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MATH 499 CAPSTONE PROJECT

NONLINEAR SCHRÖDINGER-AIRY EQUATION IN SOBOLEV SPACES OF LOW REGULARITY

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Abstract. The Nonlinear Schrödinger-Airy equation is one of the general examples of dispersive nonlinear partial differential equations. It is commonly used to characterize the nonlinear propagation of light pulses in optical fibers and is of great importance in quantum mechanics. In this Capstone Project, we perform the first steps to show that the solution satisfies a priori upper bound in terms of the H^s(Sobolev Space) size of the initial data for $-\frac{1}{8} < s < \frac{1}{4}$. The result is weaker than the well-posedness. The Capstone Project provides a general scheme of the ideas for the problem described above.

1. Introduction

In this Capstone Project we will consider the initial value problem for the Nonlinear Schrödinger-Airy equation

$$(1.1)\qquad \begin{cases} \partial_t u + i\,a\,\partial_x^2 u + b\,\partial_x^3 u + i\,c\,|u|^2 u + d\,|u|^2\partial_x u + e\,u^2\partial_x\overline{u} = 0, & x,t \in \mathbb{R},\\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $u = u(x,t)$ is a complex valued function and a, b, c, d and e are real parameters. This model was proposed by A. Hasegawa and Y. Kodama ([7, 14]) to describe the nonlinear propagation of pulses in optical fibers. Usually it is referred to as a higher-order nonlinear Schrödinger equation.

It can be noticed that the IVP (1.1) requires so much differentiability, so it is often convenient to work with the integral formulation of the equation which is a *Duhamel's formula*

$$(1.2)\quad u(x,t) = e^{-(t-t_0)(ia\partial_x^2 + b\partial_x^3)}u(x,t_0) +$$

$$+\int_{t_0}^{t} e^{-(t-s)(ia\partial_x^2 + b\partial_x^3)}(i\,c\,|u(x,s)|^2 u(x,s) + d\,|u(x,s)|^2\partial_x u(x,s) + e\,u(x,s)^2\partial_x\overline{u(x,s)})ds,$$

for every $t \in I$. In that representation $e^{-(t-t_0)(ia\partial_x^2 + b\partial_x^3)}$ is called a semigroup, which is a solution to the linear PDE. (Lemma 2.9) We assume that u is continuous, not necessarily differentiable. One can refer to such solutions as *distributional solutions* not pointwise.

Before, starting the analysis of our problem we discuss the previous proven results for our equation. G. Staffilani ([16]) showed that the initial value problem (1.1) is locally well-possed in $H^s(\mathbb{R})$(Sobolev space), for any $s \geq 1/4$. Note that the well-posedness of the problem analyzes three main concepts which are the existence, uniqueness and stability of the solution u. So, we say that the problem is well-posed if all these properties hold. If some of these properties fail to exist, we say that the problem is ill-posed. On the other hand, it was justified that the problem is ill-posed, showing that the data solution map is *not* uniformly continuous in some fixed ball H^s in [1].

We distinguish two types of well-posedness: local and global. It was mentioned above that our initial problem (1.1) is both locally and globally well-posed($s \geq 1/4, s > 1/4$) for time $0 < t < T$ and arbitrarily large time interval. Later, X. Carvajal ([2]) established the global well-posedness in $H^s(\mathbb{R})$, $s > 1/4$, provided that $c = (d-e)a/(3b)$.

Prior to starting the discussion about Nonlinear Schrödinger-Airy equation (1.1), we note that for certain choice of the parameters, we obtain very well-known equations.

First note that taking $a = -1$, $c = \mp 1$ and $b = d = e = 0$, equation (1.1) reduces to the cubic nonlinear Schrödinger equation:

$$(1.3)\qquad i\partial_t u + \partial_x^2 u \pm |u|^2 u = 0.$$

The local and global well-posedness for the NLSE in $H^s(\mathbb{R})$, $s \geq 0$, was established by Y. Tsutsumi ([20]). For all $s < 0$ it is ill-posed, in the sense that solutions fail to depend uniformly continuously on initial data in the H^s–norm ([4, 8]). However, M. Christ, J. Colliander and T. Tao ([3]) showed an a priori upper bound for the H^s–norm of the solution, when $s > -1/12$, in terms of the H^s–norm of the datum.

Similar results were independently obtained by Koch and Tataru in ([10]); these apply to the range $s \geq -1/6$. In ([11, 12]), these authors improve their previous results for $s \geq -1/4$.

Similarly, by setting $a = c = e = 0$, $b = 1$ and $d = \pm 1$ we obtain the complex modified Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u \pm |u|^2 \partial_x u = 0. \tag{1.4}$$

When u is real, (1.4) is known as the mKdV equation. Its local well-posedness in H^s, $s \geq 1/4$, was shown by C. Kenig, G. Ponce and L. Vega ([9]) and the global well-posedness for $s > 1/4$ by J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao ([6]). Furthermore, the data-to-solution map fails to be uniformly continuous on a fixed ball in $H^s(\mathbb{R})$ when $s < 1/4$ ([4, 8]). In spite of this, M. Christ, J. Holmer and D. Tataru ([5]) established that for $-1/8 < s < 1/4$ the solution satisfies global in time $H^s(\mathbb{R})$ bounds which depend only on the time and on the H^s–norm of the initial data.

Consequently, investigating particular papers about the nonlinear Schrödinger equation([11]) and modified Korteweg-de Vries equation([5]) motivated to analyze the behavior of the solution in Sobolev space for $s < 1/4$. At this stage, it is difficult to consider the question of uniqueness and stability of solution in the H^s norm for $s < 1/4$. Primarily, our main task will be to prove a priori estimate for the size of a solution of (1.1). One can understand it as a bound on the solution, where its norm is bounded by quantity consisting of some constant and initial data.

So, the goal of this note is to show the following similar result for the Schrödinger-Airy equation, which is weaker than the global well-posedness.

Theorem 1.1. *Fix $R > 0$ and $T > 0$, and let $-1/8 < s < 1/4$. There exists a constant $C = C(R,T) > 0$ such that for every initial data $u_0 \in \mathcal{S}(\mathbb{R})$ satisfying*

$$\|u_0\|_{H^s} \leq R,$$

and every solution $u \in \mathscr{C}_t^0 \mathcal{S}_x([0,T] \times \mathbb{R})$ to the IVP (1.1),

$$\|u\|_{L_t^\infty H_x^s([0,T]\times\mathbb{R})} \leq C \|u_0\|_{H^s}.$$

Note that we will show the result for H^s, $-1/8 < s < 0$ only, since the proof for $0 < s < 1/4$ can be done in a different approach.

In this way our main task was introduced. Next step is to prove this result and show that the solution to Nonlinear Schrödinger-Airy equation satisfies a priori upper bound. So, in order to prove our main theorem (1.1) we investigate and apply three theorems: basic estimates, trilinear estimates and energy bound. We will establish and prove those theorems and justify them by stating preliminary lemmas and propositions. Note that the main objective of my Capstone Project will be to propose the structure and settle the first steps to achieve the goal. The project with full details that involves all of the proofs and statements will be published later.

An outline of the Capstone Project is as follows.

In Section 2, we define all functions spaces employed in the analysis. The section will include three subsections, namely Littlewood-Paley partition, atomic decomposition of u and function spaced adapted to our PDE.

In Section 3, the basic estimate is proved. It will be about controling the linear part of our equation (1.1).

In Section 4, we examine the fundamental estimates applied in the proofs of the trilinear estimates and energy bound. These combine local smoothing, Stricharts estimates and Bernstein inequality.

Section 5 will discuss the trilinear estimates to control the nonlinear part of our equation (1.1).

In Section 6, we similarly discuss all necessary preliminaries used in the proof of energy bound. We use a varitaion of the I-method in [6, page 708] in order to construct almost conserved energy functional. Then the behavior of the energy functional will be computed.

Finally, in Section 7, we combine all the components to prove Main Theorem 1.1 This is done by defining the result for the small data. Unlike two popular equations 1.3 and 1.4 our equation does not have the property of scaling, so the main theorem is proved in a different approach.

2. Function spaces

Since the theorems and statements involve different spaces and functions we shall define them step by step. Before going into details, first define the Hilbert space.

Let H be a function space with inner product $\langle f, g\rangle$ for $f, g \in H$. We call it *Hilbert space* if it is a complete metric space with norm defined by

$$\|f\|_H = \sqrt{\langle f, f\rangle}.$$

Next we define the norm of the function space $L^2(\mathbb{R})$, which is one of the Hilbert spaces as

$$\|f\|_{L^2(\mathbb{R})} = \Big(\int_{\mathbb{R}} |f(x)|^2 dx\Big)^{1/2},$$

where $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$. Conjugate symmetry property of inner product is

$$\langle f, g\rangle = \overline{\langle g, f\rangle}$$

for all $f, g \in L^2(\mathbb{R})$. The inner product of $f \in L^2(\mathbb{R})$ with itself is stated as

$$\langle f, f\rangle = \int_{\mathbb{R}} f(x)\overline{f(x)}dx = \int_{\mathbb{R}} |f(x)|^2 dx = \|f\|^2_{L^2(\mathbb{R})}.$$

Now we are going to state lemma about Hölder inequality which will be one of the useful tools in proving theorems.

Lemma 2.1. *Let* $p, q \in [1, \infty)$ *with* $1/p + 1/q = 1$. *Then*

$$\int_a^b |f(x)g(x)|dx \le \Big(\int_a^b |f(x)|^p dx\Big)^{\frac{1}{p}}\Big(\int_a^b |g(x)|^q dx\Big)^{\frac{1}{q}} \tag{2.1}$$

The proceeding definition will define *Fourier Transform* of a function f. Let f be a function from $\mathbb{R} \to \mathbb{C}$. Then its Fourier Transform is denoted by $\hat{f}$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)dx, \qquad for\ \xi \in \mathbb{R}.$$

The Inverse Fourier Transform is defined to be

$$f(x) = \int_{\mathbb{R}} e^{2\pi i x\xi} \hat{f}(\xi)d\xi, \qquad for\ x \in \mathbb{R}.$$

The Fourier Transform of the Complex Conjugate of function f is expressed as

$$\widehat{\overline{f(x)}} = \overline{\hat{f}(-\xi)}$$

Now we state the following result that is called Plancherel's equality.

Lemma 2.2. *(Plancherel's theorem). Let* f *and* g *be square integrable functions on* $\mathbb{R}$. *Then*

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)\, dx = \int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\, d\xi = \langle \widehat{f}, \widehat{g}\rangle. \tag{2.2}$$

Next we define Schwartz space.

The Schwartz space or space of rapidly decreasing functions on the set of real numbers is the function space defined by:

$$\mathcal{S}(\mathbb{R}) = \{f \in \mathscr{C}^\infty(\mathbb{R}) : \|f\|_{\alpha,\beta} < \infty \quad for\ \ all\ \ \alpha, \beta \in \mathbb{N}\}.$$

Where $\mathscr{C}^\infty(\mathbb{R})$ is the set of all smooth functions from $\mathbb{R}$ to C, and

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}} |x^\alpha \partial^\beta f(x)|.$$

Next, we define Sobolev space. Particularly, we need Sobolev spaces with non-integer order:

$$H^s(\mathbb{R}) := \Big\{ f(x) \in L^2(\mathbb{R}) : \mathcal{F}^{-1}\Big[(1+|\xi|^2)^{\frac{s}{2}} \hat{f}(\hat{\xi})\Big] \in L^2(\mathbb{R}) \Big\}.$$

Norm H^s is defined by

$$\|\phi\|_{H^s} = \|(|\xi|^2+1)^{\frac{s}{2}} \widehat{\phi}\|_{L^2}.$$

Similarly, we can define the norm H^s_M

$$\|\phi\|_{H^s_M} = \|(|\xi|^2+M)^{\frac{s}{2}} \widehat{\phi}\|_{L^2}.$$

From the definition of Sobolev spaces, we can deduce the following properties:

(1) If $s < s'$, then $H^{s'}(\mathbb{R}) \subset H^s(\mathbb{R})$.

(2) For any $s \in \mathbb{R}$, the Schwartz space $\mathcal{S}(\mathbb{R})$ is dense in $H^s(\mathbb{R})$.

Other properties can be found in (page 46, [15]).

Finally, for a smooth positive even symbol a satisfying $|a_N(\xi)| \lesssim a(\xi)$ the following space H^a is defined as

$$\|\phi\|_{H^a} = \langle \phi, a(D)\phi \rangle.$$

2.1. **Littlewood-Paley partition of the unity.** We begin this section by defining *Littlewood-Paley partition* which will be used to define other function spaces. We now state our first definition in this subsection that was stated in [18].

Definition 2.3. *Let $\phi \in \mathscr{C}_c^\infty(\mathbb{R})$ be equal 1 on $[-1,1]$ and have its support in $[-2,2]$. Let $\psi(\xi)$ be the function*

$$\psi(\xi) \stackrel{\text{def}}{=} \phi(\xi) - \phi(2\xi).$$

Then we can define ψ_N

$$\psi_N := \phi(\frac{\xi}{N}) - \phi(2\frac{\xi}{N}),$$

where N is a dyadic number such that $N = 2^k$. This ψ is a bump function that is supported in the annulus $1/2 \le |\xi| \le 2$. And we have the following Littlewood-Paley partition of unity of ξ-space that was defined in [17, Equation 24, page 242]

$$\phi(\xi) + \sum_{k=1}^{\infty} \psi\Big(\frac{\xi}{N}\Big) = 1 \quad \text{for all} \quad \xi \in \mathbb{R}. \tag{2.3}$$

$\psi\Big(\frac{\xi}{N}\Big)$ is supported in the annulus $|\xi| \sim N$. Moreover, ψ_N is supported inside the $\{\xi \in \mathbb{R}^n : \frac{N}{2} \le |\xi| \le 2N\}$. It implies that for every ξ there are at most three nonzero terms in the sum 2.3

We now define *Littlewood-Paley projection operators.*

Definition 2.4. *$P_N f$ and $P_{\le N} f$ Fourier multiplier associated to the function ψ, then we have*

$$\widehat{P_N u(\xi,t)} = \psi\Big(\frac{\xi}{N}\Big)\widehat{u}(\xi,t), \qquad P_N u(x,t) = \mathcal{F}^{-1}\Big[\psi\Big(\frac{\xi}{N}\Big)\widehat{u}(\xi,t)\Big](x)$$

$$\widehat{P_{\le N} u(\xi,t)} = \phi\Big(\frac{\xi}{N}\Big)\widehat{u}(\xi,t), \qquad P_{\le N} u(x,t) = \mathcal{F}^{-1}\Big[\phi\Big(\frac{\xi}{N}\Big)\widehat{u}(\xi,t)\Big](x)$$

where N is a dyadic number of the form $N = 2^k$, $k \in \mathbb{Z}$.

Generally, P_N is a frequency projection in the annulus $\{|\xi| \sim N\}$, while $P_{\leq N}$ is a frequency projection to the ball $\{|\xi| \lesssim N\}$.

The Littlewood-Paley projections $P_N f$ commute with derivatives. Now we express it more precisely by the following statement from [18].

Lemma 2.5. *Let N be a dyadic number and let $f(x,t)$ be a function with support in the annulus $\{\xi \in \mathbb{R} : N/2 \leq |\xi| \leq 2N\}$. Then we have*

$$\|\partial_x f(x,t)\|_{L^p_x} \sim N\|f(x,t)\|_{L^p_x}$$

for all $1 \leq p \leq \infty$. Particularly, we have $\|\partial_x P_N f(x,t)\|_{L^p_x} \sim N\|P_N f(x,t)\|_{L^p_x}$.

2.2. **Atomic decomposition of u.**

We first state the space-time function spaces $U^2(I;H)$ (atomic-space) and $V^2(I;H)$ (space of functions of bounded p-variation) in [10]. Particularly, spaces U^2 and V^2 allow us to define Bourgain's function spaces adapted to the dispersive equations. They are defined on a time interval $I = [a,b)$, where $-\infty \leq a < b \leq +\infty$ and take values in Hilbert space $H \in \{L^2, H^s, H^a\}$. In addition, this section will define and mention some of their basic properties. Now, we state the definition from the [13].

Definition 2.6. *Given a partition $a = t_0 < t_1 < ... < t_k = b$ of I and a sequence $\{\phi_k\}_{k=0}^{K-1} \subset H$ such that $\sum_{k=1}^{K} \|\phi_{k-1}\|_H^2 = 1$, the function*

$$a(t) = \sum_{k=1}^{K} \phi_{k-1}(x)\chi_{[t_{k-1},t_k)}(t)$$

is called a $U^2(I;H)$ atom.
Let a_l be a sequence of atoms and let λ_l be a summable sequence, then

$$u(t) = \sum_{l=0}^{\infty} \lambda_l\, a_l, \quad \textit{where } a_l \textit{ are } U^p(I) \textit{ atoms.} \tag{2.4}$$

is a U^2 function. $U^2(I;H)$ is defined as the collection of functions $u(t)$ on I that has the following norm

$$\|u(t)\|_{U^2(I;H)} = \inf_{\textit{representations } (2.4)} \sum_{l=0}^{\infty} |\lambda_l|.$$

Atoms are right-continuous. Next, we define the space $V^2(I)$ as the space of all functions $v : I \to H$. It is considered as the dual space of a space U^2. Then the following norm will be finite:

$$\|v\|_{V^2(I;H)} = \sup_{\{t_k\}} \Big(\sum_{k=1}^{K-1} \|v(t_k) - v(t_{k-1})\|_H^2 \Big)^{1/2}.$$

Here, the supremum is taken over partitions $a = t_0 < ... < t_K = b$. For $I = [a,b)$, $-\infty < a < b < \infty$, we have

$$\|u\|_{U^2(I;H)} = \|\chi_I u\|_{U^2([-\infty,+\infty))}.$$

Also,

$$\|v\|_{V^2(I;H)} \leq \|\chi_I v\|_{V^2([-\infty,\infty))} \leq 2\|v\|_{V^2(I;H)}, \qquad v \in V_0^2(I) \tag{2.5}$$

$V_0^2(I)$ is the subspace of functions v in $V^2(I;H)$ such that $v(a)=0$.
Here,

$$\chi_I(x)=\begin{cases}1, & x\in I\\ 0, & x\notin I\end{cases}$$

is called *characteristic function* defined on a set I.

Next, we state the following lemma about embeddings. Please, refer to [13] for the proof in L^2 space. Now we conjecture that this lemma should also hold for any Hilbert space H.

Lemma 2.7. *(U-V embeddings). Let I be fixed interval such that $I=[a,b)$.*

1. If $1\le p\le q<\infty$, then $U^p\subset U^q(I;H)$, $V^p(I;H)\subset V^q$ and $\|u\|_{U^q(I;H)}\le \|u\|_{U^p(I;H)}$, $\|u\|_{V^q(I;H)}\le \|u\|_{V^p(I;H)}$.

2. If $1\le p<\infty$ then $U^p(I;H)\subset V^p(I;H)$ $\|u\|_{V^p(I;H)}\lesssim \|u\|_{U^p(I;H)}$.

3. If $1\le p<q<\infty$, $u(a)=0$, and $u\in V^p(I;H)$ is right-continuous, then $\|u\|_{U^q(I;H)}\lesssim \|u\|_{V^p(I;H)}$.

4. Suppose that $1\le p<q<\infty$, and T is a linear operator with the boundedness properties:

$$\|Tu\|_X\le C_q\|u\|_{U^q(I;H)},\qquad \|Tu\|_X\le C_p\|u\|_{U^p(I;H)},\qquad \text{with } 0<C_p\le C_q,$$

for some Banach space X. Then

$$\|Tu\|_X\lesssim \langle \ln\frac{C_q}{C_p}\rangle \|u\|_{V^p(I;H)},$$

with implicit constant depending only on the proximity of q and p.

Next lemma is about duality relation between two function spaces.

Lemma 2.8. *(DU-V duality). We have $(DU^2(I;H))^*=V_0^2(I;H)$ with respect to a duality relation which for $f\in H$ becomes the usual pairing $\langle f,v\rangle=\int_a^b\langle f(t),v(t)\rangle_x dt=\int_a^b\int_x f\overline{v}\,dx\,dt$.*

This lemma has an application in the proof of theorems about Trilinear Estimates in Section 5.

2.3. **Function spaces adapted to our PDE.**

In order to introduce spaces that are adapted to our PDE (1.1) we consider the following linear Schrödinger-Airy IVP

$$\begin{cases}\partial_t u+i\,a\,\partial_x^2 u+b\,\partial_x^3 u=0, & x,t\in\mathbb{R},\quad b\neq 0,\\ u(x,0)=u_0(x), & x\in\mathbb{R}.\end{cases}\tag{2.6}$$

Next lemma introduces the *semi-group* for our PDE.

Lemma 2.9. *The solution to the linear Schrödinger-Airy IVP* (2.6) *is formally given by*

$$u(x,t)=e^{-t(ia\partial_x^2+b\partial_x^3)}u_0(x),$$

which has to be interpreted as

$$\widehat{u}(\xi,t)\overset{\text{def}}{=}e^{4\pi^2 it(a\xi^2+2\pi b\xi^3)}\widehat{u}_0(\xi).$$

Proof. We start by taking the spatial Fourier transform of (2.6), obtaining that

$$\begin{cases} \partial_t \widehat{u}(\xi,t) - 4\pi^2 ia\xi^2\widehat{u}(\xi,t) - 8\pi^3 ib\xi^3\widehat{u}(\xi,t) = 0, \\ \widehat{u}(\xi,0) = \widehat{u_0}(\xi). \end{cases}$$

This is an ODE in t, and its general solution u is given by

$$\widehat{u}(\xi,t) = e^{4\pi^2 it(a\xi^2+2\pi b\xi^3)}\widehat{u}_0(\xi).$$

□

Hence, our Schrödinger-Airy semi-group is given by $e^{-t(ia\partial_x^2+b\partial_x^3)}$ and pullback by the semi-group is $e^{t(ia\partial_x^2+b\partial_x^3)}$.
Now we will define the space $DU^2(I;H)$

$$DU^2(I;H) = \{\partial_t u | u \in U^2(I;H)\}.$$

Hence, if $f \in DU^2(I;H)$ and $u \in U^2(I;H)$ then $\partial_t u = f$. The subspace U_0^2 of U^2 of functions with limit 0 at b, can be identified with the following norms:

$$\|f\|_{DU^2(I;H)} = \|u(x,t)\|_{U^2(I;H)} \quad f = \partial_t u, \quad u \in U_0^2.$$

Finally, we are now ready to define the spaces U^2_{SA}, V^2_{SA}, and DU^2_{SA}, where SA stands for Schrödinger-Airy semigroup. So, pulling back by the Schrödinger–Airy semigroup $e^{-t(ia\partial_x^2+b\partial_x^3)}$ gives the spaces

$$\|u\|_{U^2_{SA}(I;H)} \stackrel{def}{=} \|e^{t(ia\partial_x^2+b\partial_x^3)}u\|_{U^2(I;H)}, \quad \|u\|_{V^2_{SA}(I;H)} \stackrel{def}{=} \|e^{t(ia\partial_x^2+b\partial_x^3)}u\|_{V^2(I;H)},$$

$$\|u\|_{DU^2_{SA}(I;H)} \stackrel{def}{=} \|e^{t(ia\partial_x^2+b\partial_x^3)}u\|_{DU^2(I;H)}.$$

Next, we define the norm: $\|\cdot\|_{l^2L_t^\infty H^s_{M,x}([0,T]\times\mathbb{R})}$

$$\begin{aligned}\|u\|_{l^2L_t^\infty H^s_{M,x}([0,T]\times\mathbb{R})} &\stackrel{\text{def}}{=} \left(\|P_{\le M}u\|^2_{L_t^\infty H^s_{M,x}([0,T]\times\mathbb{R})} + \sum_{N>M}\|P_N u\|^2_{L_t^\infty H^s_{M,x}([0,T]\times\mathbb{R})}\right)^{1/2} \\ &= \left(\sup_{t\in[0,T]}\int_{\mathbb{R}}(|\xi|^2+M)^s|\phi(\xi)|^2|\widehat{u}(\xi,t)|^2 d\xi \right. \\ &\left. + \sum_{N>M}\sup_{t\in[0,T]}\int_{\mathbb{R}}(|\xi|^2+M)^s|\psi_N(\xi)|^2|\widehat{u}(\xi,t)|^2 d\xi\right)^{1/2}.\end{aligned}$$

Since

$$\|u\|_{L_t^\infty H^s_{M,x}([0,T]\times\mathbb{R})} \le \|u\|_{l^2L_t^\infty H^s_{M,x}([0,T]\times\mathbb{R})}$$

this norm is stronger than $L_t^\infty H^s_{M,x}([0,T]\times\mathbb{R})$, where $L_t^\infty H^s_{M,x}([0,T]\times\mathbb{R})$ is defined by

$$\|u(x,t)\|_{L_t^\infty H^s_{M,x}([0,T]\times\mathbb{R})} := \sup_{0\le t\le T}\|u(\cdot,t)\|_{H^s_{M,x}([0,T]\times\mathbb{R})}.$$

In order to attain the region below $s = \frac{1}{4}$, we introduce the slightly smaller spaces via the following norms.

It was shown by D.Tataru [13] that we can adapt Bourgain's function spaces to the dispersive equations through the U^p spaces.

To measure the solutions to the Nonlinear Schrödinger-Airy equation we define the spaces X^s_M with the norm

$$\|u\|_{X^s_M([0,T]\times\mathbb{R})} = \left(\sup_{|I|=M^{4s-1}} \|P_{\leq M}u\|^2_{U^2_{SA}(I;H^s_M)} + \sum_{N>M} \sup_{|I|=N^{4s-1}} \|P_N u\|^2_{U^2_{SA}(I;H^s_M)} \right)^{1/2}$$

where the supremum is taken over all half-open subintervals $I = [a,b] \subset [0,T]$ of length N^{4s-1}.

In order to measure the nonlinearity in Schrödinger-Airy equation we define the spaces Y^s_M with the norm

$$\|f\|_{Y^s_M([0,T]\times\mathbb{R})} = \left(\sup_{|I|=M^{4s-1}} \|P_{\leq M}f\|^2_{DU^2_{SA}(I;H^s_M)} + \sum_{N>M} \sup_{|I|=N^{4s-1}} \|P_N f\|^2_{DU^2_{SA}(I;H^s_M)} \right)^{1/2}.$$

3. Basic estimate

In this section we will prove an estimate for the linear part of the equation (1.1). In the frist place, we will state our first proposition.

Proposition 3.1. *Fix $T > 0$, and suppose that $u \in \mathscr{C}_t^0\mathcal{S}_x([0,T]\times\mathbb{R})$ and $F \in \mathscr{C}_t^0\mathcal{S}_x([0,T]\times\mathbb{R})$ solve the equation*

$$\partial_t u + ia\partial_x^2 u + b\partial_x^3 u = F. \tag{3.1}$$

Then, for every $s \in \mathbb{R}$, and every dyadic integer $M \geq 1$,

$$\|u\|_{X^s_M([0,T]\times\mathbb{R})} \lesssim \|u\|_{l^2L^\infty_t H^s_{M,x}([0,T]\times\mathbb{R})} + \|F\|_{Y^s_M([0,T]\times\mathbb{R})}. \tag{3.2}$$

Proof. Fix a dyadic frequency $N > M$, and apply P_N to the equation (3.1) to obtain that

$$\partial_t u_N + ia\partial_x^2 u_N + b\partial_x^3 u_N = F_N, \tag{3.3}$$

with $u_N = P_N u$, and $F_N = P_N F$. The same equation is satisfied by $u_{\leq M} = P_{\leq M} u$ and $F_{\leq M} = P_{\leq M} F$.

Suppose that we were able to prove that for every time interval $I = [t_0, t_1) \subseteq [0,T]$,

$$\|u_N\|_{U^2_{SA}(I;H^s_M)} \leq \|u_N(\cdot,t_1)\|_{H^s_M} + \|F_N\|_{DU^2_{SA}(I;H^s_M)}, \tag{3.4}$$

and the analogous relation for $u_{\leq M}$ and $F_{\leq M}$ (see [19, Proposition 2.12]). Then, the desired result would follow. Indeed, by (3.4) and the triangular inequality, we have that

$$\begin{aligned}
\|u\|_{X^s_M([0,T]\times\mathbb{R})} &= \left(\sup_{|I|=M^{4s-1}} \|u_{\leq M}\|^2_{U^2_{SA}(I;H^s_M)} + \sum_{\mathscr{D}\ni N>M} \left(\sup_{|I|=N^{4s-1}} \|u_N\|_{U^2_{SA}(I;H^s_M)} \right)^2 \right)^{1/2} \\
&\leq \Bigg(\left(\sup_{|I|=M^{4s-1}} \|u_{\leq M}(\cdot,t_1)\|_{H^s_M} + \sup_{|I|=M^{4s-1}} \|F_{\leq M}\|_{DU^2_{SA}(I;H^s_M)} \right)^2 \\
&\quad + \sum_{\mathscr{D}\ni N>M} \left(\sup_{|I|=N^{4s-1}} \|u_N(\cdot,t_1)\|_{H^s_M} + \sup_{|I|=N^{4s-1}} \|F_N\|_{DU^2_{SA}(I;H^s_M)} \right)^2 \Bigg)^{1/2} \\
&\leq \left(\sup_{|I|=M^{4s-1}} \|u_{\leq M}(\cdot,t_1)\|^2_{H^s_M} + \sum_{\mathscr{D}\ni N>M} \sup_{|I|=N^{4s-1}} \|u_N(\cdot,t_1)\|^2_{H^s_M} \right)^{1/2} \\
&\quad + \left(\sup_{|I|=M^{4s-1}} \|F_{\leq M}\|^2_{DU^2_{SA}(I;H^s_M)} + \sum_{\mathscr{D}\ni N>M} \sup_{|I|=N^{4s-1}} \|F_N\|^2_{DU^2_{SA}(I;H^s_M)} \right)^{1/2} \\
&\leq \left(\|u_{\leq M}\|^2_{L^\infty_t H^s_{M,x}([0,T]\times\mathbb{R})} + \sum_{\mathscr{D}\ni N>M} \|u_N\|^2_{L^\infty_t H^s_{M,x}([0,T]\times\mathbb{R})} \right)^{1/2} \\
&\quad + \|F\|_{Y^s_M([0,T]\times\mathbb{R})} \\
&\leq \left(2^2 \sum_{\mathscr{D}\ni N\leq M} \|u_N\|^2_{L^\infty_t H^s_{M,x}([0,T]\times\mathbb{R})} + \sum_{\mathscr{D}\ni N>M} \|u_N\|^2_{L^\infty_t H^s_{M,x}([0,T]\times\mathbb{R})} \right)^{1/2} \\
&\quad + \|F\|_{Y^s_M([0,T]\times\mathbb{R})}
\end{aligned}$$

$$\leq 2\left(\|u\|_{l^2L_t^\infty H^s_{M,x}([0,T]\times\mathbb{R})} + \|F\|_{Y^s_M([0,T]\times\mathbb{R})}\right).$$

To prove (3.4), note that in virtue of Duhamel's formula (1.2),

$$u_N(\cdot,t_1) = e^{-(t_1-t)(ia\partial_x^2+b\partial_x^3)}u_N(\cdot,t) + \int_t^{t_1} e^{-(t_1-s)(ia\partial_x^2+b\partial_x^3)}F_N(\cdot,s)ds,$$

for every $t\in I$.
In particular,

$$e^{t(ia\partial_x^2+b\partial_x^3)}u_N(\cdot,t) - e^{t_1(ia\partial_x^2+b\partial_x^3)}u_N(\cdot,t_1) = -\int_t^{t_1} e^{s(ia\partial_x^2+b\partial_x^3)}F_N(\cdot,s)ds,$$

and applying Leibniz's rule,

$$\partial_t(e^{t(ia\partial_x^2+b\partial_x^3)}u_N(\cdot,t) - e^{t_1(ia\partial_x^2+b\partial_x^3)}u_N(\cdot,t_1)) = e^{t(ia\partial_x^2+b\partial_x^3)}F_N(\cdot,t).$$

Hence,

$$\begin{aligned}\|F_N\|_{DU^2_{SA}(I;H^s_M)} &= \left\|e^{t(ia\partial_x^2+b\partial_x^3)}F_N\right\|_{DU^2(I;H^s_M)}\\ &= \left\|e^{t(ia\partial_x^2+b\partial_x^3)}u_N(\cdot,t) - e^{t_1(ia\partial_x^2+b\partial_x^3)}u_N(\cdot,t_1)\right\|_{U^2(I;H^s_M)},\end{aligned}$$

so

$$\begin{aligned}\|u_N\|_{U^2_{SA}(I;H^s_M)} &= \left\|e^{t(ia\partial_x^2+b\partial_x^3)}u_N\right\|_{U^2(I;H^s_M)}\\ &\leq \left\|e^{t(ia\partial_x^2+b\partial_x^3)}u_N(\cdot,t) - e^{t_1(ia\partial_x^2+b\partial_x^3)}u_N(\cdot,t_1)\right\|_{U^2(I;H^s_M)}\\ &+ \left\|e^{t_1(ia\partial_x^2+b\partial_x^3)}u_N(\cdot,t_1)\right\|_{U^2(I;H^s_M)}\\ &\leq \|F_N\|_{DU^2_{SA}(I;H^s_M)} + \|u_N(\cdot,t_1)\|_{H^s_M}.\end{aligned}$$

To obtain the last inequality, observe that

$$\begin{aligned}\lambda \stackrel{\text{def}}{=} \|u_N(\cdot,t_1)\|_{H^s_M} &= \left(\int_{\mathbb{R}}(M+|\xi|^2)^s\left|\widehat{u}_N(\xi,t_1)\right|^2d\xi\right)^{1/2}\\ &= \left(\int_{\mathbb{R}}(M+|\xi|^2)^s\left|e^{-4\pi^2 it_1(a\xi^2+2\pi b\xi^3)}\widehat{u}_N(\xi,t_1)\right|^2d\xi\right)^{1/2}\\ &= \left\|e^{t_1(ia\partial_x^2+b\partial_x^3)}u_N(\cdot,t_1)\right\|_{H^s_M},\end{aligned}$$

so

$$\mathfrak{a}(\cdot,t) \stackrel{\text{def}}{=} \lambda^{-1}e^{t_1(ia\partial_x^2+b\partial_x^3)}u_N(\cdot,t_1)\chi_I(t)$$

is a $U^2(I;H^s_M)$ atom, and

$$\chi_I(t)e^{t_1(ia\partial_x^2+b\partial_x^3)}u_N(\cdot,t_1) = \lambda\mathfrak{a}(\cdot,t).$$

Hence,

$$\left\|e^{t_1(ia\partial_x^2+b\partial_x^3)}u_N(\cdot,t_1)\right\|_{U^2(I;H^s_M)} \leq \lambda.$$

The case when $\lambda = 0$ is trivial because $u_N(\cdot,t_1)=0$ and there is nothing to do.

The same argument works for the functions $u_{\leq M}$ and $F_{\leq M}$. □

4. Useful estimates

This section of my capstone will be dedicated to preliminary lemmas that will be useful in subsequent sections.

First lemma is about Bernstein inequality that was defined in [19, Equation A.6].

Lemma 4.1. *(Bernstein inequality). For* $1 \leq p \leq q \leq \infty$,

$$\|P_N f\|_{L^q} \leq C N^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^p}. \tag{4.1}$$

Next definition introduces the region of admissibility that was also stated in [19, Theorem 2.3].

Definition 4.2. *A pair* (p,q) *of Hölder exponents will be called admissible if*

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{2},\ 2 \leq p \leq \infty,\ 4 \leq q \leq \infty. \tag{4.2}$$

In particular, it can be seen that the following pairs (p,q) *of indices are admissible:* $(\infty, 2)$, $(6,6)$, $(4,\infty)$.

Next two fundamental lemmas will be necessary to prove other theorems and lemmas in this section. Please note that these were defined for modified Korteweg-de Vries equation in [5]. The similar estimates could be implemented to our PDE. The general formula can be found in the paper by Kenig, Pouce, Vega [9]

Lemma 4.3. *(Strichartz estimates). Let* (p,q) *satisfy the admissibility condition* (4.2)*. Then*

$$\|D_x^{1/p} e^{-t(ia\partial_x^2 + b\partial_x^3)}\phi\|_{L_t^p L_x^q} \lesssim \|\phi\|_{L^2}. \tag{4.3}$$

Lemma 4.4. *(Local smoothing/maximal function estimates). If* (p,q) *satisfies the admissibility conditon* (4.2)*, then*

$$\|D_x^{1-\frac{5}{p}} e^{-t(ia\partial_x^2 + b\partial_x^3)}\phi\|_{L_x^q L_t^p} \lesssim \|\phi\|_{L^2} \tag{4.4}$$

The next two corollaries are consequences of Strichartz and Local smoothing estimates. They connect Strichartz norms to our Schrödinger-Airy norms of the form $\|\cdot\|_{U^2_{SA}}$ and $\|\cdot\|_{DU^2_{SA}}$. The main application of these corollaries are in theorems about Trilinear estimates in Section 5 to obtain a projection bound.

Corollary 4.5. *If I=[a,b) is any interval, and u = u(x,t) any function, then for (p,q) satisfying the admissibility condition* (4.2)*, we have, for* $N \geq 1$,

$$\|P_N u\|_{L_I^p L_x^q} \lesssim N^{-\frac{1}{p}} \|\chi_I u\|_{U^p_{SA} L^2}, \tag{4.5}$$

and we have the dual relation for p >2

$$\|P_N u\|_{DU^2_{SA}(I;L^2)} \lesssim N^{-\frac{1}{p}} \|u\|_{L_I^{p'} L_x^{q'}}, \tag{4.6}$$

where (p', q') *denotes the Hölder dual pair.*

The proof of (4.5) is straightforward, because it will suffice to consider U^p_{SA} from Section 2 and apply Strichartz estimate (4.3).To prove (4.6) we apply duality (Lemma 2.8), (4.5) and (3) from Lemma 2.7.

Corollary 4.6. *If* (p,q) *is admissible according to* (4.2) *and* $p, q \geq r$*, then*

$$\|P_N u\|_{L_x^p L_I^q} \lesssim N^{\frac{5}{p}-1} \|\chi_I u\|_{U^p_{SA} L^2}, \tag{4.7}$$

for any interval $I = [a, b)$. We also have the dual relation for $q > 2$,

$$\|P_N u\|_{DU^2_{SA}(I;L^2)} \lesssim N^{\frac{5}{p}-1} \|u\|_{L^{p'}_x L^{q'}_I}, \tag{4.8}$$

where (p', q') is the Hölder dual pair.

The proof of (4.7) follows from atom for u, local smoothing estimate (4.4) and triangle inequality. For the proof of (4.8) we use duality (Lemma 2.8), (4.7).

5. Trilinear estimates

5.1. **Preliminaries.**
The following lemma is about the controlling the L^2 norms of the product of projections. It was stated by using the similar idea as in paper by Christ, Holmer and Tataru [5].

Lemma 5.1. *For $N_1 << N_2$ ($N_2 \geq CN_1$, $C > 2$ a large number) and $u(x,t), v(x,t)$ are any functions, we have*

$$\|(P_{N_1}u)(P_{N_2}v)\|_{L_I^2 L_x^2} \lesssim N_2^{-1} \|\chi_I P_{N_1} u\|_{U_{SA}^2(L^2)} \|\chi_I P_{N_2} v\|_{U_{SA}^2(L^2)}, \tag{5.1}$$

and

$$\|(P_{N_1}u)(P_{N_2}v)\|_{L_I^2 L_x^2} \lesssim N_2^{-1} \left(\log \frac{N_2}{N_1}\right)^2 \|\chi_I P_{N_1} u\|_{V_{SA}^2(L^2)} \|\chi_I P_{N_2} v\|_{U_{SA}^2(L^2)}. \tag{5.2}$$

5.2. **Main Results.** In this section we are going to consider the nonlinear part f our equation (1.1). Let the nonlinear part be $f = i\,c\,|u|^2 u + d\,|u|^2 \partial_x u + e\,u^2 \partial_x \overline{u}$.

Proposition 5.2. *For all $-1/8 < s < 1/4$ and $M > 1$ we have*

$$\|\partial_x(u_1 u_2 \overline{u}_3)\|_{Y_M^s([0,T]\times\mathbb{R})} \lesssim \|u_1\|_{X_M^s([0,T]\times\mathbb{R})} \|u_2\|_{X_M^s([0,T]\times\mathbb{R})} \|u_3\|_{X_M^s([0,T]\times\mathbb{R})}. \tag{5.3}$$

Since proof of Proposition 5.2 is technical, the proof will be divided into several steps so that reader will be able to follow it easily.
Before starting the proof of the proposition, we first reduce matters to proving, for an interval $|J| = N^{4s-1}$ with $N > 1$, a bound of the type
(5.4)

$$\|P_N \partial_x(u_{N_1} u_{N_2} \overline{u}_{N_3})\|_{DU_{SA}^2(J;H^s)} \leq \alpha(N, N_1, N_2, N_3) \prod_{j=1}^{3} \sup_{|I_j| = N_j^{4s-1}} \|\chi_{I_j} u_{N_j}\|_{U_{SA}^2 H^s}.$$

It can be shown by the definition of Y_M^s norm and Hölder's inequality for all possible cases of permutation of N_1, N_2, N_3 from *Step 1. Step 2* will show that the bound above can be reduced to the following bound. So, we can write
(5.5)

$$\|P_N \partial_x(u_{N_1} u_{N_2} \overline{u}_{N_3})\|_{DU_{SA}^2(J;L^2)} \leq \alpha(N, N_1, N_2, N_3) \frac{N_1^s N_2^s N_3^s}{N^s} \prod_{j=1}^{3} \sup_{|I_j| = N_j^{4s-1}} \|\chi_{I_j} u_{N_j}\|_{U_{SA}^2 L^2}.$$

Here α has certain summability properties. As a general rule, we should have at least $|\ \alpha(N, N_1, N_2, N_3)\ | \ \lesssim 1$, and in some cases, need a slight power decay in N and N_j to insure the summation with respect to all indices. *Step 3* will discuss the proof of (5.3).

The first step of our proof will examine the derivation of all possible cases of permutation of N_1,N_2,N_3.

Step 1. The cases. We have four real numbers $N, N_1, N_2, N_3 \geq 1$. In our argument, the sub-indices of N_1, N_2, N_3 can be permuted, so we can assume that $N_1 \leq N_2 \leq N_3$. Now, we have four ways of ordering such four numbers:

(i) $N_1 \leq N_2 \leq N_3 \leq N$.
(ii) $N_1 \leq N_2 \leq N \leq N_3$.
(iii) $N_1 \leq N \leq N_2 \leq N_3$.
(iv) $N \leq N_1 \leq N_2 \leq N_3$.

We will see that cases (i) and (ii) reduce to *Case 1*, case (iii) reduces to *Case 2*, and case (iv) reduces to *Case 3* approximately.

It is known that for a function f, if $\widehat{f} = 0$, then $f = 0$. So, we are going to investigate conditions on N, N_1, N_2, N_3 in such a way that for $f = P_N(u_{N_1}u_{N_2}u_{N_3})$ we have $\widehat{f} = 0$. Next we consider

$$\widehat{f}(\xi) = \psi_k(\xi)(\widehat{u}_{N_1} * \widehat{u}_{N_2} * \widehat{u}_{N_3})(\xi),$$

with

$$S \stackrel{\text{def}}{=} \operatorname{supp}\psi_k \subseteq \{\frac{N}{2} \le |\xi| \le 2N\},$$

and for $i = 1, 2, 3$, $\operatorname{supp}\widehat{u}_{N_i} \subseteq \{\frac{N_i}{2} \le |\xi| \le 2N_i\} \stackrel{\text{def}}{=} S_i$.
So now we define

$$\operatorname{supp}(\widehat{u}_{N_1} * \widehat{u}_{N_2} * \widehat{u}_{N_3}) \subseteq \sum_{i=1}^{3}\{\frac{N_i}{2} \le |\xi| \le 2N_i\} \stackrel{\text{def}}{=} S_{1,2,3},$$

as the *Minkowski sum* of the three sets. Then we have that $\operatorname{supp}\widehat{f} \subseteq S \cap S_{1,2,3}$.

Note that since we work on $\mathbb{R}$, $S_i = [-2N_i, -\frac{N_i}{2}] \cup [\frac{N_i}{2}, 2N_i]$, and using the property

$$(A \cup B) + (C \cup D) \subseteq (A + C) \cup (A + D) \cup (B + C) \cup (B + D), \quad A, B, C, D \text{ sets},$$

we get that

$$\begin{aligned}
S_{1,2,3} \subseteq\ & [-2(N_1 + N_2 + N_3), -\frac{N_1}{2} - \frac{N_2}{2} - \frac{N_3}{2}] \cup [\frac{N_1}{2} + \frac{N_2}{2} + \frac{N_3}{2}, 2(N_1 + N_2 + N_3)] \\
& \cup [\frac{N_1}{2} - 2N_2 - 2N_3, 2N_1 - \frac{N_2}{2} - \frac{N_3}{2}] \cup [-2N_1 + \frac{N_2}{2} + \frac{N_3}{2}, -\frac{N_1}{2} + 2N_2 + 2N_3] \\
& \cup [-2N_1 + \frac{N_2}{2} - 2N_3, -\frac{N_1}{2} + 2N_2 - \frac{N_3}{2}] \cup [\frac{N_1}{2} - 2N_2 + \frac{N_3}{2}, 2N_1 - \frac{N_2}{2} + 2N_3] \\
& \cup [\frac{N_1}{2} + \frac{N_2}{2} - 2N_3, 2N_1 + 2N_2 - \frac{N_3}{2}] \cup [-2N_1 - 2N_2 + \frac{N_3}{2}, -\frac{N_1}{2} - \frac{N_2}{2} + 2N_3] \\
& \stackrel{\text{def}}{=} L_1 \cup L_2 \cup L_3 \cup L_4 \stackrel{\text{def}}{=} L \subseteq [-2(N_1 + N_2 + N_3), 2(N_1 + N_2 + N_3)] \stackrel{\text{def}}{=} T.
\end{aligned}$$

Now we will analyze each of the cases.
Under the assumptions of case (i) we have that $S \cap T = \emptyset$ if and only if

$$2(N_1 + N_2 + N_3) < \frac{N}{2}.$$

This implies that $\widehat{f} = 0$. Hence, we can assume that

$$2(N_1 + N_2 + N_3) \ge \frac{N}{2}$$

and, in particular, $N_1 \le N_2 \le N_3 \le N \le 12N_3$. Hence $N_1 \le N_2 \le N_3 \approx N$ which can be reduced to *Case 1* in *Step 4*.

In a similar fashion under the assumptions of case (ii): to get $S \cap L = \emptyset$, we need $4N+4N_1+4N_2 < N_3$, and then, $\widehat{f} = 0$. Hence, we can assume that $4N+4N_1+4N_2 \ge N_3$ and, in particular, $N_1 \le N_2 \le N \le N_3 \le 12N$, so $N_1 \le N_2 \le N_3 \approx N$ and this case also reduces to *Case 1* in *Step 4*.

Similarly, under the assumptions of case (iii), in order to get $S \cap L = \emptyset$, we need $4N+4N_1+4N_2 < N_3$, and then, $\widehat{f} = 0$.Hence, we can assume that $4N+4N_1+4N_2 \ge N_3$ and, in particular, $N_1 \le N \le N_2 \le N_3 \le 12N_2$, so $N_1 \le N \le N_2 \approx N_3$. This case reduces to *Case 2* in *Step 4*.

Lastly, under the assumptions of case (iv), in order to get $S \cap L = \emptyset$, we need $4N+4N_1+4N_2 < N_3$, and then, $\widehat{f} = 0$. Hence, we can assume that $4N+4N_1+4N_2 \geq N_3$ and, in particular, $N \leq N_1 \leq N_2 \leq N_3 \leq 12N_2$, so $N \leq N_1 \leq N_2 \approx N_3$. □

In the third step of our Proof, we perform second reduction from H^s_M to L^2.

Step 2. Second reduction. Fix $N \geq M \geq 1$. We have that

$$\|P_N u\|_{H^s_M} = \|(|\cdot|^2 + M)^{s/2}\widehat{P_N u}\|_{L^2} = \|(|\cdot|^2 + M)^{s/2}\psi_k \widehat{u}\|_{L^2},$$

with $\operatorname{supp}\psi_k \subseteq \{\frac{N}{2} \leq |\xi| \leq 2N\} \stackrel{\text{def}}{=} S$. Now, it is true that

$$N^2 \leq N^2 + M \leq N^2 + N \leq 2N^2,$$

and for $\xi \in S$. Moreover

$$\frac{N^2}{4} \leq \frac{1}{4}(N^2 + M) \leq \frac{N^2}{4} + M \leq |\xi|^2 + M \leq 4N^2 + M \leq 4(N^2 + M) \leq 8N^2,$$

so that we get the following inequality

$$8^{-|s|/2} N^s \|\psi_k \widehat{u}\|_{L^2} \leq \|P_N u\|_{H^s_M} \leq 8^{|s|/2} N^s \|\psi_k \widehat{u}\|_{L^2}.$$

Hence it can be seen that

$$\|P_N u\|_{H^s_M} \approx_s N^s \|\psi_k \widehat{u}\|_{L^2} = N^s \|P_N u\|_{L^2}.$$

So, in order to perform the second reduction we will show that

$$\|P_N u\|_{DU^2_{SA}(I;H^s_M)} =_s N^s \|P_N u\|_{DU^2_{SA}(I;L^2)}.$$

Particularly, we will prove both directions:

$$N^s \|P_N u\|_{U^2_{SA}(I;L^2)} \lesssim_s \|P_N u\|_{U^2_{SA}(I;H^s_M)}, \tag{5.6}$$

and

$$\|P_N u\|_{DU^2_{SA}(I;H^s_M)} \lesssim_s N^s \|P_N u\|_{DU^2_{SA}(I;L^2)}. \tag{5.7}$$

For the first inequality (5.6), we assume that the right-hand side is finite, and write

$$P_N u \chi_I = \sum_{\ell \geq 0} \lambda_\ell \mathfrak{a}_\ell, \tag{5.8}$$

with atom

$$\mathfrak{a}_\ell(x,t) = \sum_{j=1}^{n} \chi_{[t_{j-1},t_j)}(t) e^{-t(ia\partial_x^2 + b\partial_x^3)} \phi_{j-1}(x),$$

and $\sum_{j=1}^n \|\phi_{j-1}\|^2_{H^s_M} \leq 1$ (see [13, Page 46]). Since $\widehat{P_N u}$ is supported on S, we can assume that each ϕ_{j-1} has Fourier transform supported on S. Hence, by the previous computations, we obtain

$$8^{-|s|/2} N^s \|\phi_{j-1}\|_{L^2} \leq \|\phi_{j-1}\|_{H^s_M} \leq 8^{|s|/2} N^s \|\phi_{j-1}\|_{L^2},$$

and

$$P_N u \chi_I = \sum_{\ell \geq 0} 8^{|s|/2} N^{-s} \lambda_\ell \tilde{\mathfrak{a}}_\ell,$$

with

$$\tilde{\mathfrak{a}}_\ell(x,t) = \sum_{j=1}^{n} \chi_{[t_{j-1},t_j)}(t) e^{-t(ia\partial_x^2 + b\partial_x^3)} 8^{-|s|/2} N^s \phi_{j-1}(x),$$

and $\sum_{j=1}^{n} \|8^{-|s|/2} N^s \phi_{j-1}\|_{L^2}^2 \leq 1$. In conclusion, for every representation (5.8), we have that

$$\|P_N u\|_{U^2_{SA}(I;L^2)} \leq 8^{|s|/2} N^{-s} \sum_{\ell \geq 0} |\lambda_\ell|.$$

By taking the infimum over all such representations (5.8), we have

$$\|P_N u\|_{U^2_{SA}(I;L^2)} \leq 8^{|s|/2} N^{-s} \|P_N u\|_{U^2_{SA}(I;H^s_M)}. \tag{5.9}$$

For the second inequality (5.7), we have to show that

$$\|P_N u\|_{DU^2_{SA}(I;H^s_M)} \lesssim_s N^s \|P_N u\|_{DU^2_{SA}(I;L^2)}.$$

Without loss of generality, we can assume that the right-hand side is finite, meaning that there exists a unique function $f \in U^2_0([0,T]; L^2_x)$ such that

$$\partial_t f = e^{t(ia\partial_x^2 + b\partial_x^3)} P_N u(\cdot, t).$$

Then by definition it can be seen that

$$\|P_N u\|_{DU^2_{SA}(I;L^2)} = \|e^{t(ia\partial_x^2 + b\partial_x^3)} P_N u(\cdot,t)\|_{DU^2(I;L^2)} = \|f\|_{U^2(I;L^2)} < \infty.$$

Note that $\operatorname{supp} \widehat{f} \subseteq \{\frac{N}{2} \leq |\xi| \leq 2N\}$, so from the previous result (5.9)

$$\|f\|_{U^2(I;H^s_M)} \leq 8^{|s|/2} N^s \|f\|_{U^2(I;L^2)}.$$

Since $\|f\|_{U^2(I;H^s_M)} < \infty$, we get

$$\|P_N u\|_{DU^2_{SA}(I;H^s_M)} = \|f\|_{U^2(I;H^s_M)}.$$

Therefore, we get the desired result

$$\begin{aligned} \|P_N u\|_{DU^2_{SA}(I;H^s_M)} &= \|f\|_{U^2(I;H^s_M)} \\ &\leq 8^{|s|/2} N^s \|f\|_{U^2(I;L^2)} = \|P_N u\|_{DU^2_{SA}(I;L^2)}. \end{aligned}$$

The same argument works for $P_{\leq M}$. □

Next we are going to introduce the following estimate, which is a consequence of Lemma 2.5. This has an application in last step of the proof of 5.3.

$$\|\partial_x P_N u\|_{U^2_{SA}(I;L^2)} \lesssim N \|P_N u\|_{U^2_{SA}(I;L^2)}. \tag{5.10}$$

Proof. We can assume that the right-hand side is finite so that we write

$$P_N u \chi_I = \sum_{\ell \geq 0} \lambda_\ell \mathfrak{a}_\ell \quad (L^2 - \textit{summable}), \tag{5.11}$$

with

$$\mathfrak{a}_\ell(x,t) = \sum_{j=1}^{n} \chi_{[t_{j-1},t_j)}(t) e^{-t(ia\partial_x^2 + b\partial_x^3)} \phi_{j-1}(x)$$

and $\sum_{j=1}^{n} \|\phi_{j-1}\|_{L^2}^2 \leq 1$, and $\operatorname{supp} \widehat{\phi}_{j-1} \subseteq \{\frac{N}{2} \leq |\xi| \leq 2N\}$. Moreover, we have

$$\begin{aligned} \|\mathfrak{a}_\ell(x,t)\|_{L^2} &\leq \sum_{j=1}^{n} \chi_{[t_{j-1},t_j)}(t) \|e^{-t(ia\partial_x^2 + b\partial_x^3)} \phi_{j-1}(x)\|_{L^2} \\ &= \chi_I(t) \left(\sum_{j=1}^{n} \|\phi_{j-1}(x)\|_{L^2}^2 \right)^{1/2} \leq \chi_I(t) \leq 1, \end{aligned}$$

and

$$\|P_N u\chi_I\|_{L^2} \le \|\sum_{\ell\ge0}\lambda_\ell \mathfrak{a}_\ell\|_{L^2} \le \sum_{\ell\ge0}|\lambda_\ell|\|\mathfrak{a}_\ell\|_{L^2} \le \sum_{\ell\ge0}|\lambda_\ell| < \infty.$$

Note that $\|\partial_x \mathfrak{a}_\ell\|_{L^2} \le (\kappa N)^2$ and

$$\left\|\partial_x\left(\sum_{\ell\ge0}\lambda_\ell\mathfrak{a}_\ell\right)\right\|_{L^2} = \left(\int|\xi|^2\left|\int\sum_{\ell\ge0}\lambda_\ell\mathfrak{a}_\ell(x,t)e^{-ix\xi}dx\right|^2 d\xi\right)^{1/2}.$$

Then by differentiating (5.11), we get that

$$\chi_I\partial_x P_N u = \sum_{\ell\ge0}\kappa N\lambda_\ell {\mathfrak{a}'}_\ell,$$

with

$${\mathfrak{a}'}_\ell(x,t) = \sum_{j=1}^{n}\chi_{[t_{j-1},t_j)}(t)e^{-t(ia\partial_x^2+b\partial_x^3)}(\kappa N)^{-1}\partial_x\phi_{j-1}(x).$$

In virtue of [18, Lemma 1.1],

$$\sum_{j=1}^{n}\|(\kappa N)^{-1}\partial_x\phi_{j-1}\|_{L^2}^2 \le \sum_{j=1}^{n}\|\phi_{j-1}\|_{L^2}^2 \le 1.$$

In conclusion, for every representation (5.11), we have that

$$\|\partial_x P_N u\|_{U^2_{SA}(I;L^2)} \le \kappa N\sum_{\ell\ge0}|\lambda_\ell|,$$

and taking the infimum over all such representations (5.11),

$$\|\partial_x P_N u\|_{U^2_{SA}(I;L^2)} \le \kappa N\|P_N u\|_{U^2_{SA}(I;L^2)},$$

as desired. □

Finally, by obtaining all the necessary steps, we can now begin the proof by considering cases that were mentioned in *Step 1.*

Step 3. Proof of Theorem 5.2. In order to prove the bound (5.5) we will consider the following cases from *Step 1*:

Case 1. $N_1, N_2, N_3 \lesssim N$. We can assume that $N_1 \le N_2 \le N_3 \sim N$. In this case, all I_j have length $\ge |J|$ and can be neglected. We will then distribute the derivative, which in the worst case applies to u_{N_3}. By (4.6) and Hölder inequality in time variable (2.1), we have,

$$\begin{aligned}\|P_N(u_{N_1}u_{N_2}\partial_x\overline{u}_{N_3})\|_{DU^2_{SA}(J;L^2)} &\lesssim \|u_{N_1}u_{N_2}\partial_x\overline{u}_{N_3}\|_{L^1_JL^2_x}\\ &\lesssim \|1\|_{L^2_JL^2_x}\|u_{N_1}u_{N_2}\partial_x\overline{u}_{N_3}\|_{L^1_JL^2_x}\\ &\lesssim |J|^{\frac12}\|u_{N_1}u_{N_2}\partial_x\overline{u}_{N_3}\|_{L^2_JL^2_x}.\end{aligned}$$

Further, again by computations and by Hölder inequality (2.1) we get

$$\begin{aligned}|J|^{\frac12}\|u_{N_1}u_{N_2}\partial_x\overline{u}_{N_3}\|_{L^2_JL^2_x} &\lesssim |J|^{\frac12}\Big(\int_J\int_X u_{N_1}^2u_{N_2}^2(\partial_x\overline{u}_{N_3})^2dx\,dt\Big)^{\frac12}\\ &\lesssim N^{2s-\frac12}\Big[\sup_J\Big(\int_X u_{N_1}^4dx\Big)^{\frac14}\sup_J\Big(\int_X u_{N_2}^4dx\Big)^{\frac14}\sup_x\Big(\int_J(\partial_x\overline{u}_{N_3})^2dt\Big)^{\frac12}\Big]\end{aligned}$$

$$\lesssim N^{2s-\frac{1}{2}}\|u_{N_1}\|_{L_x^4L_J^\infty}\|u_{N_2}\|_{L_x^4L_J^\infty}\|\partial_x\overline{u}_{N_3}\|_{L_x^\infty L_J^2}.$$

Lastly, we apply (4.7) and (5.10) to the last term, to obtain the following result

$$\begin{aligned}&N^{2s-\frac{1}{2}}\|u_{N_1}\|_{L_x^4L_J^\infty}\|u_{N_2}\|_{L_x^4L_J^\infty}\|\partial_x\overline{u}_{N_3}\|_{L_x^\infty L_J^2}\\ &\lesssim N^{2s-\frac{1}{2}}N_1^{\frac{1}{4}}\|\chi_J u_{N_1}\|_{U_{SA}^2L^4}N_2^{\frac{1}{4}}\chi_J\|u_{N_2}\|_{U_{SA}^2L^4}N_3^{-1}N_3\|\chi_J u_{N_3}\|_{U_{SA}^\infty L^2}\end{aligned}$$

Finally, by Lemma (2.7) (1) we obtain that

$$\begin{aligned}N^{2s-\frac{1}{2}}N_1^{\frac{1}{4}}\|\chi_J u_{N_1}\|_{U_{SA}^2L^4}N_2^{\frac{1}{4}}\chi_J\|u_{N_2}\|_{U_{SA}^2L^4}N_3^{-1}N_3\|\chi_J u_{N_3}\|_{U_{SA}^\infty L^2}\lesssim&\\ \lesssim N^{2s-\frac{1}{2}}N_1^{\frac{1}{4}}N_2^{\frac{1}{4}}\prod_{j=1}^{3}\|\chi_J u_{N_j}\|_{U_{SA}^2L^2}.&\end{aligned}$$

Thus we have (5.4) with $\alpha = N^{2s-\frac{1}{2}}N_1^{\frac{1}{4}-s}N_2^{\frac{1}{4}-s}$, which suffices for all s.

Case 2. $N_1 \lesssim N \ll N_2 \sim N_3$. We divide J into $|J|/|I| = (N_3/N)^{1-4s} \gg 1$ intervals of size $|I| = N_3^{4s-1}$. For $u \in V_{SA}^2(J;L^2)$ we estimate by duality (Lemma 2.8)

$$\begin{aligned}\|P_N(u_{N_1}u_{N_2}\overline{u}_{N_3})\|_{DU_{SA}^2(J;L^2)} &= \Big|\int_J\int_x u_{N_1}u_{N_2}\overline{u}_{N_3}\overline{u}_N dx\,dt\Big|\\ &\le \Big(\frac{N_3}{N}\Big)^{1-4s}\sup_{\substack{I\subset J\\ |I|=N_3^{4s-1}}}\Big|\int_I\int_x u_{N_1}u_{N_2}\overline{u}_{N_3}\overline{u}_N dx\,dt\Big|.\end{aligned}$$

Then by Hölder inequality (2.1) we have

$$\begin{aligned}\Big(\frac{N_3}{N}\Big)^{1-4s}\sup_{\substack{I\subset J\\ |I|=N_3^{4s-1}}}\Big|\int_I\int_x u_{N_1}u_{N_2}\overline{u}_{N_3}\overline{u}_N dx\,dt\Big| \le&\\ \le \Big(\frac{N_3}{N}\Big)^{1-4s}\sup_{\substack{I\subset J\\ |I|=N_3^{4s-1}}}\|u_{N_1}u_{N_2}\|_{L_I^2L_x^2}\|u_Nu_{N_3}\|_{L_I^2L_x^2}.&\end{aligned}$$

Next, using (5.1), (5.2) we bound the above by

$$\Big(\frac{N_3}{N}\Big)^{1-4s}N_1^{-1}N_3^{-1}\ln^2\frac{N_3}{N}\sup_{\substack{I\subset J\\ |I|=N_3^{4s-1}}}\|\chi_I u_{N_1}\|_{U_{SA}^2L^2}\|\chi_I u_{N_2}\|_{U_{SA}^2L^2}\|\chi_I u_{N_3}\|_{U_{SA}^2L^2}\|\chi_I u_N\|_{V_{SA}^2L^2}.$$

Finally, we apply (2.5) ($\|\chi_I P_N u\|_{V_{SA}^2} \le 2\|P_N u\|_{V_{SA}^2(J)}$)

$$\Big(\frac{N_3}{N}\Big)^{1-4s}N_1^{-1}N_3^{-1}\ln^2\frac{N_3}{N}\sup_{\substack{I\subset J\\ |I|=N_3^{4s-1}}}\|\chi_I u_{N_1}\|_{U_{SA}^2L^2}\|\chi_I u_{N_2}\|_{U_{SA}^2L^2}\|\chi_I u_{N_3}\|_{U_{SA}^2L^2}\|\chi_I u_N\|_{V_{SA}^2L^2}$$

and add a factor of N to account for the derivative in (5.4) to get the coefficient

$$\alpha = N_3^{-1-7s}N^{5s}\Big(\ln\frac{N_3}{N}\Big)^2$$

so this case is handled if $s \ge -\frac{1}{7}$.

Case 3. $N \ll N_1 < N_2 = N_3$.

We again argue by duality as in *Case 2* (Lemma 2.8) and divide into subintervals of size $|I| = N_3^{4s-1}$. By $v \in V^2_{SA}(J;L^2)$ and by Hölder inequality (2.1) we have

$$\begin{aligned}\|P_N(u_{N_1}u_{N_2}\overline{u}_{N_3})\|_{DU^2_{SA}(J;L^2)} &= \Big|\int_{t\in J}\int_x u_N u_{N_2}\overline{u}_{N_1}\overline{u}_{N_3}dx\,dt\Big| \\ &\leq \left(\frac{N_3}{N}\right)^{1-4s}\sup_{I\subset J|I|=N_3^{4s-1}}\Big|\int_{t\in I}\int_x u_N u_{N_2}\overline{u}_{N_1}\overline{u}_{N_3}\,dx\,dt\Big| \\ &\leq \left(\frac{N_3}{N}\right)^{1-4s}\sup_{\substack{I\subset J\\ |I|=N_3^{4s-1}}}\|u_N u_{N_2}\|_{L^2_I L^2_x}\|u_{N_1}u_{N_3}\|_{L^2_I L^2_x}.\end{aligned}$$

We then use the estimates (5.1),(5.2) to bound the above by

$$\left(\frac{N_3}{N}\right)^{1-4s} N_3^{-2}\ln^2\frac{N_3}{N}\sup_{\substack{I\subset J\\ |I|=N_3^{4s-1}}}\|\chi_I u_{N_1}\|_{U^2_{SA}L^2}\|\chi_I u_{N_2}\|_{U^2_{SA}L^2}\|\chi_I \overline{u}_{N_3}\|_{U^2_{SA}L^2}\|\chi_I u_N\|_{V^2_{SA}L^2}.$$

Finally, we apply (2.5). Thus we have $\alpha = N_3^{-1-7s}N^{5s}\left(\ln\frac{N_3}{N}\right)^2$ which is satisfied if we have $s > -\frac{1}{7}$. □

Proposition 5.3. *For all* $-1/8 < s < 1/4$ *and* $M > 1$ *we have*

$$\|u_1u_2\overline{u}_3\|_{Y^s_M([0,T]\times\mathbb{R})} \lesssim \|u_1\|_{X^s_M([0,T]\times\mathbb{R})}\|u_2\|_{X^s_M([0,T]\times\mathbb{R})}\|u_3\|_{X^s_M([0,T]\times\mathbb{R})}.$$

Proof. The proof will be very similar to the proof of Proposition 5.2. We will again state similar bounds, but without derivatives. Therefore all steps can be considered as a consequences of the previous estimate 5.2. □

6. Energy bound

The following section of capstone project will analyze the almost conserved energy by using the adapted I-method of Colliander-Keel-Staffilani-Takaoka-Tao [6]. The main theorem of the section is

Proposition 6.1. *Let $-1/8 \leq s \leq 0$, $M > 1$ and u a solution of* (1.1). *Then,*

(6.1)
$$\|u\|_{\ell^2 L_t^\infty H^s_{M,x}([0,T]\times\mathbb{R})} \lesssim \|u(\cdot,0)\|_{H^s_{M,x}([0,T]\times\mathbb{R})} + \|u\|^2_{\ell^2 L_t^\infty H^s_{M,x}([0,T]\times\mathbb{R})} + \|u\|^3_{X^s_M([0,T]\times\mathbb{R})}.$$

6.1. **Preliminaries.** Before proving the energy bound we first study the weighted energy conservation for solutions u to (1.1).

Due to the ℓ^2 dyadic summation on the left we cannot simply obtain a uniform in time bound for the H^s norm of u. Hence, we introduce a class S_M of real smooth positive symbols $A(\xi)$ for $\epsilon > 0$:

Definition 6.2. *Let $M \geq 1$. Then S_M is the class of real smooth positive symbols with the following properties:*
(i) $A(\xi)$ is constant for $|\xi| \leq 1$.
(ii) Regularity:

$$|\partial_\xi^\alpha A(\xi)| \leq c_\alpha A(\xi)\langle\xi\rangle^{-\alpha}. \tag{6.2}$$

(iii) Decay properties

$$-\frac{1}{2} \leq \frac{d\, log\, A(\xi)}{d\, log(1+\xi^2)} \leq 0 \tag{6.3}$$

The latter property implies that $A(\xi)$ is nonincreasing but decays no faster than $|\xi|^{-\frac{1}{2}}$. For $A \in S_M$ we will prove the uniform bound

(6.4)
$$\|u\|^2_{L_t^\infty H^a([0,T]\times\mathbb{R})} \leq \|u(\cdot,0)\|^2_{H^a([0,T]\times\mathbb{R})} + c(\|u\|^2_{L_t^\infty H^s_{M,x}([0,T]\times\mathbb{R})}\|u\|^2_{L_t^\infty H^a([0,T]\times\mathbb{R})} + \\ + \|u\|^4_{X^s_M([0,T]\times\mathbb{R})}\|u\|^2_{X^a_M([0,T]\times\mathbb{R})})$$

which implies the desired bound (6.1). In order to undertand and prove it, we consider a symbol $a_N \in S_M$ for each dyadic number $N \geq 1$ such that

$$a_N(\xi) \stackrel{\text{def}}{=} \begin{cases} N^{2s} & \text{if } |\xi| \leq N \\ N^{\frac{1}{2}+2s}|\xi|^{-\frac{1}{2}} & \text{if } |\xi| \geq 2N \end{cases}.$$

6.2. **Main Results.** We inspired by the I-method to construct energy functional and investigate its behavior along the flow. So we now define the energy functional as

$$E_0(u) \stackrel{\text{def}}{=} \langle A(D)u, u\rangle = \|u\|^2_{H^a([0,T]\times\mathbb{R})}, \tag{6.5}$$

where $A(D)u = \mathcal{F}^{-1}(A(\xi)\hat{u}(\xi))(x)$. We then compute its derivative along the flow. Note that $A(D)$ is self-adjoint because $A(\xi)$ is real. Hence, by taking the derivative with respect to time, we get that

$$\begin{aligned}\frac{d}{dt}E_0(u) &= \langle \frac{d}{dt}A(D)u, u\rangle + \langle A(D)u, \partial_t u\rangle = \langle A(D)\partial_t u, u\rangle + \langle A(D)u, \partial_t u\rangle \\ &= \langle \partial_t u, A(D)u\rangle + \langle A(D)u, \partial_t u\rangle = \overline{\langle A(D)u, \partial_t u\rangle} + \langle A(D)u, \partial_t u\rangle\end{aligned}$$

$$= 2\Re\langle A(D)u, \partial_t u\rangle \stackrel{\text{def}}{=} R_4(u),$$

where we have used the conjugate symmetry of the inner product.

Thus, using equation (1.1) to compute $\partial_t u$, we obtain that

$$\begin{aligned} R_4(u) &= 2\Re\langle A(D)u, -ia\partial_x^2 u - b\partial_x^3 u - ic|u|^2u - d|u|^2\partial_x u - eu^2\partial_x\overline{u}\rangle \\ &= 2a\Re\langle iA(D)u, \partial_x^2 u\rangle - 2b\Re\langle A(D)u, \partial_x^3 u\rangle \\ &+ 2c\Re\langle iA(D)u, |u|^2u\rangle - 2d\Re\langle A(D)u, |u|^2\partial_x u\rangle - 2e\Re\langle A(D)u, u^2\partial_x\overline{u}\rangle. \end{aligned}$$

In virtue of Plancherel's theorem and the polarization identity, we have that

$$\Re\langle iA(D)u, \partial_x^2 u\rangle = \Re\langle \widehat{iA(D)u}, \widehat{\partial_x^2 u}\rangle = -4\pi^2\Re\int iA(\xi)\xi^2|\widehat{u}(\xi)|^2 d\xi = 0.$$

Similarly, we also have that

$$\Re\langle A(D)u, \partial_x^3 u\rangle = \Re\langle \widehat{A(D)u}, \widehat{\partial_x^3 u}\rangle = 8\pi^3\Re\int iA(\xi)\xi^3|\widehat{u}(\xi)|^2 d\xi = 0.$$

As a consequence, we deduce that

$$\begin{aligned} R_4(u) &= 2c\Re\langle iA(D)u, |u|^2u\rangle - 2d\Re\langle A(D)u, |u|^2\partial_x u\rangle - 2e\Re\langle A(D)u, u^2\partial_x\overline{u}\rangle \\ &\stackrel{\text{def}}{=} cR_4^I(u) - dR_4^{II}(u) - eR_4^{III}(u). \end{aligned}$$

Now, we focus on the term $R_4^I(u)$. Let us write it as a multi-linear operator in the Fourier space,

$$\begin{aligned} R_4^I(u) &= 2\Re\langle \widehat{iA(D)u}, \widehat{uu\overline{u}}\rangle = 2\Re\int iA(\xi_1)\widehat{u}(\xi_1)\overline{(\widehat{u} * \widehat{u} * \widehat{\overline{u}})(\xi_1)}d\xi_1 \\ &= 2\Re\iiint iA(\xi_1)\widehat{u}(\xi_1)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)\widehat{\overline{u}}(\xi_1 - \xi_3 - \xi_4)}d\xi_1 d\xi_3 d\xi_4 \\ &= 2\Re\iiint iA(\xi_1)\widehat{u}(\xi_1)\widehat{u}(-\xi_1 + \xi_3 + \xi_4)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\xi_1 d\xi_3 d\xi_4 \\ &= 2\Re\iiiint_{\{\xi_1+\xi_2-\xi_3-\xi_4=0\}} iA(\xi_1)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\ &= 2\Re\int_{P_4} iA(\xi_1)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi}, \end{aligned}$$

where

$$P_4 = \left\{\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : \xi_1 + \xi_2 - \xi_3 - \xi_4 = 0\right\}.$$

Note that if in this last expression for $R_4^I(u)$ we apply the change of variables given by $\xi_1 \leftrightarrow \xi_3$ and $\xi_2 \leftrightarrow \xi_4$, we realize that

$$\begin{aligned} R_4^I(u) &= 2\Re\int_{P_4} iA(\xi_3)\widehat{u}(\xi_3)\widehat{u}(\xi_4)\overline{\widehat{u}(\xi_1)\widehat{u}(\xi_2)}d\boldsymbol{\xi} \\ &= 2\Re\overline{\int_{P_4} iA(\xi_3)\widehat{u}(\xi_3)\widehat{u}(\xi_4)\overline{\widehat{u}(\xi_1)\widehat{u}(\xi_2)}d\boldsymbol{\xi}} \\ &= -2\Re\int_{P_4} iA(\xi_3)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi}. \end{aligned}$$

In a similar fashion,

$$R_4^I(u) = 2\Re\int_{P_4} iA(\xi_2)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi}$$

$$= -2\Re \int_{P_4} iA(\xi_4)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi},$$

and hence, $R_4^I(u)$ can be symmetrized as

$$R_4^I(u) = \frac{1}{2}\Re \int_{P_4} iA^I(\boldsymbol{\xi})\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi},$$

with

$$A^I(\boldsymbol{\xi}) \stackrel{\text{def}}{=} A(\xi_1) + A(\xi_2) - A(\xi_3) - A(\xi_4).$$

Next, we focus on the term $R_4^{II}(u)$. We can write

$$\begin{aligned} R_4^{II}(u) &= 2\Re\langle \widehat{A(D)u}, \widehat{u\overline{u}\partial_x u}\rangle = 2\Re \int A(\xi_1)\widehat{u}(\xi_1)\overline{(\widehat{u} * \widehat{\partial_x u} * \widehat{\overline{u}})(\xi_1)}d\xi_1 \\ &= -4\pi\Re \iiint i\xi_4 A(\xi_1)\widehat{u}(\xi_1)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)\widehat{\overline{u}}(\xi_1 - \xi_3 - \xi_4)}d\xi_1 d\xi_3 d\xi_4 \\ &= -4\pi\Re \int_{P_4} i\xi_4 A(\xi_1)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi}. \end{aligned}$$

If in this last expression we perform all the 8 possible permutations of variables that leave P_4 invariant, we get that

$$\begin{aligned} R_4^{II}(u) &= -4\pi\Re \int_{P_4} i\xi_4 A(\xi_2)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi} \\ &= -4\pi\Re \int_{P_4} i\xi_3 A(\xi_1)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi} \\ &= -4\pi\Re \int_{P_4} i\xi_3 A(\xi_2)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi} \\ &= 4\pi\Re \int_{P_4} i\xi_2 A(\xi_3)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi} \\ &= 4\pi\Re \int_{P_4} i\xi_2 A(\xi_4)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi} \\ &= 4\pi\Re \int_{P_4} i\xi_1 A(\xi_3)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi} \\ &= 4\pi\Re \int_{P_4} i\xi_1 A(\xi_4)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi}, \end{aligned}$$

and hence, $R_4^{II}(u)$ can be symmetrized as

$$R_4^{II}(u) = \frac{\pi}{2}\Re \int_{P_4} iA^{II}(\boldsymbol{\xi})\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi},$$

with

$$A^{II}(\boldsymbol{\xi}) \stackrel{\text{def}}{=} (\xi_1 + \xi_2)(A(\xi_3) + A(\xi_4)) - (\xi_3 + \xi_4)\left(A(\xi_1) + A(\xi_2)\right).$$

Finally, for the term $R_4^{III}(u)$ we have that

$$\begin{aligned} R_4^{III}(u) &= 2\Re\langle \widehat{A(D)u}, \widehat{uu\partial_x \overline{u}}\rangle = 2\Re \int A(\xi_1)\widehat{u}(\xi_1)\overline{(\widehat{u} * \widehat{u} * \widehat{\partial_x \overline{u}})(\xi_1)}d\xi_1 \\ &= -4\pi\Re \iiint i(\xi_1 - \xi_3 - \xi_4) A(\xi_1)\widehat{u}(\xi_1)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)\widehat{\overline{u}}(\xi_1 - \xi_3 - \xi_4)}d\xi_1 d\xi_3 d\xi_4 \\ &= 4\pi\Re \int_{P_4} i\xi_2 A(\xi_1)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi}. \end{aligned}$$

Once again, if in this last expression we perform all the possible permutations of variables that leave P_4 invariant, we obtain that

$$\begin{aligned}R_4^{III}(u) &= 4\pi\Re\int_{P_4} i\xi_1 A(\xi_2)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi} \\ &= -4\pi\Re\int_{P_4} i\xi_4 A(\xi_3)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi} \\ &= -4\pi\Re\int_{P_4} i\xi_3 A(\xi_4)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi},\end{aligned}$$

and hence, $R_4^{III}(u)$ can be symmetrized as

$$R_4^{III}(u) = \pi\Re\int_{P_4} iA^{III}(\boldsymbol{\xi})\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi},$$

with

$$A^{III}(\boldsymbol{\xi}) \stackrel{\text{def}}{=} \xi_1 A(\xi_2) + \xi_2 A(\xi_1) - \xi_3 A(\xi_4) - \xi_4 A(\xi_3).$$

In conclusion, $R_4(u)$ can be symmetrized as

$$R_4(u) = \frac{1}{2}\Re\int_{P_4} i\left(cA^I(\boldsymbol{\xi}) - d\pi A^{II}(\boldsymbol{\xi}) - 2e\pi A^{III}(\boldsymbol{\xi})\right)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi}.$$

Note that for $d = e = 0$ and $c = 1$, then we recover NLSE, while for $c = 0$ and $d = 2e$ and $e = \pm\frac{1}{2\pi}$ and A is even and u is real, then we recover $\mathbb{R}$-mKdV.

Inspired by the I-method of Tao et al., to estimate $R_4(u)$, we will introduce an extra term $E_1(u)$ of the form

$$E_1(u) = \int_{P_4} B(\boldsymbol{\xi})\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi},$$

being B a nice real function that we will choose later, so that $E_0(u) = E_1(u) + (E_0(u) - E_1(u))$, and $\frac{d}{dt}E_1(u) = R_4(u) + \frac{d}{dt}(E_1(u) - E_0(u))$. $\partial_t\overline{u} = \overline{\partial_t u}$.

To determine the appropriate choice for B, we compute $\frac{d}{dt}E_1(u)$ as follows:

$$\begin{aligned}\frac{d}{dt}E_1(u) &= \int_{P_4} B(\boldsymbol{\xi})\left(\partial_t\widehat{u}(\xi_1)\right)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi} \\ &+ \int_{P_4} B(\boldsymbol{\xi})\widehat{u}(\xi_1)\left(\partial_t\widehat{u}(\xi_2)\right)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi} \\ &+ \int_{P_4} B(\boldsymbol{\xi})\widehat{u}(\xi_1)\widehat{u}(\xi_2)\left(\partial_t\overline{\widehat{u}(\xi_3)}\right)\overline{\widehat{u}(\xi_4)}d\boldsymbol{\xi} \\ &+ \int_{P_4} B(\boldsymbol{\xi})\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)}\left(\partial_t\overline{\widehat{u}(\xi_4)}\right)d\boldsymbol{\xi} \\ &= 2\int_{P_4} B(\boldsymbol{\xi})\left(\partial_t\widehat{u}(\xi_1)\right)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi} \\ &+ 2\int_{P_4} B(\boldsymbol{\xi})\widehat{u}(\xi_1)\widehat{u}(\xi_2)\left(\partial_t\overline{\widehat{u}(\xi_3)}\right)\overline{\widehat{u}(\xi_4)}d\boldsymbol{\xi} \\ &= 2\int_{P_4} B(\boldsymbol{\xi})\left(\partial_t\widehat{u}(\xi_1)\right)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)}d\boldsymbol{\xi} \\ &+ \overline{2\int_{P_4} B(\boldsymbol{\xi})\overline{\widehat{u}(\xi_1)\widehat{u}(\xi_2)}(\partial_t\widehat{u}(\xi_3))\widehat{u}(\xi_4)d\boldsymbol{\xi}}\end{aligned}$$

$$= 4\Re \int_{P_4} B(\boldsymbol{\xi}) \left(\partial_t \widehat{u}(\xi_1)\right) \widehat{u}(\xi_2) \overline{\widehat{u}(\xi_3) \widehat{u}(\xi_4)} d\boldsymbol{\xi},$$

assuming that

$$B(\xi_1, \xi_2, \xi_3, \xi_4) = B(\xi_2, \xi_1, \xi_3, \xi_4) = B(\xi_1, \xi_2, \xi_4, \xi_3) = B(\xi_3, \xi_4, \xi_1, \xi_2). \tag{6.6}$$

Taking the Fourier transform of equation (1.1), we deduce that

$$\begin{aligned}
\partial_t \widehat{u}(\xi) &= -ia\widehat{\partial_x^2 u}(\xi) - b\widehat{\partial_x^3 u}(\xi) - ic\widehat{|u|^2 u}(\xi) + d\widehat{|u|^2 \partial_x u}(\xi) - e\widehat{u^2 \partial_x \overline{u}}(\xi) \\
&= 4\pi^2 ia\xi^2 \widehat{u}(\xi) + 8\pi^3 ib\xi^3 \widehat{u}(\xi) - ic\widehat{|u|^2 u}(\xi) + d\widehat{|u|^2 \partial_x u}(\xi) - e\widehat{u^2 \partial_x \overline{u}}(\xi) \\
&\stackrel{\text{def}}{=} 4\pi^2 i(a\xi^2 + 2\pi b\xi^3)\widehat{u}(\xi) + S_u(\xi),
\end{aligned}$$

and hence,

$$\begin{aligned}
\frac{d}{dt} E_1(u) &= 16\pi^2 \Re \int_{P_4} iB(\boldsymbol{\xi})(a\xi_1^2 + 2\pi b\xi_1^3)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)} d\boldsymbol{\xi} \\
&\quad + 4\Re \int_{P_4} B(\boldsymbol{\xi}) S_u(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)} d\boldsymbol{\xi} \stackrel{\text{def}}{=} J + R_6(u).
\end{aligned}$$

So, we get the following expression for R_6

$$R_6(u) \stackrel{\text{def}}{=} 4\Re \int_{P_4} B(\boldsymbol{\xi}) S_u(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)} d\boldsymbol{\xi}. \tag{6.7}$$

Using (6.6), we have that

$$\begin{aligned}
J &= 16\pi^2 \Re \int_{P_4} iB(\boldsymbol{\xi})(a\xi_2^2 + 2\pi b\xi_2^3)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)} d\boldsymbol{\xi} \\
&= -16\pi^2 \Re \int_{P_4} iB(\boldsymbol{\xi})(a\xi_3^2 + 2\pi b\xi_3^3)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)} d\boldsymbol{\xi} \\
&= -16\pi^2 \Re \int_{P_4} iB(\boldsymbol{\xi})(a\xi_4^2 + 2\pi b\xi_4^3)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)} d\boldsymbol{\xi},
\end{aligned}$$

and J can be symmetrized as

$$J = 4\pi^2 \Re \int_{P_4} iB(\boldsymbol{\xi})Q(\boldsymbol{\xi})\widehat{u}(\xi_1)\widehat{u}(\xi_2)\overline{\widehat{u}(\xi_3)\widehat{u}(\xi_4)} d\boldsymbol{\xi},$$

with

$$Q(\boldsymbol{\xi}) \stackrel{\text{def}}{=} a(\xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2) + 2\pi b(\xi_1^3 + \xi_2^3 - \xi_3^3 - \xi_4^3).$$

Now, we choose B in such a way that $J = R_4(u)$. That is,

$$4\pi^2 B(\boldsymbol{\xi})Q(\boldsymbol{\xi}) = \frac{1}{2}\left(cA^I(\boldsymbol{\xi}) - d\pi A^{II}(\boldsymbol{\xi}) - 2e\pi A^{III}(\boldsymbol{\xi})\right), \quad \boldsymbol{\xi} \in P_4.$$

If $(a, b) \neq (0, 0)$, then we can take

$$B(\boldsymbol{\xi}) \stackrel{\text{def}}{=} \frac{1}{8\pi^2} \frac{cA^I(\boldsymbol{\xi}) - d\pi A^{II}(\boldsymbol{\xi}) - 2e\pi A^{III}(\boldsymbol{\xi})}{a(\xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2) + 2\pi b(\xi_1^3 + \xi_2^3 - \xi_3^3 - \xi_4^3)}, \tag{6.8}$$

which satisfies (6.6).

Remark 6.3. *To avoid dividing by zero in the previous definition of B, note that*

$$0 \leq \int_{P_4 \cap \{Q(\boldsymbol{\xi})=0\}} d\boldsymbol{\xi} \leq \int_{\{(x,y,z)\in\mathbb{R}^3: -2(a+3b\pi(x+y))(x-z)(y-z)=0\}} dxdydz = 0,$$

since the region of integration in the second integral is a union of three planes in $\mathbb{R}^3$*. Hence, we can define* $B(\boldsymbol{\xi})$ *as before for* $\boldsymbol{\xi} \in P_4 \setminus \{Q(\boldsymbol{\xi}) = 0\}$*, and take* $B(\boldsymbol{\xi}) = 0$ *for* $\boldsymbol{\xi} \in P_4 \cap \{Q(\boldsymbol{\xi}) = 0\}$*.*

In conclusion, we have

$$\frac{d}{dt}(E_0 + E_1)(u) = R_6(u).$$

In the remaining part of this section we are going to state the useful lemmas and connect them in order to show our main inequality (6.1).

Lemma 6.4. *Let* $a \in S$ *and* B *as in* (6.8)*. Then we have*

$$E_1(u(x,t)) \lesssim \|u\|^2_{H^s_{M,x}([0,T]\times\mathbb{R})} \|u\|^2_{H^a([0,T]\times\mathbb{R})} \tag{6.9}$$

Since $\|u\|_{H^{-1/2}} \lesssim \|u\|_{H^s}$ [15, Proposition 3.1] the following corollary will imply (6.9):

Corollary 6.5. *Let* $a \in S$ *and* B *as in* (6.8)*. Then*

$$|E_1(u(x,t))| \lesssim \|u\|^2_{H^a([0,T]\times\mathbb{R})} \|u\|^2_{H^{-\frac{1}{2}}_{M,x}} \tag{6.10}$$

Given the expression of B, it can be proved by using the notations $u_N = P_N u$ for $N > 1$ and $u = P_{\leq N} u$ from Littlewood-Paley projections and Bernstein's inequality (4.1). Particularly, it will come from the behavior of derivatives and Hölder inequality (2.1).

Lemma 6.6. *Let* $a \in S$ *and* R_6 *be given as in* (6.7)*. Then*

$$\Big| \int_0^t R_6(u(t))dt \Big| \lesssim \|u\|^4_{X^s_M([0,T]\times\mathbb{R})} \|u\|^2_{X^a([0,T]\times\mathbb{R})}. \tag{6.11}$$

The proof will come from the behavior of derivatives of R_6.

Hence Lemma 6.4 and Lemma 6.6 will give us (6.4).

Proof of 6.1. It can be shown that by Fundamental Theorem of Calculus that we have

$$\begin{aligned}
\Big| \int_0^t R_6(u(x,s))ds \Big| &\leq \Big| \int_0^t \frac{d}{dt}(E_0 + E_1)(u(x,s))ds \Big| \\
&\leq |(E_0 + E_1)(u(x,t))| + |(E_0 + E_1)(u(x,0))| \\
&\leq E_0(u(x,0)) + E_1(u(x,t))
\end{aligned}$$

since we choose $E_0(u) = E_1(u) + (E_0(u) - E_1(u)) = (E_0(u) + E_1(u)) - E_1(u)$.
By (6.5) and the bound in Lemma 6.4 we get

$$\begin{aligned}
\|E_0(u)\| = \|u(\cdot,0\|_{H^a} &\leq \sup_{0\leq t\leq 1} \|u(\cdot,0\|_{H^a} = \|u\|^2_{L^\infty H^a} \\
&\lesssim \|u(0)\|^2_{H^a} + \|u\|^2_{L^\infty H^s_M} \|u\|^2_{L^\infty H^a} + \|u\|^4_{X^s_M} \|u\|^2_{X^a}.
\end{aligned}$$

□

Finally, we apply a_N to (6.4)

$$\|u\|^2_{L^\infty H^a_N} \lesssim \|u(0)\|^2_{H^a_N} + \|u\|^2_{L^\infty H^s_M} \|u\|^2_{L^\infty H^a_N} + \|u\|^4_{X^s_M} \|u\|^2_{X^a_N},$$

and by the following relations

$$\|u\|^2_{\ell^2 L^\infty H^s_M} \approx \sum_{N\geq 1} \|u\|^2_{L^\infty H^a_N},$$

$$\|u\|_{X^s_M}^2 \approx \sum_{N\geq 1} \|u\|_{X^a_N}^2$$

we get (6.1).

7. Proof of Theorem 1.1

Before starting the proof of the Main Theorem we shall introduce proposition that states the problem for small data result:

Proposition 7.1. *Fix $T > 0$, and $M \in \mathscr{D}$, and let $-1/8 \leq s < 0$. There exists $0 < \varepsilon_0 < 1$ such that for every initial data $u_0 \in \mathcal{S}(\mathbb{R})$ satisfying*

$$\|u_0\|_{H^s_M} \leq \varepsilon_0,$$

and every solution $u \in \mathscr{C}^0_t \mathcal{S}_x([0,T] \times \mathbb{R})$ to the IVP (1.1),

$$\|u\|_{L^\infty_t H^s_{M,x}([0,T]\times\mathbb{R})} \lesssim \|u_0\|_{H^s_M}.$$

Proof. To prove this result, we follow the continuity argument presented in [10, Page 5].

Let $u_0 \in \mathcal{S}(\mathbb{R})$, and let $u \in \mathscr{C}^0_t \mathcal{S}_x([0,T] \times \mathbb{R})$ be a solution to (1.1) up to time T. Also assume that the quantities in Proposition 3.1 (Basic Estimates) are finite. In particular, we have $u \in \ell^2 L^\infty_t H^s_{M,x}([0,T] \times \mathbb{R})$ and $u \in X^s_M([0,T] \times \mathbb{R})$.

We consider a small value $0 < \delta < 1$, and denote by A_δ the set

$$A_\delta \overset{\text{def}}{=} \{t \in [0,T] : \|u\|_{\ell^2 L^\infty_t H^s_{M,x}([0,t]\times\mathbb{R})} \leq 2\delta, \|u\|_{X^s_M([0,t]\times\mathbb{R})} \leq 2\delta\}.$$

Here some claims with no proof:

(1) $0 \in A_\delta$, for $0 < \delta$.
(2) The norms defining A_δ increase with t.
(3) Therefore, A_δ is an interval, possibly for $0 < \delta$ arbitrary.
(4) The norms defining A_δ are continuous with respect to t, so A_δ is closed.

We see that there exists $0 < \varepsilon_0 < 1$ such that if $\|u_0\|_{H^s} = \varepsilon \leq \varepsilon_0$, then there exists $0 < \delta < 1$ such that $A_\delta = [0,T]$.

Let $t \in A_\delta$. Suppose that our energy estimates ensure that if $\|u\|_{\ell^2 L^\infty_t H^s_{M,x}([0,t]\times\mathbb{R})} \leq 2\delta$, then

$$\|u\|_{\ell^2 L^\infty_t H^s_{M,x}([0,t]\times\mathbb{R})} \leq C_3(\|u_0\|_{H^s_M} + \|u\|^2_{\ell^2 L^\infty_t H^s_{M,x}([0,T]\times\mathbb{R})} + \|u\|^3_{X^s_M([0,t]\times\mathbb{R})}),$$

with $C_3 \geq 1$ independent of δ, t, and u_0. Then,

$$\|u\|_{\ell^2 L^\infty_t H^s_{M,x}([0,t]\times\mathbb{R})} \leq 8C_3(\varepsilon + \delta^2 + \delta^3).$$

By basic estimates,

$$\|u\|_{X^s_M([0,t]\times\mathbb{R})} \leq C_1(\|u\|_{\ell^2 L^\infty_t H^s_{M,x}([0,t]\times\mathbb{R})} + \|f\|_{Y^s_M([0,t]\times\mathbb{R})}),$$

with $C_1 \geq 1$ depending only on M. Also, our trilinear estimates give us

$$\|f\|_{Y^s_M([0,t]\times\mathbb{R})} \leq C_2\|u\|^3_{X^s_M([0,t]\times\mathbb{R})},$$

with $C_2 \geq 1$ independent of t. With these ingredients,

$$\|u\|_{X^s_M([0,t]\times\mathbb{R})} \leq C_1(8C_3(\varepsilon + \delta^2 + \delta^3) + 8C_2\delta^3) \leq 16C_1(C_2 + C_3)(\varepsilon + \delta^2 + \delta^3).$$

Hence, for $C \overset{\text{def}}{=} 16C_1(C_2 + C_3) > 1$, we have that

$$\max\{\|u\|_{\ell^2 L^\infty_t H^s_{M,x}([0,t]\times\mathbb{R})}, \|u\|_{X^s_M([0,t]\times\mathbb{R})}\} \leq C(\varepsilon + \delta^2 + \delta^3).$$

By setting $\delta = K\varepsilon$, with $K > 0$, the condition $C(\varepsilon + \delta^2 + \delta^3) \leq \delta$ is equivalent to $K^3\varepsilon^2 + K^2\varepsilon + 1 - \frac{K}{C} \leq 0$. And we choose $K > C$ such that the maximum value for ε is obtained; that is, $K_0 = \frac{C^2+\sqrt{C^4+3C^3}+6C}{4+C}$. Now we set $\varepsilon_0 = \frac{-K_0^2+\sqrt{K_0^4-4K_0^3(1-\frac{K_0}{C})}}{2K_0^3}$.

Hence, by the continuity of the norms with respect to t, it follows that a neighborhood of t is in A_δ. In conclusion, A_δ is an interval that is both open and closed in $[0, T]$, so it must be $[0, T]$.

To finish this argument, since $T \in A_\delta$, we have that

$$\|u\|_{L_t^\infty H_{M,x}^s([0,T]\times\mathbb{R})} \leq \|u\|_{\ell^2 L_t^\infty H_{M,x}^s([0,T]\times\mathbb{R})} \leq 2\delta = 2\frac{C^2 + \sqrt{C^4 + 3C^3} + 6C}{4 + C}\|u_0\|_{H^s},$$

and the desired result follows. □

Proof of Theorem 1.1. Let $-1/8 < s \leq 0$. Let $u_0 \in \mathcal{S}(\mathbb{R})$, and let $u \in \mathscr{C}_t^0\mathcal{S}_x([0, T] \times \mathbb{R})$ be a solution to (1.1) up to time T, and suppose that $\|u_0\|_{H^s} \leq R$. For every dyadic integer $M \geq 1$, we have that

$$\|u_0\|_{H_M^{-1/8}} = \|(|\cdot|^2 + M)^{-\frac{1}{16}}\widehat{u}_0(\cdot)\|_{L^2} \leq M^{-\frac{1}{16}-\frac{s}{2}}\|(|\cdot|^2 + 1)^{\frac{s}{2}}\widehat{u}_0(\cdot)\|_{L^2} \leq M^{-\frac{1}{16}-\frac{s}{2}}R.$$

Now, let M_0 be the smallest dyadic integer such that $M_0^{-\frac{1}{16}-\frac{s}{2}}R \leq \varepsilon_0$, and apply Proposition 7.1 to obtain that for every dyadic integer $M \geq M_0$,

$$\|u\|_{L_t^\infty H_{M,x}^{-1/8}([0,T]\times\mathbb{R})} \lesssim \|u_0\|_{H_M^{-1/8}}.$$

Now, we take a weighted square sum with respect to dyadic integers $M \geq M_0$,

$$\|u_0\|_{H_{M_0}^s}^2 \approx \sum_{M \geq M_0} M^{\frac{1}{8}+s}\|u_0\|_{H_M^{-1/8}}^2,$$

and we get the desired result,

$$M_0^{\frac{s}{2}}\|u\|_{L_t^\infty H_x^s([0,T]\times\mathbb{R})} \leq \|u\|_{L_t^\infty H_{M_0,x}^s([0,T]\times\mathbb{R})} \lesssim \|u_0\|_{H_{M_0}^s} \leq \|u_0\|_{H^s}.$$

□

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