of potential theory. Despite the long history and many works done in this sphere, potential theory has some shortcomings which need further developments. It needs to use modern approaches to solve them. Up to now, potential theory was applied to both one-dimensional and multi-dimensional parabolic equations. Usually, firstly one finds the fundamental solution which can be obtained by applications of transform operations such as Laplace and Fourier. The structure of this thesis is simple. In Chapter 2, we consider the elements of potential theory for one-dimensional degenerate parabolic equation. To do so, we first explain the fundamental solution to this equation. After that, different potentials will be characterised. Namely, volume potential, single-layer potential and double-layer potential. Note that discussion of this session is based on Malyshev's work [5]. In Chapter 3, we discuss the elements of potential theory for multi-dimensional degenerate parabolic equations based on work done by Karazym and Suragan in [4]. It is done by first considering the fundamental solution and Cauchy problems. Then, types of potentials such as Poisson integral and volume potential will be figured out. In Chapter 4, we discuss the gap left in one-dimensional parabolic equation. Namely, we find what boundary can be put in to make the given potential a unique solution of one-dimensional parabolic equation respect to zero source function. The analogue of this in multi-dimensional case is considered in [4]. Finally, in the last chapter, the key ideas and meaning of the findings are summarized.

Chapter 2

Elements of potential theory for one-dimensional degenerate parabolic equation

In [5], the author studied the parabolic potentials for the initial-boundary-value problems in a semi-infinite domain given by:

$$L_k y(z,\tau) := \frac{\partial y(z,\tau)}{\partial \tau} - k(\tau)\frac{\partial^2 y(z,\tau)}{\partial z^2} = g(z,\tau), \quad z > 0, \quad \tau > 0; \tag{2.1}$$

$$y(z,0) = \zeta(z), \quad z \geq 0; \tag{2.2}$$

$$y(0,\tau) = \eta(\tau), \quad \tau \geq 0. \tag{2.3}$$

The conductivity coefficient $k(\tau)$ with $\tau \in [0,T]$ is not always positive and follows one of the following two assumptions:

(a) $k(\tau) \geq 0$, the coefficient can accept zero values at isolated points.

(b) $k_1(\tau)$ given by integral

$$k_1(\tau) = \int_0^\tau k(\nu)d\nu$$

is greater than zero for any $\tau > 0$. This means, $k(\tau)$ can be negative somewhere in the interval.

It is clear that if a function fits the condition (a), it automatically fits the condition (b) too. It is very important to mention that (2.1) is never reducible to the heat operator in standard form $y_\tau - y_{zz}$ due to several reasons. The demonstration can be done by using evident substitution:

$$\gamma = \int_0^\tau k(\nu)d\nu,$$

which is under (a) refers the presence of inverse function $\tau(\gamma)$ with derivative $\tau'_\gamma = \frac{1}{k(\tau)}$ at the points where the denominator is not equal to zero. Under (b) the inversion is impossible. To avoid these difficulties, we develop the fundamental solution and potentials with characteristics. Note that discussion of this session is based on Malyshev's work [5].

2.1 Fundamental solution

The boundary N of the domain can be divided into two parts. The first part is given by $N_1 = (z \geq 0, \tau = 0)$ and the second part is given by $N_2 = (z = 0, \tau \geq 0)$. If the function satisfies (b), then the fundamental solution of (2.1) can be constructed by using Fourier transform in z in forms of:

$$\xi_k(z,\tau) = \xi(z, k_1(\tau)) = \frac{H(\tau)}{2\sqrt{\pi k_1(\tau)}}\exp(-\frac{z^2}{4k_1(\tau)}). \tag{2.4}$$

It is worth to mention that here $k_1(\tau) > 0$ and $H(\tau)$ is the Heaviside function. The fundamental solution has a following property:

$$\int_{-\infty}^{\infty} \xi_k(z,\tau)dz = 1; \;\; \xi_k(z,\tau) \to \delta(z) \;\; with \;\; z \to 0^+. \tag{2.5}$$

After assuming g, y be equal to zero for the values of z, τ out of boundary N, the initial-boundary value problem (2.1) – (2.3) can be reduced to general form

$$L_kY = G(z,\tau) + [Y]_{N_1} cos(n,e_1)\delta_{N_1} - k(\tau)\left[\frac{\partial Y}{\partial z}\right]_{N_2} cos(n,e_2)\delta_{N_2}$$
$$-\frac{\partial}{\partial z}(k(\tau)[Y]_{N_2} cos(n,e_2)\delta_{N_2}) = F(z,\tau), \tag{2.6}$$

where $[Y]_N$ is a jump of y on boundary $N = N_1 \cup N_2$, n is an external normal to boundary N and e_1, e_2 are unit vectors along τ, z axis-es.

If the operator L_k had a constant coefficient, the solution of the equation (2.6) could be of the form $y = \xi_k * F$. However, in our case it is not obvious [5].

Lemma 2.1.1 *Under (a) the distributional solution of (2.6) is unique and can be represented as a convolution of the fundamental solution ξ_α with the right-hand side of (2.6), that is $y = \xi_\alpha * F$, where, as in,*

$$\xi_\alpha(z-\epsilon, \tau-\gamma) = \xi(z-\epsilon, \alpha_1(\tau-\gamma)), \tag{2.7}$$

and

$$\alpha_1(\tau-\gamma) = \int_\gamma^\tau k(j)dj = k_1(\tau) - k_1(\gamma); \;\; \alpha_1(\tau) = k_1(\tau).$$

Namely, $k_1(\tau)$ is considered as a time variable. Apparently, α_1 is continuous and, respect to the condition (a) $\alpha_1(\tau-\gamma) > 0$ for $\tau-\gamma > 0$ [5].

Proof 2.1.1 *Suppose that (a) is true. Then $\xi_\alpha(z-\epsilon, \tau-\gamma)$ from equation (2.7) is a distributional solution of*

$$L_k\xi_\alpha(z-\epsilon, \tau-\gamma) = \frac{\partial \xi}{\partial \tau} - k(\tau)\frac{\partial^2 \xi}{\partial z^2} = \delta(z-\epsilon, \tau-\gamma) \;\; in \;\; z, \;\; \tau,$$

and

$$L_k^+\xi_\alpha(z-\epsilon, \tau-\gamma) = -\frac{\partial \xi}{\partial \gamma} - k(\gamma)\frac{\partial^2 \xi}{\partial \epsilon^2} = \delta(z-\epsilon, \tau-\gamma) \;\; in \;\; \epsilon, \;\; \gamma.$$

It can be checked by using Fourier transform strategy. Hence, integration by parts *gave us the result that* $y = \xi_\alpha * L_k y$, *then by simple derivative* $y = L_k(\xi_\alpha * y)$, *that brings to:*

$$L_k(\xi_\alpha * Y) = (L_k\xi_\alpha) * Y = \xi_\alpha * L_k Y$$

The uniqueness of distributional solution is obvious, because

$$L_k y = 0 \Rightarrow \xi_\alpha * L_k y = L_k \xi_\alpha * y = \delta * y = y = 0.$$

After generalizing the initial-boundary value problem (2.1)-(2.3), the author found the solution of the problem in terms of sum of four integrals:

$$\begin{aligned} y(z,\tau) &= \int_0^\tau d\gamma \int_0^\infty g(\epsilon,\gamma)\xi_\beta(z-\epsilon,\tau-\gamma)d\epsilon + \int_0^\infty y(\epsilon,0)\xi_\beta(z-\epsilon,\tau)d\epsilon \\ &+ \int_0^\tau k(\gamma)y(0,\gamma)\frac{\partial(\xi_\beta(z-\epsilon,\tau-\gamma))}{\partial\epsilon}\bigg|_{\epsilon=0} d\gamma - \int_0^\tau k(\gamma)\frac{\partial y}{\partial\epsilon}(0,\gamma)\xi_\beta(z,\tau-\gamma)d\gamma. \end{aligned} \tag{2.8}$$

In equation (2.8), $y(z,\tau)$ is given by the sum of four integrals where each integral is a potential called:

(i) volume potential

$$P(z,\tau) = \int_0^\tau d\gamma \int_0^\infty g(\epsilon,\gamma)\xi_\beta(z-\epsilon,\tau-\gamma)d\epsilon; \tag{2.9}$$

(ii) single-layer potential with $(z \geq 0, \tau = 0)$

$$P^1(z,\tau) = \int_0^\infty l(\epsilon)\xi_\beta(z-\epsilon,\tau)d\epsilon; \tag{2.10}$$

(iii) single-layer potential with $(\tau \geq 0, z = 0)$

$$P^2(z,\tau) = \int_0^\tau k(\gamma)m(\gamma)\xi_\beta(z,\tau-\gamma)d\gamma; \tag{2.11}$$

(iv) double layer potential with $(\tau \geq 0, z = 0)$

$$D(z,\tau) = \int_0^\tau k(\gamma)n(\gamma)\frac{\partial(\xi_\beta(z-\epsilon,\tau-\gamma))}{\partial\epsilon}\bigg|_{\epsilon=0} d\gamma[5]. \tag{2.12}$$

2.2 Volume Potential

Volume potential $P(z,\tau)$ (2.9) represented above is a partial solution of a boundary value problem related to source function $g(z,\tau)$. There is a theorem related to volume potential.

Theorem 2.2.1 *Having $k(\tau) \in L_1(0,T)$ and fit the condition (a),*

1. *for $g \in N, P(z,\tau) \in N$;*
2. *for $z \geq 0, \tau \geq 0$, $P(z,\tau)$ is a distributional solution of (2.1);*
3. *if we extend $g \in C^2 \;\; \forall z,\tau \geq 0$ and the first and second derivatives in boundary N, then $P_{zz}(z,\tau)$ is continuous in $z \geq 0, \tau \geq 0$, and P_τ exists for any z,τ is continuous in z, and its smoothness in τ is determined by that of $k(\tau)$ itself; hence, if also $k(\tau) \in C(R_+)$, then $P(z,\tau)$ satisfies (2.1) in the classical sense.*

Proof 2.2.1 *Now we express our volume potential $P(z,\tau)$ in new form. To do so, we need a new variable. Let $x(\alpha_1(\tau-\gamma) > 0 \;\; for \;\; \tau-\gamma > 0)$*

$$z - \epsilon = 2x\sqrt{\alpha_1(\tau-\gamma)},$$

for $z \geq 0, \tau \geq 0$ we write $P(z,\tau)$ as

$$P(z,\tau) = \frac{1}{\sqrt{\pi}} \int_0^\tau d\gamma \int_{-\infty}^{\frac{z}{2\sqrt{\alpha_1(\tau-\gamma)}}} g(z - 2x\sqrt{\alpha_1(\tau-\gamma)};\gamma)e^{-x^2}dx, \tag{2.13}$$

and its time-derivative ($\tau > 0$):

$$\frac{\partial P}{\partial \tau} = g(z,\tau) - \frac{k(\tau)}{\sqrt{\pi}} \int_0^\tau d\gamma \int_{-\infty}^{\frac{z}{2\sqrt{\alpha_1(\tau-\gamma)}}} g'(z - 2x\sqrt{\alpha_1(\tau-\gamma)};\gamma)\frac{x}{\sqrt{\alpha_1(\tau-\gamma)}}e^{-x^2}dx. \tag{2.14}$$

By using equations (2.13) and (2.14), it is seen that $P(z,\tau) \in C^2(z \geq 0, \tau > 0) \cap C^1(z \geq 0, \tau \geq 0)$ for g and k fitting the third part part of Theorem 2.2.1. Concurrently, volume potential P is a classical solution to $L_kP = g$ (Du Bois Reimond

Theoerem),

Next, as $g \in N$ and ξ_α satisfies (2.5),

$$|P(z,\tau)| \leq ||g|| \int_0^\tau d\gamma \int_{-\infty}^{+\infty} \xi_\alpha d\epsilon \leq \tau ||g||.$$

As the consequences we can see that $P \in N$ and fits the initial condition of zero. The remaining second part of the Theorem 2.2.1 can be derived like in Lemma 2.1.1, because

$$G = \delta * G = L_k \xi_\alpha * G = L_k(\xi_\alpha * G) = L_k P.[5]$$

2.3 Single-layer Potential

(A) Single-layer potential $P^1(z,\tau)$ as shown in equation (2.10) is a part of solution respect to the condition $y(z,0) = \zeta(z)$ with $z \geq 0$.

Theorem 2.3.1 *Suppose that (b) is satisfied. Then,*

I. *For $l \in N, P^1(z,\tau) \in N$;*

II. *P^1 is a distributional solution of the equation $L_k y = l\delta_{N_1}$ and fits the initial condition $P^1(z,\tau) \to l(z)$ $as \tau \to 0^+$ $for z > 0$;*

III. *If $l \in C^2$ and its first and second derivatives are in boundary N, then $P^1_{zz}(z,\tau)$ is continuous in $z \geq 0, \tau \geq 0$ and P^1_τ exists is continuous in z, and its smoothness in τ is determined by that of $k(\tau)$ itself;*

IV. *If, furthermore, $k \in C(R_+)$, then $P^1(z,\tau) \in C^2(z \geq 0, \tau > 0,) \cap C(z \geq 0, \tau \geq 0)$ and, it implies that $P^1(z,\tau)$ is a classical solution of the (2.1)-(2.2) where $g = 0$, because the backing of distribution δ_{N_1} is N_1.*

Proof 2.3.1 *It is same as in Theorem 2.2.1. There is a substitution of variable given by*

$$z - \epsilon = 2\sqrt{(k_1(\tau))},$$

is needed.

(B) Single-layer potential $P^2(z,\tau)$ as shown in (2.11) is a part of solution respect to the boundary values $y'_z(0,\tau)$.

Theorem 2.3.2 *Similarly, let us assume that condition (a) is satisfied. Then:*

I. For $m \in N, P^2(z,\tau) \in N$;

II. $P^2(z,\tau)$ is a distributional solution of the equation $L_k y = mk\delta_{N_2}, Z \geq 0, \tau \geq 0$; satisfies the zero initial condition as $\tau \to 0^+$;

III. If, moreover, $k \in C(R_+)$ and $m' \in N$, then $P^2(z,\tau) \in C^\infty$ in z and C^1 in τ for $z > 0, \tau \geq 0$ and is a classical solution of (2.1) with $g = m = 0$;

IV. $P^2(z,\tau)$ is continuous at $x = 0$ for any $\tau \geq 0$.

Proof 2.3.2 *Let's insert a new variable into equation (2.11):*

$$x = \frac{1}{4\alpha_1(\tau - \gamma)}. \tag{2.15}$$

Our new variable brings an implicit function

$$\gamma = \gamma(\tau, x) \ \ with \ \ \frac{1}{4\alpha_1(\tau)} \leq x < +\infty \ \ and \ \ \gamma = 0 \ \ for \ \ x = \frac{1}{4\alpha_1(\tau)},$$

because $x'_\gamma \geq 0$ (is zero only at isolated points). Therefore, (2.11) can be edited as below, because $k_1(\tau) = \alpha_1(\tau)$:

$$P^2(z,\tau) = \frac{1}{4\pi}\int_{0.25k_1(\tau)}^{\infty} m(\gamma(\tau,x))x^{-1.5}e^{-z^2 x}dx. \tag{2.16}$$

(I) can be shown easily from equation (2.16), because

$$|P^2(z,\tau)| \leq \frac{1}{\sqrt{\pi}}||m||(k_1(\tau))^{1/2}; \ \ (k_1(0) = 0).$$

(II) can be shown as in Theorem 2.2.1.

$$(P^2(z,\tau))'_\tau = 1/2\pi^{-1/2}m(0)(k_1(\tau))^{-1/2}k(\tau)e^{\frac{-z^2}{4k_1(\tau)}} + P^2(z,\tau;m'_\tau). \tag{2.17}$$

Here $P^2(z,\tau;m'_\tau)$ is the potential given by (2.16) with density $[m(\gamma(\tau,x))]'_\tau$.
(III) can be derived from (2.16)-(2.17). $P^2(z,\tau)$ satisfies the equation $L_k P^2 = 0$ for any $z > 0$, because the support of $mk\delta_{N_2}$ is N_2.
(IV) can be reached by comparison of the convergent integral

$$P^2(0,\tau) = \frac{1}{4\pi}\int_{1/(4k_1(\tau))}^{\infty} m(\cdot)x^{-3/2}dx,$$

with $P^2(z,\tau)$ for all z close to 0.
As a result, we obtain this estimate:

$$|P^2(z,\tau) - P^2(0,\tau)| \leq \frac{1}{2}||m|||z|(1 - \Psi(|x^*|/2(k_1(\tau))^{1/2}),$$

where $0 \leq x^* \leq x$ and Ψ is the probability integral [5].

2.4 Double-layer Potential

Double layer potential $D(z,\tau)$ as shown in equation (2.12) is a part of a solution respect to the boundary condition

$$y(0,\tau) = \eta(\tau), \;\; \tau \geq 0.$$

There is a theorem related to double-layer potential:

Theorem 2.4.1 *Suppose that (a) is satisfied. Then,*

I. For $\eta \in N$, $D(z,\tau) \in N$;

II. $D(z,\tau)$ is a distributional solution of the equation $L_k y = -(k\eta\delta_{N_2})'_z$ and satisfies zero initial condition as $\tau \to 0^+$;

III. For $z > 0, \tau \geq 0$ if $k, \eta \in C(R)_+$ and $\eta' \in N$, then $D(z,\tau) \in C^\infty$ in z, and C^1 in τ, and it is a classical solution of (2.1)-(2.2) with $g = \zeta = 0$;

IV. Given that $\eta(\tau) \in C^1(R_+)$, $D(z,\tau)$ satisfies the following "jump formulae":

$$\lim_{z\to\pm 0} D(z,\tau) = \pm\frac{1}{2}\eta(\tau). \tag{2.18}$$

Proof 2.4.1 *(I)-(III) can be proved as in Theorem 2.2.1. To do so, we need to create a new variable. Let it be (2.15). Then, $D(z,\tau)$ can be reformed as:*

$$D(z,\tau = \frac{z}{2\sqrt{\pi}}\int_{1/4k_1(\tau)}^{\infty}\eta(\gamma)x^{-1/2}e^{-z^2x}dx, \tag{2.19}$$

and the derivative of this equation is:

$$\frac{\partial D}{\partial \tau} = \frac{z}{\sqrt{\pi}}\eta(0)(k_1(\tau))^{-3/2}k(\tau)exp(-z^2/4(k_1(\tau)) + D(z,\tau;\eta'_(\tau))), \tag{2.20}$$

where $D(z,\tau;\eta'_\tau)$ is potential (2.19) with density $[\eta(\gamma(\tau,x))]'_\tau$. Hence, (II) can be easily shown as in Theorem 2.3.1. Similarly, (I) and (III) can be derived from (2.19)-(2.20) as shown in Theorem 2.3.2.
Part (IV) needs more efforts to put in. We start with equating $\eta(\tau) = \eta(\gamma)$ for any γ satisfying $0 \le \gamma \le \tau$, and the corresponding double-layer potential is denoted by D_0.Then (2.19) and (2.13) implies that in case $z \neq 0$:

$$D_0 = \frac{z}{2\sqrt{\pi}}\eta(\tau)\int_{1/4k_1(\tau)}^{\infty}x^{-1/2}e^{z^{-2}}xdx = \pm\frac{\eta(\tau)}{2}\left(1-\Psi\left(\frac{z}{2\sqrt{k(\tau)}}\right)\right) \tag{2.21}$$

here the sign of z is placed instead of $\pm$, also,

$$\lim_{z\to 0^\pm} D_0(z,\tau) = \pm\frac{1}{2}\eta(\tau),$$

because of $\Psi(0) = 0$. After that we deal with $D_0 - D$ for $z > 0$, by applying integration with moves, namely over $(0,\tau-\Delta)$ and $(\tau-\Delta,\tau)$. We consider the cases where τ is "regular" and "irregular" individually. To begin with, suppose that

$$D(z,\tau) - D_0(z.\tau) = J_1 + J_2,$$

where

$$J_1 = \frac{z}{4\sqrt{\pi}} \int_0^{\tau-\Delta} (\eta(\tau) - \eta(\gamma)) \frac{k(\gamma)}{\alpha_1^{3/2}(\tau-\gamma)} exp\left(-\frac{z^2}{4\alpha_1(\tau-\gamma)}\right) d\gamma,$$

$$J_2 = \frac{z}{4\sqrt{\pi}} \int_{\tau-\Delta}^{\tau} (\eta(\tau) - \eta(\gamma)) \frac{k(\gamma)}{\alpha_1^{3/2}(\tau-\gamma)} exp\left(-\frac{z^2}{4\alpha_1(\tau-\gamma)}\right) d\gamma,$$

and, similarly in (21), each τ without looking at its regularity,

$$|J_1| \leq ||\eta|| \left[\Psi\left(\frac{z}{2\sqrt{k(\tau) - k(\tau-\Delta)}}\right) - \Psi\left(\frac{z}{2\sqrt{k(\tau)}}\right)\right] \to 0$$

where $z \to 0$ and $\Delta > 0$ is arbitrary and fixed too. I_2 needs to be treated separated for different values of τ. When τ is regular meaning $k(\tau) > 0$, we can choose sufficiently small Δ. Hence, it makes $k(\gamma) > 0$ over interval $[\tau - \Delta, \tau]$, Therefore, substitution of variables $x = (\tau-\gamma)^{-1}$ and $\alpha_1(\tau-\gamma) = \alpha(\gamma)(\tau-\gamma)$ in interval $[\tau-\Delta,\tau]$ help us to get:

$$\begin{aligned} |I_2| &\leq \frac{||k|||z|||\eta'||}{4\sqrt{\pi}k_\Delta^{3/2}} \int_{1/\Delta}^{\infty} x^{-3/2} exp\left(-\frac{z^2}{4k((\tau) - k(\tau-\Delta))}\right) dx \\ &= \frac{||k|||z|||\eta'||}{2\sqrt{\pi}k_\Delta^{3/2}} \sqrt{\Delta} exp\left(-\frac{z^2}{4(k(\tau) - k(\tau-\Delta))}\right). \end{aligned}$$

In the line above, $0 < k_\Delta = \min_{\gamma\in[\tau-\Delta,\tau]} |k(\gamma^*)| \to k(\tau)$ with $\Delta \to 0$. This implies that $I_2 \to 0$ with either z or $\Delta \to 0$. When τ is not regular, in other words, $k(\tau)$=0, it needs another approach. Equation (2.15) allows us to show that

$$\begin{aligned} I_2 &\leq \frac{1}{2} \max_{\gamma\in[\tau-\Delta,\tau]} |\eta(\tau) - \eta(\gamma)| \left(1 - \Psi\left(\frac{z}{2\sqrt{k(\tau) - k(\tau-\Delta)}}\right)\right) \\ &\leq \frac{1}{2} \max_{\gamma\in[\tau-\Delta,\tau]} |\eta(\tau) - \eta(\gamma)| < \varepsilon, \end{aligned}$$

where ε is arbitrary small number. These steps imply that $D_0 - D \to 0$ with $z \to 0$, thus the equation (2.18) [5].

Chapter 3

Elements of potential theory for multi-dimensional degenerate parabolic equations

Now, we move on to construct the potential theory for the multi-dimensional version of the degenerate parabolic equation (2.1). The next step will be an analysis on the consequences. As considered in Chapter 2, we find the fundamental solution by applying Fourier transform in z. It helps us to develop degenerate potential theory. This chapter covers the degenerate versions of volume potential, single-layer potentials (Poisson integral), double layer potential with their properties and applications. To be precise, we first discuss Cauchy problems for multi-dimensional degenerate parabolic equation with its fundamental solution. Secondly, we study the layer potentials. Note that discussion of this session is based on the work [4].

3.1 Fundamental Solution and Cauchy problems

We write our equation (2.1) in new form for multi-dimensional case:

$$\Diamond_k y(z,\tau) := \frac{\partial y(z,\tau)}{\partial \tau} - k(\tau)\Delta_z y(z,\tau) = g(z,\tau), \tag{3.1}$$

Here $(z,\tau) \in \Omega \times (0,T)$, where T is finite and Ω is bounded in R^n with $n > 1$ and Lyapunov boundary $\partial\Omega \in C^{1+\sigma}$, $0 < \sigma < 1$, g is a some function of z and τ. The conditions (a)-(b) on $k(\tau)$ are same as in Chapter 2. To ease our calculation, we will use the function $\beta(\tau,\gamma)$ which is given by the formula

$$\beta(\tau,\gamma) = \int_\gamma^\tau k(v)dv = k_1(\tau) - k_1(\gamma), \ \ \beta(\tau,0) = k_1(\tau).$$

It is worth to mention that satisfying (a) means $\beta(\tau,\gamma)$ is greater than zero for any γ satisfying $\tau > \gamma > 0$.

Lemma 3.1.1 *Under condition (b), the fundamental solution of equation (3.1) is given as*

$$\xi_{n,\beta}(z,\tau) = \xi_n(z,k_1(\tau)) = \frac{H(\tau)exp\left(\frac{|z|^2}{4k_1(\tau)}\right)}{(4\pi k_1(\tau))^{n/2}}, \ \ (z,\tau) \in R^n \times R, \tag{3.2}$$

where ξ_n is a standard fundamental solution of the diffusion operator, H is a Heaviside function and $|z|$ is the Euclidean norm.

Proof 3.1.1

$$\frac{\partial\xi(z,\tau)}{\partial\tau} - k(\tau)\Delta_z\xi(z,\tau) = \delta(z)\delta(\tau). \tag{3.3}$$

By δ we represent the Dirac distribution. After applying the Fourier transform in z, the achieved equation is:

$$\frac{\partial\overline{\xi}(\epsilon,\tau)}{\partial\tau} + k(\tau)|\epsilon|^2\Delta_z\overline{\xi}(\epsilon,\tau) = 1(\epsilon)\delta(\tau), \ \ (\epsilon,\tau) \in R^n \times R, \tag{3.4}$$

where

$$\overline{\xi}(\epsilon,\tau) = G_z[\xi](\epsilon,\tau) = \int_{R^n} \xi(z,\tau)e^{i\langle\epsilon,z\rangle}dz, \ \ (i^2 = -1),$$

$1(\epsilon)$ *is the identity function in R^n and the dot product in R^n is represented as $\langle\cdot,\cdot\rangle$. Hence, solution of the last equation (3.4) is*

$$\overline{\xi}(\epsilon,\tau) = H(\tau)exp(-|\epsilon|^2 k_1(\tau)), \ \ (\epsilon,\tau) \in R^n \times R.$$

The application of inverse Fourier transform with its properties completes the proof.

Now we want to check whether

$$\xi_{n,k}(z-\epsilon,\tau-\gamma) := \xi_n(z-\epsilon,\beta(\tau,\gamma))$$

satisfies the equation

$$\frac{\partial \xi_{n,k}(z-\epsilon,\tau-\gamma)}{\partial \tau} - k(\tau)\Delta_z \xi_{n,k}(z-\epsilon,\tau-\gamma) = \delta(z-\epsilon)\delta(\tau-\gamma)$$

under condition (a) for all values of $\tau,\gamma \in R$ and $z,\epsilon \in R^n$.

Now, we make substitution of the variable $\epsilon_i = \frac{z_i}{2\sqrt{k_1(\tau)}}$ for $i = 1,2,...n$. We will obtain

$$\int_{R^n} \xi_{n,\beta}(z,\tau)dz = \frac{1}{(4\pi k_1(\tau))^{n/2}} \int_{R^n} exp\left(-\frac{|x|^2}{4k_1(\tau)}\right) dz = \prod_{i=1}^{n} \frac{1}{\pi^{n/2}} \int_{-\infty}^{\infty} exp(-\epsilon_i^2)d\epsilon_i = 1, \ \ \tau > 0. \tag{3.5}$$

In addition, $\xi_{n,\beta}(z,\tau)$ has a property such as

$$\xi_{n,\beta}(z,\tau) \longrightarrow \delta(z) \ \ \tau \longrightarrow 0^+, \ \ \forall z \in R^n. \tag{3.6}$$

This property can be shown. Let us take a function μ which is differentiable many times in R^n with compact support. By application of polarization formula, we obtain

$$\int_{R_n} \overline{g}(|z|)dz = \omega_n \int_0^{\infty} \overline{g}(\eta)\eta^{n-1}d\eta,$$

where $\overline{g}$ is some integrable function in R^n, $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. Then by applying MVT, we get

$$\left| \int_{R^n} \xi_{n,\beta}(z,\tau)(\mu(z)-\mu(0))dz \right| \leq \frac{A}{(4\pi k_1(\tau))^n/2} \int_{R^n} exp\left(-\frac{|z|^2}{4k_1(\tau)}\right)|z|dz$$

$$= \frac{A\omega_n}{(4\pi k_1(\tau))^n/2} \int_0^{\infty} exp\left(-\frac{\eta^2}{4k_1(\tau)}\right)\eta^n d\eta = \frac{2A\omega_n\sqrt{k_1(\tau)}}{\pi^{n/2}} \int_0^{\infty} exp(-y^2)y^n dy$$

$$= 2A\sqrt{k_1(\tau)},$$

Here A is some positive constant. Because of the $k_1(\tau)$ is continuous and not negative in interval $[0, T]$, by using (3.5), we can derive (3.6) as below

$$(\xi_{n,\beta}, \mu(z)) = \int_{R^n} \xi_{n,\beta}(z,\tau)\mu(z)dz = \mu(0)\int_{R^n} \xi_{n,\beta}(z,\tau)dz$$

$$+\int_{R^n} \xi_{n,\beta}(z,\tau)(\mu(z) - \mu(0))dz \longrightarrow (\delta(z), \mu(z)) = \mu(0), \;\; as \;\; \tau \longrightarrow 0^+.$$

As we have $\beta(\tau, \gamma) > 0$ for all γ such that $\tau > \gamma > 0$ under condition (a), we can check that

$$\int_{R^n} \xi_{n,k}(z - \epsilon, \tau - \gamma)d\epsilon = 1, \;\; \tau > \gamma > 0, \;\; z \in R^n \;\; [4]. \tag{3.7}$$

3.2 Types of potentials

3.2.1 Poisson potential

The Poisson potential is given by equation below.

$$(Pl)(z,\tau) = \int_{\Omega} \xi_{n,\beta}(z - \epsilon, \tau)l(\epsilon)d\epsilon, \;\; z \in \Omega, \;\; 0 < \tau < T, \tag{3.8}$$

Here l is a bounded function in range R^n with supp$l \subset \Omega$ and $\xi(n, \beta)(z - \epsilon, \tau) = \xi_n(z - \epsilon, k_1(\tau))$. There is a theorem related to Poisson potential.

Theorem 3.2.1 *Assume that coefficient $k(\tau)$ fits the condition (b). Function l is bounded as written in sentence above. Then, Poisson potential (3.8) accepts the estimation*

$$|(Pl)(z,\tau)| \le \sup_{\epsilon \in \Omega} |l(\epsilon)|, \;\; z \in \Omega, \;\; 0 < \tau < T, \tag{3.9}$$

and is a solution of the equation

$$\diamondsuit_k y = 0, \;\; in \;\; \Omega \times (0, T). \tag{3.10}$$

Furthermore, the Poisson potential Pl will be the element of C^∞ class and satisfies *the initial condition*

$$y(\cdot,0) = l, \ \ in \ \ \Omega, \tag{3.11}$$

if the function l is continuous and bounded in R^n with $suppl \subset \Omega$. Also, it provides its continuity in $\Omega \times [0,T]$.

Proof 3.2.1 *It is evident that*

$$(Pl)(z,\tau) = \int_{\Omega} \xi_{n,\beta}(z-\epsilon,\tau) l(\epsilon) d\epsilon$$

$$(Pl)(z,\tau) = \int_{R^n} \xi_{n,\beta}(z-\epsilon,\tau) l(\epsilon) d\epsilon \ \ (z,\tau) \in \Omega \times (0,T).$$

Moreover, we own the estimate

$$|(Pl)(z,\tau)| \leq \sup_{\epsilon \in R^n} |l(\epsilon)| \int_{R^n} \xi_{n,\beta}(z-\epsilon,\tau) d\epsilon = \sup_{\epsilon \in Omega} |l(\epsilon)|.$$

We must check that Pl satisfies the equation (3.10), because for all values of z and τ in their domain, integration and differentiation can be interchanged in equation (3.8). Let function l be bounded as before. It is clear that, by using (3.6), it can be observed that Pl satisfies (3.11). Now, it is turn to make substitution. Let $\epsilon = z + 2\sqrt{k_1(\tau)}v$. Then, we get

$$Pl(z,\tau) = \frac{1}{\pi^{n/2}} \int_{R^n} l(z + 2\sqrt{k_1(\tau)}v) exp(-|v|^2) dv.$$

The supposition for function l grants the continuity and boundedness. Allow N_l be the upper bound for l. Because of the fact that l is uniformly continuous, for every $\xi > 0$, there exist $\delta > 0$ such that $|l(z) - l(\epsilon)| < \xi/2$ for all values of z and $\epsilon \in R^n$ with $|z-\epsilon| < \delta$. Hence, for arbitrary $\xi > 0$, we can select $\eta > 0$ such that

$$\frac{1}{\pi^{n/2}} \int_{|v| \geq \eta} exp(-|v|^2) dv \leq \frac{\xi}{4N_l}.$$

For arbitrary $r > 0$, there exists $\delta_r > 0$ such that $|k_1(\tau)| < r$ for all values $\tau \in [0,T]$ with $\tau < \delta_r$, because $k_1(\tau)$ is continuous in $[0,T]$. Putting $r = \frac{\delta^2}{4\eta^2}$ and applying the *information* $|v| \leq \eta$ *and* $\tau < \delta_r$*, we obtain* $2\sqrt{k_1(\tau)}v < 2\sqrt{r}\eta = \delta$*. We reduce these facts to*

$$\left|\frac{1}{(4\pi k_1(\tau))}\int_{R^n} exp\left(-\frac{|z-\epsilon|^2}{4k_1(\tau)}\right)l(\epsilon)d\epsilon - l(z)\right|$$

$$\left|\frac{1}{\pi^{n/2}}\int_{R^n}(l(z+2\sqrt{k_1(\tau)}v)-l(z))exp(-|v|^2dv)\right|$$

$$< \frac{\xi}{2\pi^{n/2}}\int_{|v|\leq\eta} exp(-|v|^2)dv + \frac{2N_l}{\pi^{n/2}}\int_{|v|\geq\eta} exp(-|v|^2)dv < \xi,$$

for every $z \in R^n$ *and* $\tau < \delta_r$*. It involves the* $Pl's$ *continuity at* $\tau = 0$ *and* $Pl(\cdot, 0) = l$ *in* Ω *[4].*

3.2.2 Volume Potential

The volume potential is written by equation:

$$(Pg)(z,\tau) = \int_0^\tau \int_\Omega \xi_{n,k}(z-\epsilon, \tau-\gamma)g(\epsilon,\gamma)d\epsilon d\gamma, \;\; z\in\Omega, \;\; 0<\tau<T. \tag{3.12}$$

Here the function g is bounded in $\Omega \times [0,T]$ with supp$g(\cdot,\tau) \subset \Omega$, $\forall \tau \in [0,T]$.

Theorem 3.2.2 *Let* $k(\tau)$ *hold the condition (a) and the function* g *be bounded in* $\Omega \times [0,T]$ *with supp* $g(\cdot,\tau) \subset \Omega$ $\forall\tau \in [0,T]$*. Then the potential* Pg *with density* g *accepts the estimation*

$$|(Pg)(z,\tau)| \leq \tau \sup_{(\epsilon,\gamma)\in\Omega\times[0,\tau]} |g(\epsilon,\gamma)|, \;\; z\in\Omega, \;\; 0<\tau<T, \tag{3.13}$$

and solves (3.1) with zero initial condition

$$y(\cdot,\tau) \longrightarrow 0 \;\; as \;\; \tau \longrightarrow 0+, \;\; in \;\; \Omega. \tag{3.14}$$

Proof 3.2.2 *We know that it is evident*

$$(Pg)(z,\tau) = \int_0^\tau \int_\Omega \xi_{n,k}(z-\epsilon,\tau-\gamma)g(\epsilon,\gamma)d\epsilon d\gamma$$

$$= \int_0^{\tau} \int_{R^n} \xi_{n,k}(z-\epsilon, \tau-\gamma) g(\epsilon,\gamma) d\epsilon d\gamma, \;\; z \in \Omega, \;\; \tau \in (0,T),$$

because supp $g(\cdot,\tau) \subset \Omega, \forall \tau$ *such that* $0 \leq \tau \leq T$*. Therefore, by using equation (3.7), we get equation (3.13)*

$$|(Pg)(z,\tau)| \leq \sup_{(\epsilon,\gamma)\in R^n \times [0,\gamma]} |g(\epsilon,\gamma)| \int_0^{\tau} \int_{R^n} \xi_{n,k}(z-\epsilon, \tau-\gamma) g(\epsilon,\gamma) d\epsilon d\gamma$$

$$= t \sup_{(\epsilon,\gamma)\in \Omega \times [0,\gamma]} |g(\epsilon,\gamma)|, \;\; (z,\tau) \in \Omega \times (0,T).$$

By calculating directly, it can be seen that Pg *satisfies our initial equation (3.1) [4].*

Chapter 4

Focus of the thesis

The main focus of this thesis is to demonstrate the multi-dimensional analysis from Chapter 3 to one-dimensional case. Note that calculation in one-dimensional case is different than those in the multi-dimensional case which can be found in [4]. We consider a potential and a differential equation, and that potential must be a solution to the given equation. But, we want to make it unique solution of the problem. To do so, we try to find the lateral boundary conditions. Consider

$$y(z,\tau) = \int_0^1 \xi_\beta(z-\epsilon,\tau) l(\epsilon) d\epsilon, \tag{4.1}$$

$$y(z,0) = l(z), \tag{4.2}$$

where $\xi_\beta(z,\tau)$ is the fundamental solution of the Cauchy problem of the equation

$$\frac{\partial y}{\partial \tau} - k(\tau)\frac{\partial^2 y}{\partial z^2} = 0. \tag{4.3}$$

Conditions on $k(\tau)$

Conditions on $k(\tau)$ are as follows:

(a) $k(\tau) \geq 0$ so, the coefficient can accept zero value.

(b) $k_1(\tau) = \int_0^\tau k(s)ds$ and it is greater than zero when $\tau > 0$. So, $k_1(0) = 0$ and $k_1(t)$ can accept negative values.

It can be noticed that (a) is the special case of (b). So, satisfying first condition automatically means that it satisfies the second one too. When the coefficient $k(\tau)$ is positive, it is easy to solve the given equation. However, when the coefficient is negative and we know that it can accept negative values, the obstacle called degeneracy is observed. Therefore, it needs more effort to handle this problem. By applying Fourier transform in z and the second condition, the fundamental solution of the problem can be obtained by substituting $\xi_\beta(z,\tau)$.

$$\xi_\beta(z,\tau) = \xi(z, k_1(\tau)) = \frac{H(\tau)}{2\sqrt{\pi k_1(\tau)}} \exp(-z^2/4k_1(\tau)). \tag{4.4}$$

The fundamental solution contains $H(\tau)$ function which is called the Heaviside function. It is usually defined as an integral of the Dirac delta function. Now, the remaining task is to give lateral boundary conditions of the problem.

4.1 Simple example

Here we discuss the main result from [7].

As written above, potentials are the elements of potential theory and they are the solutions for differential equations. For instance, let us look at following potential in one dimension, where $\Omega = (0,1)$. By t we denote the time in our equation and it satisfies $t \in \Omega$.

$$y(t) = \int_0^1 -\frac{1}{2} \mid t - \tau \mid f(\tau) d\tau. \tag{4.5}$$

Here f can be integrated in the interval (0,1). The fundamental solution of the equation below (4.6) is the kernel of the potential (4.5)

$$-\partial_t^2 \xi(t-\tau) = \delta(t-\tau), \tag{4.6}$$

where

$$\xi(t-\tau) = -\frac{1}{2}(t-\tau).$$

and δ is the Dirac distribution. Therefore, the potential (4.5) fits the equation

$$-\partial_t^2 y(t) = f(t), \quad t \in \Omega. \tag{4.7}$$

The next task is to find the lateral boundary conditions of the above equation. By using integration by parts from Calculus, we obtain

$$y(t) = \int_0^1 -\frac{1}{2} \mid t - \tau \mid f(\tau) d\tau = \int_0^1 \frac{1}{2} \mid t - \tau \mid \partial_\tau^2 y(\tau) d\tau$$

$$= \int_0^t \frac{1}{2}(t - \tau)\partial_\tau^2 y(\tau) d\tau + \int_0^1 \frac{1}{2}(\tau - t)\partial_\tau^2 y(\tau) d\tau$$

$$= y(t) - t\frac{y'(0) + y'(1)}{2} - \frac{-y'(1) + y(0) + y(1)}{2}, \quad \forall t \in (0, 1),$$

after canceling $y(t)$ from both sides, we are left with,

$$t(y'(0) + y'(1)) + (-y'(1) + y(0) + y(1)) = 0, \quad \forall t \in (0, 1).$$

Then, boundary conditions for the potential (4.5) are found after canceling t.

$$y'(0) + y'(1) = 0 \ \textit{and} \ - y'(1) + y(0) + y(1) = 0. \tag{4.8}$$

Now, it is time to construct our boundary value problem with obtained results.

$$-\partial_t^2 y(t) = f(t), \quad t \in \Omega,$$

$$y'(0) + y'(1) = 0 \ \textit{and} \ - y'(1) + y(0) + y(1) = 0,$$

$$y(t) = \int_0^1 -\frac{1}{2} \mid t - \tau \mid f(\tau) d\tau. \ [7]$$

Now, our problem has become a BVP with a unique solution. Above strategy works best with ODE, but it is tiresome for partial differential equations. This type of problems first appeared in Kac's works, who was a Polish-American mathematician.

He developed these BVPs with further applications [1]. (see also [6])

4.2 Main result

Here we use some techniques from [2] and [3].

We have equations (4.1)-(4.3) with $0 < x < 1$. And we know that solution of our equation is of the form after substituting $\xi_\beta(z,\tau)$ with domain $D = 0 < z < 1, \tau > 0$

$$y(z,\tau) = \frac{1}{2\sqrt{\pi k_1(\tau)}} \int_0^1 \exp\left(\frac{-(z-\epsilon)^2}{4k_1(\tau)}\right) l(\epsilon) d\epsilon. \tag{4.9}$$

Theorem 4.2.1 *For any $k(\tau) \in C^{\frac{\alpha}{2}}(0,T)$ and $g(z,\tau) \in C_{z,\tau}^{\alpha,\frac{\alpha}{2}}(\Omega)$ the generalised heat potential is a unique solution of the equation (4.1)-(4.3) in $C_{z,\tau}^{2+\alpha,\frac{1+\alpha}{2}}$ with boundary conditions*

$$I_y(z,\tau)|_{z=1} = 0, \ I_y(z,\tau)|_{z=0} = 0. \tag{4.10}$$

where

$$I_y(z,\tau) := -\frac{y(z,\tau)k(\tau)}{2}$$

$$+ \int_0^\tau \left[\frac{\partial \xi_\beta(z-\epsilon,\tau-\gamma)}{\partial \epsilon} k(\tau) y(\epsilon,\gamma) - \epsilon_\beta(z-\epsilon,\tau-\gamma)k(\gamma)\frac{\partial y(\epsilon,\gamma)}{\partial \epsilon} \right] \Bigg|_{\epsilon=0}^{\epsilon=1} d\gamma.$$

Proof 4.2.1 *We know that*

$$y(z,\tau) = \lim_{\delta \to 0} y_\delta(z,\tau),$$

and

$$y_\delta(z,\tau) = \int_0^{\tau-\delta} d\gamma \int_0^1 \xi_k(z-\epsilon,\tau-\gamma) g(\epsilon,\gamma) d\epsilon$$

In addition, it clear that

$$\begin{aligned} \frac{dy(z,\tau)}{d\tau} &= \frac{\partial y(z,\tau)}{\partial z}\frac{\partial z}{\partial \tau} + \frac{\partial y(z,\tau)}{\partial \tau}\frac{\partial \tau}{\partial \tau} = \frac{\partial y(z,\tau)}{\partial \tau}, \\ \frac{dy(z,\tau)}{dz} &= \frac{\partial y(z,\tau)}{\partial z}\frac{\partial z}{\partial z} + \frac{\partial y(z,\tau)}{\partial \tau}\frac{\partial \tau}{\partial z} = \frac{\partial y(z,\tau)}{\partial z}. \end{aligned} \tag{4.11}$$

After applying the properties of fundamental solution (4.4), we have:

$$0 = \lim_{\delta\to 0}\int_0^{\tau-\delta} d\gamma \int_0^1 0y(\epsilon,\gamma)d\epsilon = \lim_{\delta\to 0}\int_0^{\tau-\gamma} d\gamma \int_0^1 \delta(z-\epsilon,\tau-\gamma)y(\epsilon,\gamma)d\epsilon$$

$$= \lim_{\delta\to 0}\int_0^{\tau-\delta} d\gamma \int_0^1 \left(-\frac{\partial\xi_\beta(z-\epsilon,\tau-\gamma}{\partial\gamma}) - k(\tau)\frac{\partial^2\xi_\beta(z-\epsilon,\tau-\gamma}{\partial\epsilon^2}\right) y(\epsilon,\gamma)d\epsilon,$$

also

$$\begin{gathered} \lim_{\delta\to 0}\int_0^1 \xi_\beta(z-\epsilon,\delta)y(\epsilon,\tau-\delta)d\epsilon \\ = \lim_{\delta\to 0}\int_0^1 \delta(z-\epsilon))y(\epsilon,\tau-\delta)d\epsilon = y(z,\tau), \;\; z\in(0,1). \end{gathered} \tag{4.12}$$

Taking into account that $y(z,\tau) \in C_{z,\tau}^{2+\alpha,1+\frac{\alpha}{2}}(\Omega)$ *and by considering equations (4.9), (4.8), (4.3) and (4.2) for* $\forall(z,\tau)\in(0,1)\times(0,T)$ *a straight computation gives:*

$$0 = \int_0^\tau d\gamma \int_0^\infty \xi_\beta(z-\epsilon,\tau-\gamma)g(\epsilon,\gamma)d\epsilon$$

$$= \int_0^\tau d\gamma \int_0^\infty \xi_\beta(z-\epsilon,\tau-\gamma)\diamond_\beta y(z,\tau)d\epsilon$$

$$= \lim_{\delta\to 0}\int_0^1 d\epsilon \int_0^{\tau-\delta} \xi_\beta(z-\epsilon,\tau-\gamma)\frac{\partial y(\epsilon,\gamma)}{\partial\gamma}d\gamma$$

$$-\lim_{\delta\to 0}\int_0^{\tau-\delta} d\gamma \int_0^1 \xi_\beta(z-\epsilon,\tau-\gamma)k(\gamma)\frac{\partial^2 y(\epsilon,\gamma)}{\partial\epsilon^2}d\epsilon$$

$$= \lim_{\delta\to 0}\int_0^1 d\epsilon \int_0^{\tau-\delta} \xi_\beta(z-\epsilon,\tau-\gamma)dy(\epsilon,\gamma)$$

$$-\lim_{\delta\to 0}\int_0^{\tau-\delta} d\gamma \int_0^1 \xi_\beta(z-\epsilon,\tau-\gamma)k(\gamma)dy_\epsilon(\epsilon,\gamma)$$

$$= \lim_{\delta\to 0}\int_0^1 [\xi_\beta(z-\epsilon,\tau-\gamma)y(\epsilon,\gamma)]\big|_{\gamma=0}^{\gamma=\tau-\delta}d\epsilon$$

$$-\lim_{\delta\to 0}\int_0^{\tau-\delta} d\gamma \int_0^1 \frac{\partial\xi_\beta(z-\epsilon,\tau-\gamma)}{\partial\gamma}y(\epsilon,\gamma)d\epsilon$$

$$-\lim_{\delta\to 0}\int_0^{\tau-\delta} k(\gamma)[\xi_\beta(z-\epsilon,\tau-\gamma)\frac{\partial y(\epsilon,\gamma)}{\partial \epsilon}-\frac{\partial \xi_\beta(z-\epsilon,\tau-\gamma)}{\partial \epsilon}y(\epsilon,\gamma)]\big|_{\epsilon=0}^{\epsilon=1}d\gamma$$

$$-\lim_{\delta\to 0}\int_0^{\tau-\delta}[\frac{\partial^2 \xi_\beta(z-\epsilon,\tau-\gamma)}{\partial \epsilon^2}k(\gamma)y(\epsilon,\gamma)]d\epsilon$$

$$=\lim_{\delta\to 0}\int_0^{\tau-\delta} d\gamma\int_0^1\left(-\frac{\partial \xi_\beta(z-\epsilon,\tau-\gamma)}{\partial \gamma}-k(\gamma)\frac{\partial^2 \xi_\beta(z-\epsilon,\tau-\gamma)}{\partial \epsilon^2}\right)y(\epsilon,\gamma)d\epsilon$$

$$+\lim_{\delta\to 0}\int_0^1 \xi_\beta(z-\epsilon,\delta)y(\epsilon,\tau-\delta)-\xi_\beta(z-\epsilon,\tau)y(\epsilon,0)d\epsilon$$

$$-\lim_{\delta\to 0}\int_0^{\tau-\delta} k(\gamma)[\xi_\beta(z-\epsilon,\tau-\gamma)\frac{\partial y(\epsilon,\gamma)}{\partial \epsilon}-\frac{\partial \xi_\beta(z-\epsilon,\tau-\gamma)}{\partial \epsilon}y(\epsilon,\gamma)]\big|_{\epsilon=0}^{\epsilon=1}d\gamma$$

$$=0+\lim_{\delta\to 0}\int_0^1 \xi_\beta(z-\epsilon,\delta)y(\epsilon,\tau-\delta)d\epsilon$$

$$-\lim_{\delta\to 0}\int_0^1 \xi_\beta(z-\epsilon,\tau)y(\epsilon,0)d\epsilon$$

$$-\lim_{\delta\to 0}\int_0^{\tau-\delta} k(\gamma)[\xi_\beta(z-\epsilon,\tau-\gamma)\frac{\partial y(\epsilon,\gamma)}{\partial \epsilon}-\frac{\partial \xi_\beta(z-\epsilon,\tau-\gamma)}{\partial \epsilon}y(\epsilon,\gamma)]\big|_{\epsilon=0}^{\epsilon=1}d\gamma$$

$$=y(z,\tau)-y(z,\tau)-\lim_{\delta\to 0}\int_0^{\tau-\delta} k(\gamma)[\xi_\beta(z-\epsilon,\tau-\gamma)\frac{\partial y(\epsilon,\gamma)}{\partial \epsilon}-\frac{\partial \xi_\beta(z-\epsilon,\tau-\gamma)}{\partial \epsilon}y(\epsilon,\gamma)]\big|_{\epsilon=0}^{\epsilon=1}d\gamma$$

$$=\lim_{\delta\to 0}\int_0^{\tau-\delta} k(\gamma)[\xi_\beta(z-\epsilon,\tau-\gamma)\frac{\partial y(\epsilon,\gamma)}{\partial \epsilon}-\frac{\partial \xi_\beta(z-\epsilon,\tau-\gamma)}{\partial \epsilon}y(\epsilon,\gamma)]\big|_{\epsilon=0}^{\epsilon=1}d\gamma.$$

It yields

$$\int_0^{\tau} k(\gamma)[\xi_\beta(z-\epsilon,\tau-\gamma)\frac{\partial y(\epsilon,\gamma)}{\partial \epsilon}-\frac{\partial \xi_\beta(z-\epsilon,\tau-\gamma)}{\partial \epsilon}y(\epsilon,\gamma)]\big|_{\epsilon=0}^{\epsilon=1}d\gamma=0,\quad (z,\tau)\in\Omega. \tag{4.13}$$

In case if $z\longrightarrow 1-0$ *and* $z\longrightarrow 0+0$, *the two integrals below brings the jump relations.*

$$\int_0^{\tau}\left[\frac{\partial \xi_k(z-\epsilon,\tau-\gamma)}{\partial \epsilon}k(\gamma)y(\epsilon,\gamma)\right]\bigg|_{\epsilon=1}d\gamma,$$

$$\int_0^{\tau}\left[\frac{\partial \xi_k(z-\epsilon,\tau-\gamma)}{\partial \epsilon}k(\gamma)y(\epsilon,\gamma)\right]\bigg|_{\epsilon=0}d\gamma.$$

As a result, when $z\longrightarrow 1-0$ *and* $z\longrightarrow 0+0$, *we achieve the lateral boundary* *conditions*

$$I_y(z,\tau)\Big|_{z=1} = 0, \;\; \textit{and} \;\; I_y(z,\tau)\Big|_{z=0} = 0,$$

where

$$I_y(z,\tau) := -\frac{y(z,\tau)k(\tau)}{2}$$

$$+\int_0^\tau \left[\frac{\partial\xi_\beta(z-\epsilon,\tau-\gamma)}{\partial\epsilon}k(\tau)y(\epsilon,\gamma) - \epsilon_\beta(z-\epsilon,\tau-\gamma)k(\gamma)\frac{\partial y(\epsilon,\gamma)}{\partial\epsilon}\right]\Bigg|_{\epsilon=0}^{\epsilon=1} d\gamma.$$

It completes the proof.

Chapter 5

Conclusion

To conclude with, this thesis paper studied the elements of potential theory and their applications in one-dimensional and multi-dimensional degenerate parabolic equations. Researchers have been studying potential theory and have applied it to solve initial-boundary value problems. Elements of potential theory such as volume potentials, Poisson integrals, single and double layer potentials can be solutions for different problems differing in source function and initial conditions. To do so, one finds first fundamental solution by applying transform operations. In this thesis, we considered the potential which is a solution for degenerate type parabolic equation in one dimension with zero source function. Here the word degenerate refers to the conductivity coefficient given with two conditions. The aim of the thesis was to find boundary conditions that make given potential a unique solution for given equation. Before finding them, we gave a simple example which can easily explain the task. As a result, we found desired boundary conditions by using properties of fundamental solution, potential theory and simple integration techniques. The multi-dimensional analogue of this result can be found in [4].

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