WELL-POSEDNESS OF THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. The Nonlinear Schrödinger equation (NLSE) is a prototypical example of nonlinear partial differential equation. It is commonly used to describe propagation of light in nonlinear optical fibers and is of great importance in quantum mechanics. In this Capstone Project, we provide a complete proof of well-posedness, that is existence of a unique solution of the NLSE using one of the major mathematical techniques: the Banach fixed-point theorem. Both local and global results for inital data in $L^2(\mathbb{R})$ are obtained. Moreover, we briefly discuss possible extensions of the topic in terms of different function spaces, general nonlinearities and higher dimension.

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1. INTRODUCTION

There exist various states of matter. Some of them such as gas, liquid, solid, and plasma we can see in everyday life. Others are considered to exist theoretically whereas some states of matter are only possible under special conditions, such as extremely high energy, extreme density, or extreme cold. Example of the latter is Bose-Einstein condensate (BEC), which have been studied since XX century. The idea of BEC was first proposed by Albert Einstein and Satyendra Nath Bose in 1924. However, the first evidence of the state of matter was produced experimentally only in 1995 by the research groups of Eric Cornell and Carl Wieman.

BEC is a state of matter discribed as a dilute gas of elementary particles called bosons which are cooled to temperatures very close to absolute zero. Under the condition of extreme cold, all the particles occupy the lowest quantum state and cannot be distinguished from one another. They start behaving like a macroscopic fluid with new properties.

The Gross-Pitaevskii equation (GPE) describes the ground state of a quantum system of identical bosons. It means that the solution of the equation is a wave-function of the system. Let us look at the following time-dependent form of the equation which characterizes the dynamics of the Bose-Einstein condensate

$$ih\,\partial_t\Psi(x,t) = -\frac{h^2}{2m}\Delta\Psi(x,t) + V(x)\Psi(x,t) + g\,|\Psi(x,t)|^2\,\Psi(x,t)$$

where m is the mass of boson particles, $V(x)$ is the external potential, h is the reduced Planck constant, and g is the representative of inter-particle interactions. Remarkably, GPE is evolved from the nonlinear Schrödinger equation. In general, we can state that the NLSE plays significant role in quantum world. Therefore, we would like to derive the equation in order to understand its origins.

The following derivation of the NLSE from the wave equation is presented by W. Bao in [1]. We reproduce it with a slight modification (taking $\mathbb{R}^2$ instead of $\mathbb{R}^3$) for the reader's convenience.

Consider a wave equation

$$\frac{1}{c^2}\partial_{tt}u(\mathbf{x},t) - \Delta u(\mathbf{x},t) = 0, \quad \mathbf{x}\in\mathbb{R}^2,$$

where $\mathbf{x}=(x,y)$ is a Cartesian coordinate, t is time, and $c=c(\mathbf{x},|u|)$ is propagation speed. It has a solution of the form

$$u(\mathbf{x},t) = e^{iwt}v(\mathbf{x})$$

where w represents angular frequency. If we now substitute it into the equation, we get the Helmholtz or reduced wave equation of the form

$$\Delta v(\mathbf{x}) + \frac{w^2}{c^2}v = 0, \quad x\in\mathbb{R}^2. \tag{1}$$

In other words, we have reduced our problem into time-independent one. Next, let c_0 be a uniform reference speed, $k_0 = w/c_0$ be the wave number, and $n(\mathbf{x},|u|) = c_0/c(\mathbf{x},|u|)$ be the index of refraction. Then (1) becomes

$$\Delta v(\mathbf{x}) + k_0^2 n^2(\mathbf{x},|u|)v = 0. \tag{2}$$

We look for solution of the following form

$$v(x,y) = e^{ik_0 y}\psi(x,y).$$

If we insert v into the reduced wave equation (2), we get

$$2ik_0\partial_y\psi + \Delta\psi + k_0^2\mu(x,y,|\psi|)\psi + \partial_y^2\psi = 0, \tag{3}$$

where Δ is a Laplacian in x, and $\mu(x,y,|\psi|) = n^2(x,y,|\psi|) - 1$ represents fluctuation in the refractive index. Also, note that variable y plays the role of time, and $-k_0^2\mu(x,y,|\psi|)$

is a potential. Next, in order to remove all the unwanted constants from the equation, we introduce change of variables such that

$$X = \frac{x}{r_0}, \qquad \tau = \frac{y}{k_0 r_0^2}, \qquad \psi(X,\tau) = \frac{\psi(x,y)}{\psi_s},$$

where r_0 is a dimensionless length unit and ψ_s is a dimensionless unit for ψ to be determined later. After changing the variables in (3), multiplying everything by $r_0^2/2$, the following equation is obtained

$$i\partial_\tau \psi = -\frac{1}{2}\partial_{XX}\psi + f(x,t,|\psi|)\psi - \frac{\delta}{2}\partial_\tau^2\psi \tag{4}$$

where $\delta = 1/r_0^2 k_0^2$ and the real-valued function f depends on μ. Moreover, since $r_0 >> \lambda = 2\pi/k_0$,

$$\frac{\delta}{2} = \lambda^2/8\pi^2 r_0^2 << 1$$

which gives us the possibility to remove $\partial_\tau^2\psi$ from the equation. Consequently, it will result in Nonlinear Schrödinger equation

$$i\partial_\tau \psi = -\frac{1}{2}\partial_{XX}\psi + f(X,\tau,|\psi|)\psi. \tag{5}$$

Let us continue with a little explanation of a physical background. As it was mentioned, $n(\mathbf{x},|u|)$ is the refraction index which is a dimensionless number that describes how fast light propagates through a material or medium. If the medium is uniform, $n(\mathbf{x},|u|) = 1$, and if it is linear, then $n(\mathbf{x},|u|) = n(\mathbf{x})$. In addition, a medium can have a nonlinear effect called Kerr effect. It occurs when an electrical field is applied to the medium, and as a result, the refractive index is modified. In such cases, $n(\mathbf{x},|u|) = \sqrt{1 + 4n_2|u|^2/n_0}$ where n_0 is a linear index of refraction and n_2 is a nonlinear refractive index or Kerr coefficient.

Consider a uniform medium with $n = 1$, then $\mu(x,y,|\psi|) = n^2(x,y,|\psi|) - 1 = 0$. This, in return, will remove the $f(X,\tau,|\psi|)$ term in (4), collapsing to the free Schrödinger equation

$$i\partial_\tau \psi = -\frac{1}{2}\partial_{XX}\psi.$$

Next, in a linear medium with $n(\mathbf{x},|u|) = n(\mathbf{x})$, we conclude that $\mu(x,y,|\psi|) = \mu(x,y)$, and the equation (4) becomes a linear Schrödinger equation with a potential $f(X,\tau,|\psi|) = V(X,\tau)$

$$i\partial_\tau \psi = -\frac{1}{2}\partial_{XX}\psi + V(X,\tau)\psi. \tag{6}$$

In addition, if we observe Kerr effect in some medium, then n is taken to be

$$n(\mathbf{x},|u|) = \sqrt{1 + 4n_2|u|^2/n_0}$$

so that $\mu(x,y,|\psi|) = 2n_2 r_0^2 k_0^2 |\psi|^2/n_0$. Also, let us choose $\psi_s = \sqrt{n_0}/r_0 k_0 \sqrt{2n_2}$ which results in $f(X,\tau,\psi|) = -|\psi|^2$. Therefore, the equation (4) transforms into NLSE with cubic nonlinearity.

$$i\partial_\tau \psi = -\frac{1}{2}\partial_{XX}\psi - |\psi|^2\psi. \tag{7}$$

It is an extremely difficult task to find an explicit solution of the following initial value problem for the Nonlinear Schrödinger equation

$$\begin{cases} i\,\partial_t u(x,t) + \Delta u(x,t) + \lambda\,|u(x,t)|^2\,u(x,t) = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases} \tag{8}$$

where $\lambda \in \mathbb{R}$ is a fixed constant, and Δ is the Laplacian with respect to x (see [14, §5]).

Before even trying to find the solution, at least we should guarantee its existence and uniqueness, that is well-posedness of the initial value problem for NLSE. This is the main idea of the project. Primarily, we prove that a unique solution can exist up to a particular time T (local result, Section 4.1). Moreover, the well-posedness result will be obtained for an arbitrarily large time interval (global result, Section 4.2). In order to do so, the key mathematical tool to be used is the Banach fixed-point theorem (BFPT) (Section 2.1). It states that every contraction mapping on a complete metric space has a unique fixed point. However, since the NLSE is not in the form of a fixed-point problem, the theorem cannot be directly applied to the equation. The first necessary step will be interchanging the PDE with an integral equation. Consequently, we will convert our problem into a fixed-point one and then we will be able to use the BFPT.

Before proceeding to the major question about well-posedness of the NLSE (Theorem 4.2), we need to review some basic concepts. Therefore, firstly, we will consider preliminary results concerning function spaces (Section 2.2). In addition, several significant lemmas about properties of functions such as Hölder's, Minkowski's inequalities, duality principle, Hardy-Littlewood-Sobolev and Riesz-Thorin theorems are introduced.

The third section examines Linear Schrödinger equations (LSE). Particularly, we deduce explicit solutions for both homogeneous (Section 3.1) and non-homogeneous (Section 3.2) initial value problems. Noteworthy, the latter is crucial in the analysis of the main subject of the paper. In addition, some important inequalities related to the solution of LSE are proved (Section 3.3). Those inequalities will be essential for further discussion.

The forth section studies the main research topic which is local and global well-posedness of the Nonlinear Schrödinger Equation.

Finally, the last part of the paper (Section 5) discusses possible extensions of the results in terms of higher dimension, different function spaces, or more general forms of nonlinearities. Also, we include a brief overview on various partial differential equations that are somehow connected to the NLSE.

Primary reference of the project has been the book "Introduction to Nonlinear Dispersive Equations" by F. Linares and G. Ponce [14].

2. PRELIMINARIES

2.1. **Banach fixed-point theorem.**

Banach fixed-point theorem is the main mathematical tool that is used to prove the uniqueness and existence of our results. In order to satisfy the conditions for the theorem, we should have a complete normed space and a contraction mapping on that space. Since the conditions are quite general, the theorem is frequently used in context of different spaces and functions. Before going into details, we should introduce some definitions. The first definition is important in understanding the concept of a complete space.

Definition 2.1. *Let $(X, \|\cdot\|_X)$ be a normed space. A sequence $\{x_n\}_{n=0}^{\infty} \in X$ is a Cauchy sequence if for any given $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for $m, n \geq K$*

$$\|x_n - x_m\|_X < \varepsilon.$$

Definition 2.2. *A normed space $(X, \|\cdot\|_X)$ is said to be complete if every Cauchy sequence in X is convergent in X.*

Next, we give a definition of a contraction mapping.

Definition 2.3. *Let $(X, \|\cdot\|_X)$ be a normed space. An operator $T : X \mapsto X$ is called a contraction mapping if there exists $0 \leq K < 1$ such that*

$$\|T(x) - T(y)\|_X \leq K\|x - y\|_X, \qquad x, y \in X.$$

Moreover, the following definition is about the concept of a fixed point which is crucial for the Banach fixed-point theorem.

Definition 2.4. *Let f(x) to be a function. A point x^* is called a fixed point of the function $f(x)$ if $f(x^*) = x^*$.*

Finally, for the following Banach fixed-point theorem and its proof refer to the paper by R. S. Palais [15].

Theorem 2.5. *Let $(X, \|\cdot\|_X)$ be a complete normed space and $T : X \to X$ a contraction mapping. Then T has a unique fixed point $x^* \in X$.*

Proof. First of all, the existence of a fixed-point $x^* \in X$ will be proved. To proceed, define a sequence $\{x_n\}_{n=0}^{\infty}$ such that starting from an arbitrary point $x_0 \in X$

$$x_n = T(x_{n-1}) = T^n(x_0), \qquad n \geq 1.$$

We will show that the sequence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence whose limit x^* is a fixed point for T.Since $T : X \to X$ is a contraction mapping, for any $x_0, x_1 \in$ X,

$$\|T(x_0) - T(x_1)\|_X \leq K\|x_0 - x_1\|_X, \qquad 0 \leq K < 1.$$

Moreover, by triangle inequality, for all $x, y \in X$

$$\|x - y\|_X \leq \|x - T(x)\|_X + \|T(x) - T(y)\|_X + \|y - T(y)\|_X$$
$$\|x - y\|_X - \|T(x) - T(y)\|_X \leq \|x - T(x)\|_X + \|y - T(y)\|_X.$$

Because

$$-\|T(x) - T(y)\|_X \geq -K\|x - y\|_X$$

we have that

$$(1 - K)\|x - y\|_X \leq \|x - T(x)\|_X + \|y - T(y)\|_X.$$

Remembering that $K < 1$, we deduce the following inequality

$$\|x - y\|_X \leq \frac{1}{1 - K}\Big(\|x - T(x)\|_X + \|y - T(y)\|_X\Big). \tag{9}$$

Furthermore, we claim that

$$\|T^n(x_1) - T^n(x_0)\|_X \leq K^n\|x_1 - x_0\|_X, \qquad n \in \mathbb{N} \tag{10}$$

which can also be understood

$$\|x_{n+1} - x_n\|_X \leq K^n \|x_0 - x_1\|_X, \qquad n \in \mathbb{N}.$$

In order to prove the inequality (10), we will use induction. Firstly, for $n = 1$

$$\|x_2 - x_1\|_X = \|T(x_1) - T(x_0)\|_X \leq K\|x_0 - x_1\|_X.$$

Next, assume that the statement holds for some $k \in \mathbb{N}$, that is

$$\|x_{k+1} - x_k\|_X \leq K^k \|x_1 - x_0\|_X.$$

Let us show that it also holds for $k + 1$,

$$\begin{aligned}
\|x_{(k+1)+1} - x_{k+1}\|_X &= \|x_{k+2} - x_{k+1}\|_X \\
&= \|T(x_{k+1}) - T(x_k)\|_X \\
&\leq K\|x_{k+1} - x_k\|_X \\
&= K^{k+1}\|x_1 - x_0\|_X.
\end{aligned}$$

Now we are in position of proving that the sequence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. According to (9) and (10) (taken with $x = T^n(x_0)$ and $y = T^m(x_0)$)

$$\begin{aligned}
\|x_n - x_m\|_X &= \|T^n(x_0) - T^m(x_0)\|_X \\
&\leq \frac{1}{1-K}\Big(\|T^n(x_0) - T^n(T(x_0))\|_X + \|T^m(x_0) - T^m(T(x_0))\|_X\Big) \\
&\leq (K^n + K^m)\frac{\|x_0 - T(x_0)\|_X}{1-K} \to 0, \qquad n, m \to \infty
\end{aligned}$$

because $K < 1$. Moreover, since X is complete, there exists $x^* \in X$ such that $x^* = \lim_{n\to\infty}(x_n)$. Now, we need to show that x^* is a fixed-point of T. Suppose x^* is not a fixed-point, then $0 < \delta = \|T(x^*) - x^*\|_X$. Since $x_n \to x^*$, there is $N \in \mathbb{N}$ such that $\|x_n - x^*\|_X < \delta/2$, for all $n \geq N$. Then, using T is a contraction, $n \to \infty$,

$$\begin{aligned}
\delta = \|T(x^*) - x^*\|_X &\leq \|T(x^*) - T(x_N)\|_X + \|x^* - x_{N+1}\|_X \\
&\leq K\|x^* - x_N\|_X + \|x^* - x_{N+1}\|_X \\
&< \delta/2 + \delta/2 = \delta,
\end{aligned}$$

which is a contradiction. Thus, x^* is a fixed-point of T.

Next, we prove the uniqueness of the fixed-point. If we assume that both x and y are fixed-points of T, then $T(x) = x$ and $T(y) = y$. By using (9) we deduce that $\|x-y\|_X = 0$. In other words, $x = y$. Consequently, it leads us to the conclusion that a contraction mapping has a unique fixed point $x^* \in X$.

□

2.2. **Function Spaces.**

Definition 2.6. *Let X be a function space with an inner product $<f,g>$ for $f, g \in X$. If X is a complete metric space where norm is defined by*

$$\|f\|_X = \sqrt{<f,f>}$$

then X is a Hilbert space.

The norm of the function space $L^2(\mathbb{R})$ has the following form

$$\|f\|_{L^2(\mathbb{R})} = \left(\int_{-\infty}^{\infty} |f(x)|^2 dx\right)^{1/2}$$

where $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$. Moreover, the inner product of the function $f \in L^2(\mathbb{R})$ with itself is expressed as

$$\begin{aligned} < f, f > &= \int_{-\infty}^{\infty} f(x)\overline{f(x)}dx \\ &= \int_{-\infty}^{\infty} |f(x)|^2 dx \\ &= \|f\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

As we can see, the definition of the norm can be rewritten as an inner product of the functions. Therefore, since $L^2(\mathbb{R})$ is a complete metric space where the norm is induced by the inner product, it is also a Hilbert space.

Definition 2.7. *Let function f be in $L^2(\mathbb{R})$. Then its Fourier Transform denoted by $\widehat{f}$ is defined to be*

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx \qquad for \quad \xi \in \mathbb{R}.$$

The norm for a function $f \in L^p(\mathbb{R})$ is obtained using the following definition

$$\|f\|_{L^p(\mathbb{R})} := \begin{cases} \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p}, & 0 < p < \infty, \\ \\ \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)|, & p = \infty. \end{cases}$$

Let us review some properties of norm in $L^p(\mathbb{R})$ spaces. Having the norm of some function $f \in L^p(\mathbb{R})$ to be equal to zero is the same as saying that the function is zero itself, thus

$$\|f\|_{L^p(\mathbb{R})} = 0 \Leftrightarrow f(x) = 0.$$

Now, let us take another function $g(x)$ such that

$$g(x) = \begin{cases} 0, & x \neq 7, \\ 7, & x = 7. \end{cases}$$

Taking the $L^2(\mathbb{R})$ norm of the function $g(x)$ will result in

$$\|g\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |g(x)|^2 dx \right)^{1/2} = 0,$$

which means that $g(x) = 0$ itself. However, we have defined the function $g(x)$ to be different from zero when $x = 7$. This kind of contradiction can be resolved by introducing the concept of equality almost everywhere. In the example above, we say that the given functions $f(x)$ and $g(x)$ are equal in $L^2(\mathbb{R})$ a.e. (almost everywhere) $x \in \mathbb{R}$. Moreover, the set of points where $f \neq g$ has measure zero. Therefore, the point $x = 7$ has zero measure under the "glasses" of $L^2(\mathbb{R})$. For $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$, measure corresponds to length, area, and volume respectively.

There are some useful results regarding the properties of functions in L^p spaces. Let us consider some of them in the following discussion. The following lemma is about Hölder's inequality for integrals.

Lemma 2.8. *Let $p, q > 1$ with $1/p + 1/q = 1$. Then*

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^p dx \right)^{1/p} \left(\int_a^b |g(x)|^q dx \right)^{1/q} = \|f(x)\|_{L^p(a,b)} \|g(x)\|_{L^q(a,b)}$$

with equality when $|g(x)| = c|f(x)|^{p-1}$.

Proof. We will use the following Young's inequality which states that for $1 < p,q < \infty$, and $1/p+1/q=1$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad a,b>0.$$

First, let us justify it in a few steps. Remembering that the mapping e^x is convex, we deduce that

$$\begin{aligned} ab = e^{ln(ab)} = e^{ln(a)+ln(b)} &= e^{\frac{1}{p}ln(a^p)+\frac{1}{q}ln(b^q)} \\ &\leq \frac{1}{p}e^{ln(a^p)} + \frac{1}{q}e^{ln(b^q)} \\ &= \frac{a^p}{p} + \frac{b^q}{q}. \end{aligned}$$

Now, let us continue proving Hölder's inequality. Take $a = |f(x)|$ and $b = |g(x)|$. Using Young's inequality for p,q>1, and for any $x \in \mathbb{R}$ we have

$$|f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q$$

Let us assume that $\|f(x)\|_{L^p(a,b)} = \|g(x)\|_{L^q(a,b)} = 1$. Then

$$\begin{aligned} \int_a^b |f(x)g(x)|dx &\leq \frac{1}{p}\int_a^b |f(x)|^p dx + \frac{1}{q}\int_a^b |g(x)|^q dx \\ &\leq \frac{1}{p}\|f(x)\|^p_{L^p(a,b)} + \frac{1}{q}\|g(x)\|^q_{L^q(a,b)} \\ &\leq \frac{1}{p} + \frac{1}{q} \leq 1. \end{aligned}$$

Next, let $f \in L^p, g \in L^q$. Define

$$\tilde{f}(x) := \frac{f(x)}{\|f(x)\|_{L^p}}, \quad \tilde{g}(x) := \frac{g(x)}{\|g(x)\|_{L^q}}.$$

Therefore,

$$\|\tilde{f}(x)\|_{L^p(a,b)} = \left(\int_a^b |\tilde{f}(x)|^p dx\right)^{1/p} = \frac{1}{\|f(x)\|_{L^p(a,b)}}\left(\int_a^b |f(x)|^p dx\right)^{1/p} = \frac{\|f(x)\|_{L^p(a,b)}}{\|f(x)\|_{L^p(a,b)}} = 1.$$

Using the above procedure, we get that $\|\tilde{g}(x)\|_{L^q(a,b)} = 1$. Thus,

$$\int_a^b |\tilde{f}(x)\tilde{g}(x)|dx \leq 1.$$

By substituting $\tilde{f}(x), \tilde{g}(x)$, we obtain

$$\int_a^b \left| \frac{f(x)}{\|f(x)\|_{L^p(a,b)}} \frac{g(x)}{\|g(x)\|_{L^q(a,b)}} \right| dx \leq 1.$$

Therefore, we prove that

$$\int_a^b |f(x)g(x)|dx \leq \|f(x)\|_{L^p(a,b)}\|g(x)\|_{L^q(a,b)}.$$

□

Next, let us consider Minkowski's inequality which states the following.

Lemma 2.9. *For* $1 < p < \infty$, *and* $u,v \in L^p(\mathbb{R})$

$$\|u+v\|_{L^p(\mathbb{R})} \leq \|u\|_{L^p(\mathbb{R})} + \|v\|_{L^p(\mathbb{R})}.$$

Proof. First, we will apply triangle inequality

$$\begin{aligned}\|u+v\|^p_{L^p(\mathbb{R})} &= \int_{\mathbb{R}} |u+v|^p dx = \int_{\mathbb{R}} |u+v||u+v|^{p-1} dx \\ &\leq \int_{\mathbb{R}} |u+v|^{p-1}\big(|u|+|v|\big) dx \\ &= \int_{\mathbb{R}} |u+v|^{p-1}|u| dx + \int_{\mathbb{R}} |u+v|^{p-1}|v| dx. \qquad (11)\end{aligned}$$

Now, use Hölder's inequality for each of the integrals above. As an example,

$$\begin{aligned}\int_{\mathbb{R}} |u+v|^{p-1}|u| dx &\leq \Big(\int_{\mathbb{R}} \big(|u+v|^{p-1}\big)^{p'} dx\Big)^{1/p'} \Big(\int_{\mathbb{R}} |u|^p dx\Big)^{1/p} \\ &= \Big(\int_{\mathbb{R}} |u+v|^p dx\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}} |u|^p dx\Big)^{1/p} \\ &= \|u+v\|^{p-1}_{L^p(\mathbb{R})} \|u\|_{L^p(\mathbb{R})}.\end{aligned}$$

After using the same procedure to the second integral in (11), we get

$$\|u+v\|^p_{L^p(\mathbb{R})} \leq \|u+v\|^{p-1}_{L^p(\mathbb{R})} \Big(\|u\|_{L^p(\mathbb{R})} + \|v\|_{L^p(\mathbb{R})}\Big).$$

Now, divide both sides by $\|u+v\|^{p-1}_{L^p(\mathbb{R})}$. Thus, we can conclude that

$$\|u+v\|_{L^p(\mathbb{R})} \leq \|u\|_{L^p(\mathbb{R})} + \|v\|_{L^p(\mathbb{R})}.$$

□

Next definition introduces the Lipschitz continuity.

Definition 2.10. *Let X and Y be normed spaces. A mapping $F : X \to Y$ is said to be Lipschitz continuous if there exists a real non-negative constant $C \geq 0$ such that*

$$\|F(x) - F(y)\|_Y \leqslant C\|x-y\|_X$$

for all x,$y \in X$.

Our next lemma allows to compute the $L^p(\mathbb{R})$ norm of the function using the duality principle.

Lemma 2.11.

$$\|f\|_{L^p(\mathbb{R})} = \sup_{g\in L^{p'}} \Big| \int f(x)g(x)dx \Big|$$

where $\|g\|_{L^{p'}} = 1$.

Proof. First, let us prove the right direction. Remembering that $g \in L^{p'}$ and using Hölder's inequality, we obtain that

$$\sup_{\|g\|_{L^{p'}}=1} \Big| \int f(x)g(x)dx \Big| \leq \sup_{\|g\|_{L^{p'}}=1} \Big(\|f\|_{L^p}\|g\|_{L^{p'}}\Big) = \|f\|_{L^p}.$$

Next, for the left direction we should have that

$$\sup_{\|g\|_{L^{p'}}=1} \Big| \int f(x)g(x)dx \Big| \geq \|f\|_{L^p}.$$

Begin with the fact that for any $G \in L^{p'}$ and $\|G\|_{L^{p'}} = 1$

$$\sup_{\|g\|_{L^{p'}}=1} \Big| \int f(x)g(x)dx \Big| \geq \Big| \int f(x)G(x)dx \Big|.$$

Take

$$G(x) = sgn(f(x))\Big(\frac{|f(x)|}{\|f\|_{L^p}}\Big)^{p/p'}.$$

Let us verify that $\|G\|_{L^{p'}} = 1$.

$$\|G\|_{L^{p'}} = \Big(\int_{\mathbb{R}} |G(x)|^{p'}\Big)^{1/p'} = \Big(\int sgn(f(x))|f(x)|^{p'p/p'}\Big)^{\frac{1}{p}\frac{p}{p'}} \frac{1}{\|f\|_{L^p}^{p/p'}} = \frac{\|f\|_{L^p}^{p/p'}}{\|f\|_{L^p}^{p/p'}} = 1.$$

Also recall that $p - 1 = \dfrac{p}{p'}$, then the last inequality becomes

$$\sup_{\|g\|_{L^{p'}}=1} \Big|\int f(x)g(x)dx\Big| \geq \Big|\int f(x)sgn(f(x))|f(x)|^{p-1}dx\Big| \frac{1}{\|f\|_{L^p}^{p-1}}$$
$$= \frac{\|f\|_{L^p}^{p}}{\|f\|_{L^p}^{p-1}} = \|f\|_{L^p}$$

which was needed to show.

□

The proceeding Minkowski's integral inequality states that

Lemma 2.12. *For 1 $\leq p \leq \infty$*

$$\Big(\int_{\mathbb{R}} \Big|\int_{\mathbb{R}} u(x,t)dx\Big|^p dt\Big)^{1/p} \leq \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |u(x,t)|^p dt\Big)^{1/p} dx.$$

Proof. In order to prove, we will use duality first, then interchange the order of integration using Fubini's theorem, and after, we apply the Hölder's inequality.

$$\Big(\int_{\mathbb{R}} \Big|\int_{\mathbb{R}} u(x,t)dx\Big|^p dt\Big)^{1/p} = \Big\|\int_{\mathbb{R}} u(x,\cdot)dx\Big\|_{L_t^p(\mathbb{R})}$$
$$= \sup_{\|g\|_{L_t^{p'}(\mathbb{R})}=1} \Big|\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} u(x,t)dx\Big)g(t)dt\Big| \leq \sup_{\|g\|_{L_t^{p'}(\mathbb{R})}=1} \Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} |u(x,t)|dx\Big)|g(t)|dt\Big)$$
$$= \sup_{\|g\|_{L_t^{p'}(\mathbb{R})}=1} \Big(\int_{\mathbb{R}}\int_{\mathbb{R}} |u(x,t)||g(t)|dtdx\Big)$$
$$\leq \sup_{\|g\|_{L_t^{p'}(\mathbb{R})}=1} \Big[\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} |u(x,t)|^p dt\Big)^{1/p}\Big(\int_{\mathbb{R}} |g(t)|^{p'}dt\Big)^{1/p'} dx\Big]$$
$$= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} |u(x,t)|^p dt\Big)^{1/p} dx = \int_{\mathbb{R}} \|u(x,\cdot)\|_{L_t^p(\mathbb{R})} dx.$$

□

Further discussion is about other important preliminary result called Hardy-Littlewood-Sobolev inequality. However, since its proof is a bit technical, we need to consider several lemmas and definitions first.

Definition 2.13. *Let the Riesz potential of order α be denoted as I_α. For $0 < \alpha < 1$ its definition is as following*

$$I_\alpha f(x) = c_\alpha \int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\alpha}} dy = k_\alpha * f(x)$$

where c_α is a constant.

Definition 2.14. *For a given* $f \in L^1_{loc}(\mathbb{R})$*, the Hardy-Littlewood maximal function* $\mathcal{M}f$ *associated to* f *is defined as following*

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)| dy.$$

Next lemma introduces Hardy-Littlewood inequality. It states that $L^p(\mathbb{R})$ norm of a maximal function associated to some f is bounded by $L^p(\mathbb{R})$ norm of the function f itself. For the proof please refer to [14, Theorem 2.5].

Lemma 2.15. *For* $1 < p \leq \infty$*,* $\mathcal{M}$ *is a sublinear operator such that* $\mathcal{M} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$*. Specifically, there exists* c_p *such that*

$$\|\mathcal{M}f\|_{L^p(\mathbb{R})} \leq c_p \|f\|_{L^p(\mathbb{R})},$$

for any $f \in L^p(\mathbb{R})$*.*

Lemma 2.16. *Let* $\psi \in L^1(\mathbb{R})$ *be even, positive, and non-increasing function of* $z > 0$*. Then*

$$\sup_{t>0} \left| \psi_t * f(x) \right| = \sup_{t>0} \left| \int_{\mathbb{R}} \frac{\psi(t^{-1}(x-y)}{t} f(y) dy \right| \leq \|\psi\|_{L^1(\mathbb{R})} \mathcal{M}f(x).$$

Now, having introduced necessary components, let us prove the Hardy-Littlewood-Sobolev inequality itself.

Proposition 2.17. *For*

$$0 < \alpha < 1 \quad \text{and} \quad 1 < p < q < \infty$$

with

$$\frac{1}{q} = \frac{1}{p} - \alpha$$

if $p > 1$*, then* I_α *satisfies*

$$\|I_\alpha(f)\|_q \leq c\|f\|_p$$

where c is dependent on p*,* α*, and* n*.*

Proof. First of all, let the kernel $k_\alpha(x)$ be redefined as following

$$k_\alpha(x) = \frac{c_\alpha}{|x|^{1-\alpha}} = k_\alpha^0(x) + k_\alpha^\infty(x) \tag{12}$$

where

$$k_\alpha^0(x) = \begin{cases} k_\alpha(x), & |x| \leq \varepsilon, \\ 0, & |x| > \varepsilon, \end{cases}$$

$$k_\alpha^\infty(x) = \begin{cases} 0, & |x| \leq \varepsilon, \\ k_\alpha(x), & |x| > \varepsilon. \end{cases}$$

Here, ε is some positive constant that will be obtained later. Thus, using the definitions of Riesz potential (2.13) and (12),as well as triangle inequality, we can write that

$$|I_\alpha f(x)| \leq |k_\alpha^0 * f(x)| + |k_\alpha^\infty * f(x)| = I + II.$$

$$\begin{aligned} I &= |k_\alpha^0 * f(x)| \\ &= \left| \int_{-\infty}^{\infty} k_\alpha^0(x-y) f(y) dy \right| = \left| \int_{x-\varepsilon}^{x+\varepsilon} k_\alpha(x-y) f(y) dy \right| \\ &= \left| \int_{x-\varepsilon}^{x+\varepsilon} \frac{c_\alpha}{|x-y|^{1-\alpha}} f(y) dy \right| = \left| \int_{-\infty}^{\infty} \frac{c_\alpha}{|x-y|^{1-\alpha}} \mathbb{1}_{(-\varepsilon,\varepsilon)}(x-y) f(y) dy \right| \end{aligned}$$

$$= \varepsilon^{\alpha}\left|\int_{-\infty}^{\infty} \frac{c_\alpha}{\left|\frac{x-y}{\varepsilon}\right|^{1-\alpha}} \frac{1}{\varepsilon}\mathbb{1}_{(-1,1)}\Big(\frac{x-y}{\varepsilon}\Big) f(y)dy\right| = \varepsilon^{\alpha}\left|\int_{-\infty}^{\infty} \frac{1}{\varepsilon}\phi\big(\frac{x-y}{\varepsilon}\big) f(y)dy\right|$$

where

$$\mathbb{1}_A(y) = \begin{cases} 1, & y \in A \\ 0, & otherwise \end{cases}$$

and

$$\phi(z) := \frac{c_\alpha}{|z|^{1-\alpha}}\mathbb{1}_{(-1,1)}(z).$$

So, we can see that ϕ satisfies the conditions of being even, positive, and decreasing in lemma 2.16. In addition,we need to check that $\phi \in L^1(\mathbb{R})$. Thus

$$\|\phi\|_{L^1(\mathbb{R})} = \int_{|y|<\varepsilon} \frac{1}{|y|^{1-\alpha}} = \frac{2}{\alpha}\varepsilon^{\alpha}.$$

Therefore, by lemma 2.16

$$I \le \frac{2}{\alpha}\varepsilon^{\alpha}\mathcal{M}f(x).$$

Next,

$$\begin{aligned} II &= |k_\alpha^\infty * f(x)| \\ &\approx \int_{\mathbb{R}} \frac{|f(y)|}{|x-y|^{1-\alpha}}\mathbb{1}_{|x-y|>\varepsilon}(y)dy \\ &\le \|f\|_{L^p(\mathbb{R})}\Big(\int_{|x-y|>\varepsilon|} \frac{dy}{|x-y|^{(1-\alpha)p'}}\Big)^{1/p'} \\ &= \|f\|_{L^p(\mathbb{R})}\Big[\int_{-\infty}^{x-\varepsilon} \frac{dy}{|x-y|^{(1-\alpha)p'}} + \int_{x+\varepsilon}^{\infty} \frac{dy}{|x-y|^{(1-\alpha)p'}}\Big]^{1/p'}. \end{aligned}$$

If we change variables such that $z = x - y$, then we obtain

$$\begin{aligned} II &= \|f\|_{L^p(\mathbb{R})}\Big[\int_{\infty}^{\varepsilon} \frac{-dz}{|z|^{(1-\alpha)p'}} + \int_{-\varepsilon}^{-\infty} \frac{-dz}{|z|^{(1-\alpha)p'}}\Big]^{1/p'} \\ &= \|f\|_{L^p(\mathbb{R})}\Big(\int_{|z|>\varepsilon} \frac{dz}{|z|^{(1-\alpha)p'}}\Big)^{1/p'} = 2^{1/p'}\|f\|_{L^p(\mathbb{R})}\Big(\int_{\varepsilon}^{\infty} \frac{dz}{|z|^{(1-\alpha)p'}}\Big)^{1/p'} \\ &\approx \|f\|_{L^p(\mathbb{R})}\Big(z^{-(1-\alpha)p'+1}\Big]_{\varepsilon}^{\infty}\Big)^{1/p'} = \|f\|_{L^p(\mathbb{R})}\varepsilon^{-(1-\alpha)+1/p'}. \end{aligned}$$

For the last equality to be true, we need the following restriction

$$-(1-\alpha)p' + 1 < 0$$

which is the same as saying

$$1 < (1-\alpha)\frac{p}{p-1}$$

equivalently

$$-1 < -\alpha * p$$

thus

$$\alpha < \frac{1}{p}.$$

To sum up the results from solving I and II, we can conclude that

$$\begin{aligned} |I_\alpha f(x)| &\le I + II \\ &\le \varepsilon^{\alpha}\mathcal{M}f(x) + \varepsilon^{\alpha-1/p}\|f\|_{L^p(\mathbb{R})}. \end{aligned} \tag{13}$$

If to analyze the result obtained above, we can see that two terms will have different weight depending on the choice of ε. For example,

$$\textit{if } \varepsilon \to 0, \varepsilon^{\alpha} \to 0 \text{ and } \varepsilon^{\alpha-1/p} \to \infty$$

whereas

$$\textit{if } \varepsilon \to \infty, \varepsilon^{\alpha} \to \infty \text{ and } \varepsilon^{\alpha-1/p} \to 0.$$

Therefore, we need to balance those two terms which means we need to find the most suitable ε. It will ensure two terms having the same value in the inequality obtained above. For this, let

$$\begin{aligned}\varepsilon^{\alpha}\mathcal{M}f(x) &= \varepsilon^{\alpha-1/p}\|f\|_{L^p(\mathbb{R})}\\ \mathcal{M}f(x) &= \varepsilon^{-1/p}\|f\|_{L^p(\mathbb{R})}.\end{aligned}$$

This results in ε such that

$$\varepsilon = \varepsilon(x) = \left(\frac{\mathcal{M}f(x)}{\|f\|_{L^p(\mathbb{R})}}\right)^{-p}.$$

Consequently, (13) becomes

$$|I_\alpha f(x)| \le \Big(\mathcal{M}f(x)\Big)^{-p\alpha+1}\|f\|^{p\alpha}_{L^p(\mathbb{R})}.$$

As a last step, let us take the L^q norm

$$\begin{aligned}\|I_\alpha f\|_{L^q(\mathbb{R})} &= \left(\int_{\mathbb{R}} |I_\alpha f(x)|^q dx\right)^{1/q} \le \left(\int_{\mathbb{R}} |\mathcal{M}f(x)|^{q-qp\alpha} dx\right)^{1/q}\|f\|^{p\alpha}_{L^p(\mathbb{R})}\\ &= \left(\int_{\mathbb{R}} |\mathcal{M}f(x)|^{q(1-p\alpha)} dx\right)^{[1/q(1-p\alpha)][q(1-p\alpha)/q]}\|f\|^{p\alpha}_{L^p(\mathbb{R})}\\ &= \|\mathcal{M}f\|^{1-p\alpha}_{L^{q(1-p\alpha)}(\mathbb{R})}\|f\|^{p\alpha}_{L^p(\mathbb{R})}.\end{aligned}$$

If we restrict $q(1-p\alpha) > 1$, Hardy-Littlewood theorem (2.15) can be used, so that

$$\|\mathcal{M}f\|_{L^{q(1-p\alpha)}} \le \|f\|_{L^{q(1-p\alpha)}}.$$

In addition, if we let $q(1-p\alpha) = p$, this will lead us to the following conclusion

$$\begin{aligned}\|I_\alpha f\|_{L^q(\mathbb{R})} &\le \|\mathcal{M}f\|^{1-p\alpha}_{L^{q(1-p\alpha)}(\mathbb{R})}\|f\|^{p\alpha}_{L^p(\mathbb{R})}\\ &\le \|f\|^{1-p\alpha}_{L^{q(1-p\alpha)}(\mathbb{R})}\|f\|^{p\alpha}_{L^p(\mathbb{R})} = \|f\|^{1-p\alpha}_{L^p(\mathbb{R})}\|f\|^{p\alpha}_{L^p(\mathbb{R})} = \|f\|_{L^p(\mathbb{R})}.\end{aligned}$$

It is important to note that we have imposed several restrictions as proceeded along the proof. Those are the following

(1) $q(1-p\alpha) = p$
(2) $q(1-p\alpha) > 1$
(3) $\dfrac{1}{p} > \alpha$.

From the first condition we should have

$$\frac{1}{q} = \frac{1}{p} - \alpha.$$

Then assuming that the first condition is satisfied, from the second requirement we obtain

$$q(1-p\alpha) = \frac{p}{1-p\alpha}(a - p\alpha) = p$$

which was initially chosen to be grater than 1, $p > 1$. Moreover, the third statement is the same as imposing $q > 0$. Since $1/q = 1/p - \alpha$

$$\alpha = \frac{1}{p} - \frac{1}{q},$$

and remembering that $\alpha > 0$, we can conclude

$$p < q.$$

□

Next lemma about Young's inequality allows us to establish a property of the convolution operator. Basically, it states that the norm of the convolution of two functions is bounded by the norms of those functions. The reader can refer to [14, Theorem 2.2] for the proof of the inequality.

Lemma 2.18. *For* $1 \le p, q \le \infty$ *with* $\frac{1}{p} + \frac{1}{q} \ge 1$, *let* $f \in L^p(\mathbb{R})$ *and* $g \in L^q(\mathbb{R})$. *Then* $f * g \in L^r(\mathbb{R})$ *where* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. *In addition,*

$$\|f * g\|_{L^r(\mathbb{R})} \le \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}.$$

Next lemma states the Riesz -Thorin theorem which is about interpolation of linear operator. Using the theorem we can show that if a linear operator is bounded on two different $L^p(\mathbb{R})$ spaces then it is also bounded on any $L^q(\mathbb{R})$ space in between those two. For the proof please refer to [7, Theorem 2.1].

Lemma 2.19. *Let* T *be a bounded linear operator that maps* L^{p_0} *into* L^{q_0} *with norm* $\mathcal{M}_0$ *and maps* L^{p_1} *into* L^{q_1} *with norm* $\mathcal{M}_1$ *where* $p_0 \ne p_1$, and $q_0 \ne q_1$ *such that*

$$T : L^{p_0(\mathbb{R})} \mapsto L^{q_0(\mathbb{R})}, \qquad \|Tf\|_{L^{q_0}(\mathbb{R})} \le \mathcal{M}_0 \|f\|_{L^{p_0}(\mathbb{R})},$$
$$T : L^{p_1(\mathbb{R})} \mapsto L^{q_1(\mathbb{R})}, \qquad \|Tf\|_{L^{q_1}(\mathbb{R})} \le \mathcal{M}_1 \|f\|_{L^{p_1}(\mathbb{R})}.$$

Then for any $0 < \theta < 1$, T *maps* L^{p_θ} *into* L^{q_θ} *with norm* $\mathcal{M}_\theta$ *that satisfies*

$$\mathcal{M}_\theta \le \mathcal{M}_0^{1-\theta} \mathcal{M}_1^{\theta}$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$
$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

3. LINEAR SCHRÖDINGER EQUATION

3.1. Homogeneous equation.

This section is aimed at solving the initial value problems (IVP) for the Linear Schrödinger equation (LSE). The reason for obtaining the solution form is to use the results in the main discussion of examining well-posedness of the NLSE.

Lemma 3.1. *The following homogeneous IVP for the linear Schrödinger equation*

$$\begin{cases} i\,\partial_t u(x,t) + \Delta u(x,t) = 0, & x \in \mathbb{R},\, t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R} \end{cases} \tag{14}$$

has a solution of the form

$$u(x,t) = \frac{1}{\sqrt{4\pi i t}} e^{-(x)^2/4it} * f(x). \tag{15}$$

Proof. Taking Fourier transform in x, results in the following ODE

$$\begin{cases} i\,\partial_t \widehat{u}(\xi,t) = \xi^2 \widehat{u}(\xi,t) \\ \widehat{u}(\xi,0) = \widehat{f}(\xi). \end{cases} \tag{16}$$

Fix some $\xi \in \mathbb{R}$, and let $\mathcal{V}(t) = \widehat{u}(\xi,t)$. Then (16) takes the form

$$\begin{cases} i\,\mathcal{V}'(t) = \xi^2 \mathcal{V}(t) \\ \mathcal{V}(0) = \widehat{f}(\xi). \end{cases} \tag{17}$$

As a result, we get the separable differential equation (17) which has the solution of the following form

$$\mathcal{V}(t) = e^{-i\xi^2 t}\widehat{f}(\xi).$$

Since $\mathcal{V}(t) = \widehat{u}(\xi, t)$, we use the Inverse Fourier Transform Theorem in order to obtain $u(x, t)$. Therefore, the solution $u(x, t)$ is expressed as following

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}e^{-i\xi^2 t}\widehat{f}(\xi)d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}e^{-i\xi^2 t}\Big[\int_{\mathbb{R}} e^{-iy\xi}f(y)dy\Big]d\xi \qquad (18)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i(y-x)\xi}e^{-i\xi^2 t}f(y)dyd\xi = \frac{1}{2\pi}\int_{\mathbb{R}}\Big[\int_{\mathbb{R}} e^{-i(y-x)\xi}e^{-i\xi^2 t}d\xi\Big]f(y)dy\\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{e^{-i\xi^2 t}}(y-x)f(y)dy. \qquad (19)
\end{aligned}$$

In order to continue, let us find the Fourier transform of the complex Gaussian function $e^{-i\xi^2 t}(z)$. For a fixed t let

$$f(\xi) = e^{-i\xi^2 t}$$

. We need to find $\widehat{f}(z)$. First, solve for derivative of $f(\xi)$

$$\frac{df}{d\xi} = -2i\xi t e^{-i\xi^2 t} = -2it\xi f(\xi).$$

Take the Fourier transform of both sides

$$\widehat{\frac{df}{d\xi}}(z) = -2it\widehat{\xi f}(z).$$

According to [8, Theorem 7.5], we can rewrite

$$iz\widehat{f}(z) = -2iti\frac{d\widehat{f}}{dz}.$$

Let us introduce notation $g(z) := \widehat{f}(z)$. Then we will get a separable differential equation

$$izg(z) = 2t\frac{dg}{dz}.$$

After solving the above equation we obtain the solution of the form

$$\widehat{f}(z) = g(z) = ke^{\frac{-z^2}{4it}}.$$

Now we need to find constant k which is the value of the Fourier transform of the function f at $z = 0$, that is

$$\begin{aligned}
k = \widehat{f}(0) &= \int_{\mathbb{R}} e^{-i\xi 0}f(\xi)d\xi = \int_{\mathbb{R}} f(\xi)d\xi = \int_{\mathbb{R}} e^{-i\xi^2 t}d\xi\\
&= \int_{\mathbb{R}}\big[\cos(\xi^2 t) - i\sin(\xi^2 t)\big]d\xi = 2\int_0^\infty \cos(\xi^2 t)d\xi - 2i\int_0^\infty \sin(\xi^2 t)d\xi.
\end{aligned}$$

In order to solve the integrals, let us make change of variables such that

$$u = \xi\sqrt{t} \qquad du = \sqrt{t}d\xi.$$

Thus, continuing the process of finding k, we rewrite the integrals above

$$k = \widehat{f}(0) = \frac{1}{\sqrt{t}}\Big[2\int_0^\infty \cos(u^2)du - 2i\int_0^\infty \sin(u^2)du\Big].$$

According to [5, Exercise 12, p. 276], we will use the values of the following Fresnel integrals

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Therefore,

$$k=\widehat{f}(0)=\frac{1}{\sqrt{t}}\big[2\sqrt{\frac{\pi}{8}}-2i\sqrt{\frac{\pi}{8}}\big]=\sqrt{\frac{\pi}{t}}\Big[\frac{1-i}{\sqrt{2}}\Big].$$

Let us rewrite the result obtained above such that

$$\frac{1-i}{\sqrt{2}}=\frac{2}{\sqrt{2}}\frac{1-i}{1^2+1^2}=\frac{2}{\sqrt{2}}\frac{1-i}{1-i}\frac{1}{1+i}=\frac{2}{\sqrt{2}}\frac{1}{1+i}.$$

Moreover,

$$\sqrt{i}=\sqrt{e^{i\pi/2}}=e^{\frac{\pi}{4}i}=\cos(\frac{\pi}{4})+i\sin(\frac{\pi}{4})=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}(1+i).$$

Thus, combining the computations, we can conclude that

$$\frac{1-i}{\sqrt{2}}=\frac{1}{\sqrt{i}}.$$

Returning to the coefficient k, we get

$$k=\sqrt{\frac{\pi}{it}}.$$

Finally, the Fourier transform of the complex Gaussian function is determined to be

$$\widehat{f}(z)=\widehat{e^{-i\xi^2t}}=\sqrt{\frac{\pi}{it}}e^{\frac{-z^2}{4it}}.$$

Consequently, the equality (19) can be continued in the following way

$$\begin{aligned}u(x,t)&=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-(y-x)^2/4it}\sqrt{\frac{\pi}{it}}f(y)dy=\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi it}}e^{-(x-y)^2/4it}f(y)dy\\&=\frac{1}{\sqrt{4\pi it}}e^{-(x)^2/4it}*f(x).\end{aligned}$$

□

Here we introduce the following notation for the solution of the problem (14)

$$e^{it\Delta}f(x):=\frac{1}{\sqrt{4\pi it}}e^{-|\cdot|^2/4it}*f(x)$$

where $e^{it\Delta}$ acts as an operator on a given function.

3.2. **Non-homogeneous equation.**

Now,we will consider inhomogeneous linear Schrödinger equation. Specifically, we will obtain the solution form of the IVP for the LSE in order to use it in further discussion. The primary method to be applied is called "Duhamel's Principle". The key idea underlying the method is to find a solution form of the equation with help of solutions of homogeneous LSE.

Lemma 3.2. *The following inhomogeneous IVP for linear Schrödinger equation*

$$\begin{cases} i\,\partial_t u(x,t)+\Delta u(x,t)=F(x,t), & x\in\mathbb{R},\ t>0,\\ u(x,0)=f(x), & x\in\mathbb{R}\end{cases}\tag{20}$$

has a solution of the form

$$u(x,t)=e^{it\Delta}f(x)-i\int_0^t e^{i(t-\tau)\Delta}F(x,\tau)d\tau.$$

Proof. In order to solve the IVP, we first take the Fourier transform of the system (20) in x. Therefore,

$$\begin{cases} i\,\partial_t \widehat{u}(\xi,t) - \xi^2 \widehat{u}(\xi,t) = \widehat{F}(\xi,t) \\ \widehat{u}(\xi,0) = \widehat{f}(\xi). \end{cases} \tag{21}$$

As in the solution of the homogeneous IVP for Linear Schrödinger equation, fix some $\xi \in \mathbb{R}$, and let $\mathcal{V}(t) = \widehat{u}(\xi,t)$. Consequently, the system becomes

$$\begin{cases} i\,\mathcal{V}'(t) - \xi^2 \mathcal{V}(t) = \widehat{F}(\xi,t) \\ \mathcal{V}(0) = \widehat{f}(\xi). \end{cases} \tag{22}$$

Specifically, (22) is the IVP for the first-order non-separable linear differential equation. It is solved by introducing an integration factor μ. Thus, (22) has the following solution

$$\mathcal{V}(t) = e^{-i\xi^2 t}\widehat{f}(\xi) - i\int_0^t e^{-i\xi^2(t-\tau)}\widehat{F}(\xi,\tau)d\tau.$$

By [8, The Fourier Inversion Theorem, p. 218]

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\left[e^{-i\xi^2 t}\widehat{f}(\xi) - i\int_0^t e^{-i\xi^2(t-\tau)}\widehat{F}(\xi,\tau)d\tau\right]d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}e^{-i\xi^2 t}\widehat{f}(\xi)d\xi - \frac{i}{2\pi}\int_{\mathbb{R}}\int_0^t e^{ix\xi}e^{-i\xi^2(t-\tau)}\widehat{F}(\xi,\tau)d\tau d\xi \end{aligned}$$

where the first integral is the solution of the homogeneous IVP (14), therefore

$$\begin{aligned} u(x,t) &= e^{it\Delta}f(x) - \frac{i}{2\pi}\int_{\mathbb{R}}\int_0^t e^{ix\xi}e^{-i\xi^2(t-\tau)}\widehat{F}(\xi,\tau)d\tau d\xi \\ &= e^{it\Delta}f(x) - \frac{i}{2\pi}\int_{\mathbb{R}}\int_0^t\int_{\mathbb{R}} e^{ix\xi}e^{-i\xi^2(t-\tau)}e^{-iy\xi}F(y,\tau)dyd\tau d\xi \\ &= e^{it\Delta}f(x) - i\int_{\mathbb{R}}\int_0^t F(y,\tau)\left[\int_{\mathbb{R}}\frac{1}{2\pi}e^{i(x-y)\xi}e^{-i\xi^2(t-\tau)}d\xi\right]d\tau dy. \end{aligned}$$

Using the Inverse Fourier Transform [8, Proposition 9, p. 223] and taking $a = 1/2i(t-\tau)$

$$\begin{aligned} u(x,t) &= e^{it\Delta}f(x) - i\int_0^t\int_{\mathbb{R}} F(y,\tau)e^{-(x-y)^2/4i(t-\tau)}\frac{1}{\sqrt{4\pi i(t-\tau)}}dyd\tau \\ &= e^{it\Delta}f(x) - i\int_0^t \frac{e^{-x^2/4i(t-\tau)}}{\sqrt{4\pi i(t-\tau)}} * F(x,\tau)d\tau \\ &= e^{it\Delta}f(x) - i\int_0^t e^{i(t-\tau)\Delta}F(x,\tau)d\tau. \end{aligned}$$

□

3.3. **Useful estimates.**

The following lemma explains the regularity of solutions of the LSE in the space variable.

Lemma 3.3. *If* $t \neq 0$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, *and* $p' \in [1,2]$, *then* $e^{it\Delta}\colon L^{p'}(\mathbb{R}) \to L^p(\mathbb{R})$ *is continuous and*

$$\left\|e^{it\Delta}f\right\|_{L^p(\mathbb{R})} \leq c|t|^{-n/2(1/p'-1/p)}\|f\|_{L^{p'}(\mathbb{R})}$$

Proof. Firstly, we will show that

$$\left\|e^{it\Delta}f\right\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$$

which indicates that the initial data and the solution for the LSE initial value problem (14) have equal $L^2(\mathbb{R})$ norms. By using Plancherel theorem twice, which states that the $L^2(\mathbb{R})$ norm of a function is equal to the $L^2(\mathbb{R})$ norm of its Fourier transform, we get

$$\begin{aligned}\left\|e^{it\Delta}f\right\|_{L^2(\mathbb{R})} &= \left\|e^{-i\xi^2 t}\widehat{f}\right\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} \left|e^{-i\xi^2 t}\widehat{f}(\xi)\right|^2 d\xi\right)^{1/2} \\ &= \left(\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi\right)^{1/2} = \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^{1/2} = \|f\|_{L^2(\mathbb{R})}.\end{aligned}$$

Thus,

$$\left\|e^{it\Delta}f\right\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.$$

Moreover, using Young's inequality, we get that

$$\left\|e^{it\Delta}f\right\|_{L^\infty(\mathbb{R})} = \left\|\frac{e^{i|\cdot|^2/4t}}{\sqrt{(4\pi it)^n}} * f\right\|_{L^\infty(\mathbb{R})} \leq \left\|\frac{e^{i|\cdot|^2/4t}}{\sqrt{(4\pi it)^n}}\right\|_{L^\infty(\mathbb{R})} \|f\|_{L^1(\mathbb{R})} \leq c|t|^{-n/2}\|f\|_{L^1(\mathbb{R})}$$

since

$$\sup_{x\in\mathbb{R}}\left|\frac{e^{i|x|^2/4t}}{\sqrt{(4\pi it)^n}}\right| = \frac{c}{|t|^{n/2}}.$$

In our case, using the above information with the Riesz-Thorin theorem (Lemma 2.19), we have that

$$T : L^1(\mathbb{R}) \to L^\infty(\mathbb{R}), \quad \|Tf\|_{L^\infty(\mathbb{R})} \leq \mu_0 \|f\|_{L^1(\mathbb{R})}$$

and

$$T : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \|Tf\|_{L^2(\mathbb{R})} \leq \mu_1 \|f\|_{L^2(\mathbb{R})}$$

where $p_0 = 1, q_0 = \infty, \mu_0 = c|t|^{-n/2}$, and $p_1 = 2, q_1 = 2, \mu_1 = 1$. Therefore, for any $0 < \theta < 1$ we can obtain

$$T : L^{p_\theta}(\mathbb{R}) \to L^{q_\theta}(\mathbb{R}), \quad \|Tf\|_{L^{q_\theta}(\mathbb{R})} \leq \mu_\theta \|f\|_{L^{p_\theta}(\mathbb{R})}$$

with

$$\mu_\theta \leq \mu_0^{1-\theta}\mu_1^\theta.$$

Thus, our μ_θ can be expressed as

$$\mu_\theta \leq \left(c|t|^{-1/2}\right)^{1-\theta}.$$

In order to find $1-\theta$, remember from Riesz-Thorin theorem (Lemma 2.19) that

$$\begin{aligned}\frac{1}{p_\theta} &= \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \\ \frac{1}{q_\theta} &= \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{p_\theta} &= 1-\theta+\frac{\theta}{2} \\ \frac{1}{q_\theta} &= \frac{1-\theta}{\infty} + \frac{\theta}{2} = \frac{\theta}{2}.\end{aligned}$$

Thus,

$$1-\theta = \frac{1}{p_\theta} - \frac{1}{q_\theta}.$$

Take $p_\theta = p' \in [1,2]$. We can find q_θ.

$$\frac{1}{p_\theta} + \frac{1}{q_\theta} = 1-\theta+\frac{\theta}{2}+\frac{\theta}{2}$$

$$\frac{1}{p_\theta} + \frac{1}{q_\theta} = 1$$
$$\frac{1}{q_\theta} = 1 - \frac{1}{p_\theta} = 1 - \frac{1}{p'}$$
$$\frac{1}{q_\theta} = \frac{1}{p}.$$

As a result, we get that $q_\theta = p$ such that

$$e^{it\Delta} : L^{p'}(\mathbb{R}) \mapsto L^p(\mathbb{R}), \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Moreover,

$$\left\|e^{it\Delta} f\right\|_{L^p(\mathbb{R})} \leq (c|t|^{-1/2})^{1-\theta} \|f\|_{L^{p'}(\mathbb{R})} = c|t|^{-1/2(1/p'-1/p)} \|f\|_{L^{p'}(\mathbb{R})}$$

where

$$\frac{1}{p} = \frac{\theta}{2}, \quad 1-\theta = 1 - \frac{2}{p} = \frac{1}{p'} - \frac{1}{p}, \quad \theta \in [0,1].$$

□

Next subject of our discussion is several significant inequalities concerning the properties of the LSE solutions. Those inequalities will be a key-step in proving well-posedness of the NLSE (Theorem 4.1).

Note that in the following proposition 3.4, $c = c(p,n)$ is a constant that depends on p and n, and in our case $n = 1$. In addition, the constants p, p', q and q' must satisfy

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1.$$

Proposition 3.4. *For*

$$2 \leq p \leq \infty \quad \text{and} \quad \frac{2}{q} = \frac{1}{2} - \frac{1}{p}$$

the family of operators $\left\{e^{it\Delta} f\right\}_{t=-\infty}^{\infty}$ *satisfies*

$$\left(\int_{-\infty}^{\infty} \left\|e^{it\Delta} f\right\|_p^q dt\right)^{1/q} \leq c\|f\|_2, \tag{23}$$

$$\left\|\int_{-\infty}^{\infty} e^{it\Delta} g(\cdot,t)dt\right\|_2 \leq c\left(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{p'}^{q'} dt\right)^{1/q'}, \tag{24}$$

$$\left\|\int_0^t e^{i(t-\tau)\Delta}\big(g(\cdot,\tau)\big)(x)d\tau\right\|_{L^2(\mathbb{R})} \leq c\left(\int_0^t \|g(\cdot,\tau)\|_{L^{p'}(\mathbb{R})}^{q'} d\tau\right)^{1/q'}, \tag{25}$$

$$\left(\int_{-\infty}^{\infty} \left\|\int_{-\infty}^{\infty} e^{i(t-\tau)\Delta} g(\cdot,\tau)d\tau\right\|_p^q dt\right)^{1/q} \leq c\left(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{p'}^{q'} dt\right)^{1/q'}, \tag{26}$$

$$\left(\int_{-\infty}^{\infty} \left\|\int_0^t e^{i(t-\tau)\Delta} g(\cdot,\tau)d\tau\right\|_p^q dt\right)^{1/q} \leq c\left(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{p'}^{q'} dt\right)^{1/q'}. \tag{27}$$

Proof of the Proposition 3.4 is quite technical, therefore it will be divided into several steps in order to make it more understandable for a reader. First we will show that the inequality (26) holds. Then the equivalence relation between (23) and (24) will be proved. After that we determine that (24) and (26) are also equivalent. Thus, it will give us justification for both (23) and (24). Moreover, we explicitly prove the inequalities (24) and (25) to be true.

Proof of (26). Remembering that $p \geq 2$, we are allowed to use the Minkowski's inequality

$$\left\| \int_{-\infty}^{\infty} e^{i(t-t')\Delta} g(\cdot, t')dt' \right\|_p \leq \int_{-\infty}^{\infty} \| e^{i(t-t')\Delta} g(\cdot, t')\|_p dt'.$$

According to the relation $p' = \dfrac{p}{p-1}$, and since $p \geq 2$, the values for p' will be in the interval $[1, 2]$. Therefore, we can apply Lemma 4.1 for the next step

$$\int_{-\infty}^{\infty} \| e^{i(t-t')\Delta} g(\cdot, t')\|_p dt' \leq c \int_{-\infty}^{\infty} \frac{1}{|t-t'|^{\alpha}} \|g(\cdot, t')\|_{p'} dt'$$

where $\alpha = \frac{1}{2}(\frac{1}{p'} - \frac{1}{p})$. Then by using Hardy-Littlewood-Sobolev theorem we get

$$\begin{aligned} \left(\int_{-\infty}^{\infty} \left\| \int_{-\infty}^{\infty} e^{i(t-t')\Delta} g(\cdot, t')dt' \right\|_p^q dt \right)^{1/q} &\leq c \left\| \int_{-\infty}^{\infty} \frac{1}{|t-t'|^{\alpha}} \|g(\cdot, t')\|_{p'} dt' \right\|_q \\ &\leq c \left(\int_{-\infty}^{\infty} \|g(\cdot, t)\|_{p'}^{q'} dt \right)^{1/q'} \end{aligned}$$

with $\frac{1}{q'} = \frac{1}{q} + (1-\alpha)$. The latter results in the following

$$\begin{aligned} \frac{1}{q'} &= \frac{1}{q} + (1-\alpha) \\ 1 - \frac{1}{q} &= \frac{1}{q} + 1 - \frac{1}{2}(\frac{1}{p'} - \frac{1}{p}) \\ \frac{2}{q} &= \frac{1}{2}(\frac{1}{p'} - \frac{1}{p}) \\ \frac{2}{q} &= \frac{1}{2}(1 - \frac{1}{p} - \frac{1}{p}) \\ \frac{2}{q} &= \frac{1}{2} - \frac{1}{p}. \end{aligned}$$

□

Proof of equivalence relation between (23) *and* (24). Firstly, let us prove the right direction, so by assuming the inequality (23) we should be able to show that the inequality (24) holds. By using duality we obtain that

$$\left\| \int_{-\infty}^{\infty} e^{it\Delta} g(\cdot, t)dt \right\|_2 = \sup_{\substack{f \in L^2 \\ \|f\|_{L^2(\mathbb{R})}=1}} \left| \int_{\mathbb{R}} \left[\int_{-\infty}^{\infty} e^{it\Delta} g(\cdot, t)dt \right] f(x)dx \right|. \tag{28}$$

Using Fubini's theorem

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{\mathbb{R}} e^{it\Delta} f(x) g(x,t)dxdt &= \int_{-\infty}^{\infty} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi it}} e^{-(x-y)^2/4it} f(y)dy \right] g(x,t)dxdt \quad (29) \\ &= \int_{\mathbb{R}} f(y) \int_{-\infty}^{\infty} \left[\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi it}} e^{-(x-y)^2/4it} g(x,t)dx \right] dtdy \\ &= \int_{\mathbb{R}} f(y) \int_{-\infty}^{\infty} e^{it\Delta} g(\cdot, t)(y)dtdy \\ &= \int_{\mathbb{R}} \left[\int_{-\infty}^{\infty} e^{it\Delta} g(\cdot, t)(x)dt \right] f(x)dx. \quad (30) \end{aligned}$$

Therefore, (28) can be rewritten as

$$\Big\| \int_{-\infty}^{\infty} e^{it\Delta} g(\cdot,t)dt \Big\|_2 = \sup_{\substack{f\in L^2 \\ \|f\|_{L^2(\mathbb{R})}=1}} \left| \int_{-\infty}^{\infty} \int_{\mathbb{R}} e^{it\Delta} f(x) g(x,t) dx dt \right|.$$

Then by using Hölder's inequality twice we get that

$$\begin{aligned}
\Big\| \int_{-\infty}^{\infty} e^{it\Delta} g(\cdot,t)dt \Big\|_2 &\le \sup_{\substack{f\in L^2 \\ \|f\|_{L^2(\mathbb{R})}=1}} \left| \int_{-\infty}^{\infty} \left[\int_{\mathbb{R}} \left(e^{it\Delta} f(x)\right)^p dx \right]^{1/p} \left[\int_{\mathbb{R}} \left(g(x,t)\right)^{p'} dx \right]^{1/p'} dt \right| \\
&= \sup_{\substack{f\in L^2 \\ \|f\|_{L^2(\mathbb{R})}=1}} \left| \int_{-\infty}^{\infty} \|e^{it\Delta} f\|_{L^p(\mathbb{R})} \|g(\cdot,t)\|_{L^{p'}(\mathbb{R})} dt \right| \\
&\le \sup_{\substack{f\in L^2 \\ \|f\|_{L^2(\mathbb{R})}=1}} \left| \left[\int_{-\infty}^{\infty} \left(\|e^{it\Delta} f\|_{L^p(\mathbb{R})} \right)^q dt \right]^{1/q} \left[\int_{-\infty}^{\infty} \left(\|g(\cdot,t)\|_{L^{p'}(\mathbb{R})} \right)^{q'} dt \right]^{1/q'} \right| \\
&= \sup_{\substack{f\in L^2 \\ \|f\|_{L^2(\mathbb{R})}=1}} \left(\|e^{it\Delta} f\|_{L^q(\mathbb{R};L^p(\mathbb{R}))} \|g\|_{L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}))} \right). \qquad (31)
\end{aligned}$$

By applying the inequality (23) to the (31) we obtain that

$$\begin{aligned}
\Big\| \int_{-\infty}^{\infty} e^{it\Delta} g(\cdot,t)dt \Big\|_2 &\le \sup_{\substack{f\in L^2 \\ \|f\|_{L^2(\mathbb{R})}=1}} \left(c\|f\|_{L^2(\mathbb{R})} \|g\|_{L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}))} \right) \\
&= c\|g\|_{L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}))}
\end{aligned}$$

since the norm of any $f(x)$ in $L^2(\mathbb{R})$ is assumed to be equal to 1. Therefore,

$$\Big\| \int_{-\infty}^{\infty} e^{it\Delta} g(\cdot,t)dt \Big\|_2 \le c \left(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{L^{p'}(\mathbb{R})}^{q'} dt \right)^{1/q'}.$$

Now, let us prove the left direction. We will assume that the inequality (24) is true. The goal is to prove (23) by implying (24). First step is to use the duality, thus

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} \|e^{it\Delta} f\|_p^q dt \right)^{1/q} &= \|e^{it\Delta} f\|_{L^q(\mathbb{R};L^p(\mathbb{R}))} \\
&= \sup_{\substack{w\in L^{q'}(\mathbb{R};L^{p'}(\mathbb{R})) \\ \|w\|_{L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}))}=1}} \left| \int_{-\infty}^{\infty} \int_{\mathbb{R}} e^{it\Delta} f(x) w(x,t) dx dt \right|.
\end{aligned}$$

Next, using the equality (29) derived from Fubini's theorem, as well as using Hölder's inequality, we get

$$\left(\int_{-\infty}^{\infty} \|e^{it\Delta} f\|_p^q dt \right)^{1/q} = \sup_{\substack{w\in L^{q'}(\mathbb{R};L^{p'}(\mathbb{R})) \\ \|w\|_{L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}))}=1}} \left| \int_{\mathbb{R}} \left[\int_{-\infty}^{\infty} e^{it\Delta} w(\cdot,t)(x) dt \right] f(x) dx \right| \qquad (32)$$

$$\le \sup_{\substack{w\in L^{q'}(\mathbb{R};L^{p'}(\mathbb{R})) \\ \|w\|_{L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}))}=1}} \left| \left[\int_{\mathbb{R}} \left(\int_{-\infty}^{\infty} e^{it\Delta} w(\cdot,t)(x) dt \right)^{m'} dx \right]^{1/m'} \left[\int_{\mathbb{R}} f(x)^m dx \right]^{1/m} \right| \qquad (33)$$

$$= \sup_{\substack{w\in L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}))\\ \|w\|_{L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}))}=1}} \left(\left\| \int_{-\infty}^{\infty} e^{it\Delta} w(\cdot,t)dt \right\|_{L^{m'}(\mathbb{R})} \|f\|_{L^m(\mathbb{R})} \right). \tag{34}$$

If we let $m'=2$, we obtain inequality (24), giving the following result

$$\left(\int_{-\infty}^{\infty} \|e^{it\Delta} f\|_p^q dt \right)^{1/q} = \sup_{\substack{w\in L^{q'}(\mathbb{R};L^{2}(\mathbb{R}))\\ \|w\|_{L^{q'}(\mathbb{R};L^{2}(\mathbb{R}))}=1}} \left(\left\| \int_{-\infty}^{\infty} e^{it\Delta} w(\cdot,t)dt \right\|_{L^{2}(\mathbb{R})} \|f\|_{L^2(\mathbb{R})} \right)$$

$$\leq \sup_{\substack{w\in L^{q'}(\mathbb{R};L^{2}(\mathbb{R}))\\ \|w\|_{L^{q'}(\mathbb{R};L^{2}(\mathbb{R}))}=1}} \left(c\|w\|_{L^{q'}(\mathbb{R};L^2(\mathbb{R}))} \|f\|_{L^2(\mathbb{R})} \right).$$

Since $\|w\|_{L^{q'}(\mathbb{R};L^2(\mathbb{R}))} = 1$

$$\left(\int_{-\infty}^{\infty} \|e^{it\Delta} f\|_p^q dt \right)^{1/q} \leq c\|f\|_{L^2(\mathbb{R})}.$$

The following equivalence by P.Thomas will be applied in further discussion.

$$\begin{aligned} &\int_{\mathbb{R}} \left(\int_{-\infty}^{\infty} e^{it\Delta}\big(w(\cdot,t)\big)(x)dt \right) \overline{\left(\int_{-\infty}^{\infty} e^{i\tau\Delta}\big(g(\cdot,\tau)\big)(x)d\tau \right)} dx \\ &= \int_{\mathbb{R}} \left(\int_{-\infty}^{\infty} e^{it\Delta}\big(w(\cdot,t)\big)(x)dt \right) \left(\int_{-\infty}^{\infty} e^{-i\tau\Delta}\big(\overline{g(\cdot,\tau)}\big)(x)d\tau \right) dx \\ &= \int_{\mathbb{R}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[e^{it\Delta}\big(w(\cdot,t)\big)(x) \right] e^{-i\tau\Delta}\big(\overline{g(\cdot,\tau)}\big)(x) dt d\tau dx \\ &= \int_{\mathbb{R}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi i t}} e^{-(x-y)^2/4it} w(y,t)dy \right] e^{-i\tau\Delta}\big(\overline{g(\cdot,\tau)}\big)(x) dt d\tau dx \\ &= \int_{\mathbb{R}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(y,t) \left[\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi i t}} e^{-(x-y)^2/4it} e^{-i\tau\Delta}\big(\overline{g(\cdot,\tau)}\big)(x) dx \right] dt d\tau dy \\ &= \int_{\mathbb{R}} \int_{-\infty}^{\infty} w(x,t) \int_{-\infty}^{\infty} e^{it\Delta}\big[e^{-i\tau\Delta}\big(\overline{g(\cdot,\tau)}\big)\big](x) d\tau dt dx \\ &= \int_{\mathbb{R}} \int_{-\infty}^{\infty} w(x,t) \left(\int_{-\infty}^{\infty} e^{i(t-\tau)\Delta}\big(\overline{g(\cdot,\tau)}\big)(x) d\tau \right) dt dx. \end{aligned} \tag{35}$$

Moreover, if we take $w(x,t)=g(x,t)$ then the following equivalence will be also true

(36)

$$\begin{aligned} \left\| \int_{-\infty}^{\infty} e^{it\Delta}\big(g(\cdot,t)\big)(x)dt \right\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} \left| \int_{-\infty}^{\infty} e^{it\Delta}\big(g(\cdot,t)\big)(x)dt \right|^2 dx \\ &= \int_{\mathbb{R}} \left(\int_{-\infty}^{\infty} e^{it\Delta}\big(g(\cdot,t)\big)(x)dt \right) \overline{\left(\int_{-\infty}^{\infty} e^{i\tau\Delta}\big(g(\cdot,\tau)\big)(x)d\tau \right)} dx \\ &= \int_{\mathbb{R}} \int_{-\infty}^{\infty} g(x,t) \left(\int_{-\infty}^{\infty} e^{i(t-\tau)\Delta}\big(\overline{g(\cdot,\tau)}\big)(x) d\tau \right) dt dx. \end{aligned}$$

Let us clarify the following equality

$$e^{it\Delta}\big[e^{-i\tau\Delta}h\big] = e^{i(t-\tau)\Delta}h$$

using the remark that

$$e^{it\Delta}f(x) = (e^{-i\xi^2 t}\widehat{f})^{\vee}(x).$$

Thus,

$$\begin{aligned} \widehat{e^{it\Delta}\big[e^{-i\tau\Delta}h\big]}(\xi) &= e^{-i\xi^2 t}\widehat{e^{-i\tau\Delta}h} = e^{-i\xi^2 t}e^{i\xi^2\tau}\widehat{h}(\xi) \\ &= e^{-i\xi^2(t-\tau)}\widehat{h}(\xi) = \widehat{e^{i(t-\tau)\Delta}h}(\xi). \end{aligned}$$

□

Proof of equivalence relation between (24) *and* (26). To start with, assuming that the inequality (24) holds, we need to prove (26). Rewriting

$$\left(\int_{-\infty}^{\infty}\left\|\int_{-\infty}^{\infty}e^{i(t-\tau)\Delta}\big(g(\cdot,\tau)\big)(x)d\tau\right\|_p^q dt\right)^{1/q} = \left\|\int_{-\infty}^{\infty}e^{i(t-\tau)\Delta}\big(g(\cdot,\tau)\big)(x)d\tau\right\|_{L^q(\mathbb{R};L^p(\mathbb{R}))}.$$

Using duality, the latter can be expressed as following

$$\begin{aligned} &\left\|\int_{-\infty}^{\infty}e^{i(t-\tau)\Delta}\big(g(\cdot,\tau)\big)(x)d\tau\right\|_{L^q(\mathbb{R};L^p(\mathbb{R}))} \\ &= \sup_{\substack{w\in L^{q'}(\mathbb{R};L^{p'}(\mathbb{R})) \\ \|w\|_{L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}))}=1}}\left|\int_{\mathbb{R}}\int_{-\infty}^{\infty}w(x,t)\left(\int_{-\infty}^{\infty}e^{i(t-\tau)\Delta}\big(g(\cdot,\tau)\big)(x)d\tau\right)dtdx\right|. \end{aligned}$$

Next, applying the equivalence relation by P.Thomas (35), we get that

$$\begin{aligned} &\left\|\int_{-\infty}^{\infty}e^{i(t-\tau)\Delta}\big(g(\cdot,\tau)\big)(x)d\tau\right\|_{L^q(\mathbb{R};L^p(\mathbb{R}))} \\ &= \sup_{\substack{w\in L^{q'}(\mathbb{R};L^{p'}(\mathbb{R})) \\ \|w\|_{L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}))}=1}}\left|\int_{\mathbb{R}}\left(\int_{-\infty}^{\infty}e^{it\Delta}\big(w(\cdot,t)\big)(x)dt\right)\overline{\left(\int_{-\infty}^{\infty}e^{i\tau\Delta}\big(\overline{(g(\cdot,\tau)}\big)(x)d\tau\right)}dx\right|. \end{aligned}$$

Furthermore, we apply Hölder's inequality to obtain the following

$$\begin{aligned} &\sup_{\substack{w\in L^{q'}(\mathbb{R};L^{p'}(\mathbb{R})) \\ \|w\|_{L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}))}=1}}\left|\int_{\mathbb{R}}\left(\int_{-\infty}^{\infty}e^{it\Delta}\big(w(\cdot,t)\big)(x)dt\right)\overline{\left(\int_{-\infty}^{\infty}e^{i\tau\Delta}\big(\overline{(g(\cdot,\tau)}\big)(x)d\tau\right)}dx\right| \\ &\le \sup_{\substack{w\in L^{q'}(\mathbb{R};L^{p'}(\mathbb{R})) \\ \|w\|_{L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}))}=1}}\left\|\int_{-\infty}^{\infty}e^{it\Delta}\big(w(\cdot,t)\big)(x)dt\right\|_{L^m(\mathbb{R})}\left\|\int_{-\infty}^{\infty}e^{i\tau\Delta}\big(\overline{(g(\cdot,\tau)}\big)(x)d\tau\right\|_{L^{m'}(\mathbb{R})}. \end{aligned}$$

If you take $m = 2$, and thus $m' = 2$, then (24) can be applied, therefore,

$$\begin{aligned} &\sup_{\substack{w\in L^{q'}(\mathbb{R};L^{p'}(\mathbb{R})) \\ \|w\|_{L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}))}=1}}\left\|\int_{-\infty}^{\infty}e^{it\Delta}\big(w(\cdot,t)\big)(x)dt\right\|_{L^2(\mathbb{R})}\left\|\int_{-\infty}^{\infty}e^{i\tau\Delta}\big(\overline{(g(\cdot,\tau)}\big)(x)d\tau\right\|_{L^2(\mathbb{R})} \\ &\le \sup_{\substack{w\in L^{q'}(\mathbb{R};L^{p'}(\mathbb{R})) \\ \|w\|_{L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}))}=1}}\left[c\Big(\int_{-\infty}^{\infty}\|w(\cdot,t)\|_{L^{p'}(\mathbb{R})}^{q'}dt\Big)^{1/q'}c\Big(\int_{-\infty}^{\infty}\|g(\cdot,t)\|_{L^{p'}(\mathbb{R})}^{q'}dt\Big)^{1/q'}\right] \end{aligned}$$

$$= c\Big(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{L^{p'}(\mathbb{R})}^{q'} dt\Big)^{1/q'}$$

since $\|w\|_{L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}))} = 1$. Thus, we have that

$$\bigg(\int_{-\infty}^{\infty}\bigg\|\int_{-\infty}^{\infty} e^{i(t-\tau)\Delta}\big(g(\cdot,\tau)\big)(x)d\tau\bigg\|_p^q dt\bigg)^{1/q} \le c\Big(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{L^{p'}(\mathbb{R})}^{q'} dt\Big)^{1/q'}.$$

Now, let us prove the other direction, showing that the inequality (24) holds if to assume (26). Recalling the expression in (36), and using the Hoḯder's inequality twice, we get

$$\|\int_{-\infty}^{\infty} e^{it\Delta}\big(g(\cdot,t)\big)(x)dt\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty}\int_{\mathbb{R}} g(x,t)\bigg(\int_{-\infty}^{\infty} e^{i(t-\tau)\Delta}\big(\overline{g(\cdot,\tau)}\big)(x)d\tau\bigg)dxdt$$

$$= \int_{-\infty}^{\infty}\Big(\int_{\mathbb{R}} g(x,t)^{p'}dx\Big)^{1/p'}\bigg(\int_{\mathbb{R}}\Big[\int_{-\infty}^{\infty} e^{i(t-\tau)\Delta}\big(\overline{g(\cdot,\tau)}\big)(x)d\tau\Big]^p dx\bigg)^{1/p} dt$$

$$= \Big(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{L^{p'}(\mathbb{R})}^{q'} dt\Big)^{1/q'}\bigg(\int_{-\infty}^{\infty}\bigg\|\int_{-\infty}^{\infty} e^{i(t-\tau)\Delta}\big(\overline{g(\cdot,\tau)}\big)(x)d\tau\bigg\|_{L^p(\mathbb{R})}^q dt\bigg)^{1/q}. \tag{37}$$

For the second integral in (37) we apply the inequality (26) such that

$$\Big(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{L^{p'}(\mathbb{R})}^{q'} dt\Big)^{1/q'}\bigg(\int_{-\infty}^{\infty}\bigg\|\int_{-\infty}^{\infty} e^{i(t-\tau)\Delta}\big(\overline{g(\cdot,\tau)}\big)(x)d\tau\bigg\|_{L^p(\mathbb{R})}^q dt\bigg)^{1/q}$$

$$\le \Big(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{L^{p'}(\mathbb{R})}^{q'} dt\Big)^{1/q'} c\Big(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{L^{p'}(\mathbb{R})}^{q'} dt\Big)^{1/q'}$$

$$= c\Big(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{L^{p'}(\mathbb{R})}^{q'} dt\Big)^{2/q'}.$$

Thus, the result is

$$\|\int_{-\infty}^{\infty} e^{it\Delta}\big(g(\cdot,t)\big)(x)dt\|_{L^2(\mathbb{R})}^2 \le c\Big(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{L^{p'}(\mathbb{R})}^{q'} dt\Big)^{2/q'}$$

which is the same as saying

$$\|\int_{-\infty}^{\infty} e^{it\Delta}\big(g(\cdot,t)\big)(x)dt\|_{L^2(\mathbb{R})} \le c\Big(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{L^{p'}(\mathbb{R})}^{q'} dt\Big)^{1/q'}.$$

□

Proof of (25). For fixed $t > 0$, let

$$\bigg\|\int_0^t e^{i(t-\tau)\Delta}\big(g(\cdot,\tau)\big)(x)d\tau\bigg\|_{L^2(\mathbb{R})} = \bigg\|\int_{-\infty}^{\infty} e^{i(t-\tau)\Delta}\big(g(\cdot,\tau)\big)(x)\mathbb{1}_{(0,t)}(\tau)d\tau\bigg\|_{L^2(\mathbb{R})}.$$

After changing variables such that $s = t - \tau$, and $ds = -d\tau$, we get

$$\bigg\|\int_{-\infty}^{\infty} e^{is\Delta}\big(g(\cdot,t-s)\big)(x)\mathbb{1}_{(0,t)}(t-s)ds\bigg\|_{L^2(\mathbb{R})} = \bigg\|\int_{-\infty}^{\infty} e^{is\Delta}\big(g(\cdot,t-s)\mathbb{1}_{(0,t)}(t-s)\big)(x)ds\bigg\|_{L^2(\mathbb{R})}.$$

Next, using the inequality (24), and remembering the change of variables, the following can be obtained:

$$\bigg\|\int_{-\infty}^{\infty} e^{is\Delta}\big(g(\cdot,t-s)\mathbb{1}_{(0,t)}(t-s)\big)(x)ds\bigg\|_{L^2(\mathbb{R})} \le \Big(\int_{-\infty}^{\infty} \|g(\cdot,t-s)\mathbb{1}_{(0,t)}(t-s)\|_{L^{p'}(\mathbb{R})}^{q'} ds\Big)^{1/q'}$$

$$= \left(\int_{-\infty}^{\infty} \|g(\cdot, t-s)\|^{q'}_{L^{p'}(\mathbb{R})} \mathbb{1}_{(0,t)}(t-s) ds \right)^{1/q'} = \left(\int_{-\infty}^{\infty} \|g(\cdot, \tau)\|^{q'}_{L^{p'}(\mathbb{R})} \mathbb{1}_{(0,t)}(\tau) d\tau \right)^{1/q'}$$

$$= \left(\int_0^t \|g(\cdot, \tau)\|^{q'}_{L^{p'}(\mathbb{R})} d\tau \right)^{1/q'}.$$

□

For the proof of the inequality (27), we will use the following lemma. You can refer to [14, Lemma 4.2] for its proof.

Lemma 3.5. *Let*

$$Tf(x,t) := \int_{\mathbb{R}} K(x,y;t,\tau) f(y,\tau) d\tau$$

where T is a bounded mapping from $L^{q'}\big((\mathbb{R}); L^{p'}(\mathbb{R})\big)$ into $L^q\big((\mathbb{R}); L^p(\mathbb{R})\big)$ such that

$$\|Tf\|_{L^q\left((\mathbb{R});L^p(\mathbb{R})\right)} \le c\|f\|_{L^{q'}\left((\mathbb{R});L^{p'}(\mathbb{R})\right)}$$

where $1 < q' < q < \infty$. Then

$$\tilde{T}f(x,t) = \int_0^t K(x,y;t,\tau) f(y,\tau) d\tau$$

is also a mapping from $L^{q'}\big((\mathbb{R}); L^{p'}(\mathbb{R})\big)$ into $L^q\big((\mathbb{R}); L^p(\mathbb{R})\big)$.

Proof of (27). Let

$$K(x,y;t,\tau) = \frac{e^{-(x-y)^2/4i(t-\tau)}}{\sqrt{4\pi i(t-\tau)}}.$$

Using lemma (3.5) we define

$$\begin{aligned} Tg(x,t) &= \int_{\mathbb{R}} \frac{e^{-(x-y)^2/4i(t-\tau)}}{\sqrt{4\pi i(t-\tau)}} g(y,\tau) d\tau \\ &= \int_{\mathbb{R}} e^{i(t-\tau)\Delta} g(y,\tau) d\tau \end{aligned}$$

Then according to (26), T is a bounded mapping from $L^{q'}((\mathbb{R}); L^{p'}(\mathbb{R}))$ into $L^q((\mathbb{R}); L^p(\mathbb{R}))$ such that

$$\left(\int_{-\infty}^{\infty} \left\| \int_{-\infty}^{\infty} e^{i(t-\tau)\Delta} g(\cdot,\tau) d\tau \right\|_p^q dt \right)^{1/q} \le c \left(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{p'}^{q'} dt \right)^{1/q'}$$

Therefore, by Lemma 3.5,

$$\begin{aligned} \tilde{T}g(x,t) &= \int_0^t \frac{e^{-(x-y)^2/4i(t-\tau)}}{\sqrt{4\pi i(t-\tau)}} g(y,\tau) d\tau \\ &= \int_0^t e^{i(t-\tau)\Delta} g(y,\tau) d\tau \end{aligned}$$

is also a mapping from $L^{q'}((\mathbb{R}); L^{p'}(\mathbb{R}))$ into $L^q((\mathbb{R}); L^p(\mathbb{R}))$ such that

$$\left(\int_{-\infty}^{\infty} \left\| \int_0^t e^{i(t-\tau)\Delta} g(\cdot,\tau) d\tau \right\|_p^q dt \right)^{1/q} \le c \left(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{p'}^{q'} dt \right)^{1/q'}$$

which is what we wanted to show. □

4. NONLINEAR SCHRÖDINGER EQUATION

4.1. **Local well-posedness in $L^2(\mathbb{R})$.**

In further discussion, let $(X, \|\cdot\|_X)$ be a normed function space of initial data f. Let $\mathbb{F}$ be a continuous map of X into Y where $(Y, \|\cdot\|_Y)$ is a normed function space of solutions u.

The concept of well-posedness of an equation consists of three fundamental questions which are the existence, uniqueness, and stability of a solution u. The latter can be understood in terms of necessary continuity of the the data-solution mapping $\mathbb{F}$. Precisely, $\mathbb{F}: X \to Y$ is continuous at f if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|f(x) - g(x)\|_X < \delta \Rightarrow \|\mathbb{F}(f) - \mathbb{F}(g)\|_Y < \varepsilon.$$

In this project, we are studying the well-posedness of the following initial value problem for the Nonlinear Schrödinger equation (NLSE) in the function space $L^2(\mathbb{R})$

$$\begin{cases} i\,\partial_t u(x,t) + \Delta u(x,t) + \lambda\,|u(x,t)|^2\,u(x,t) = 0, & x \in \mathbb{R},\, t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases} \tag{38}$$

where $\lambda \in \mathbb{R}$ is a fixed constant, and Δ is the Laplacian with respect to x (see [14, §5]).

Also consider the following integral equation

$$u(x,t) = e^{it\Delta} f(x) + i\lambda \int_0^t e^{i(t-\tau)\Delta}(|u|^2 u)(x) d\tau. \tag{39}$$

As it was mentioned before, we are going to interchange the NLSE (38) with the integral equation (39). This step will give us an opportunity to convert the problem into a fixed-point one. Consequently, we will be able to use the Banach fixed-point theorem to prove well-posedness of (39). Then according to [17, Definition 3.4, p. 125], the initial value problem for NLSE is well-posed if there exists a unique solution to the integral equation. Let us present the formal definition.

Definition 4.1. *The problem* (38) *is said to be locally well-posed in the function space $L^2(\mathbb{R})$ if for every initial data $f \in L^2(\mathbb{R})$ there exists $T > 0$ and an open ball B in $L^2(\mathbb{R})$ containing f, and a subset Y of $C([0,T); L^2(\mathbb{R}))$, such that for each $f \in B$ there exist a unique solution $u \in Y$ to the integral equation* (39)*, and furthermore the map $f \mapsto u$ is continuous from B to Y.*

Therefore according to the Defintion 4.1, well-posedness of the integral equation (39) guarantees well-posedness of the NLSE (38).

Let us clarify that for a function $u(x,t)$ to be in $C([0,T); X)$ means that for some fixed t, the solution will belong to the same space as initial data, that is, $u(\cdot, t) \in X$, and if to fix some x, then the solution will be continuous in time, $u(x, \cdot) \in C([0,T))$.

Now, let us proceed to the major theorem of the paper. The theorem below states the well-posedness of the integral equation (39) in $L^2(\mathbb{R})$ function space.

Theorem 4.2. *For each $f \in L^2(\mathbb{R})$ there exists $T = T(\|f\|, \lambda) > 0$ and the integral equation has the unique solution u in the time interval $[0,T]$ such that*

$$u \in C([0,T); L^2(\mathbb{R})) \cap L^8([0,T); L^4(\mathbb{R})).$$

Also, for all $T' < T$ there is a neighborhood V of f in $L^2(\mathbb{R})$ such that

$$\begin{array}{rcl} F: V & \longrightarrow & C([0,T'); L^2(\mathbb{R})) \cap L^8([0,T'); L^4(\mathbb{R})) \\ \tilde{f} & \mapsto & \tilde{u}(t), \end{array}$$

is Lipschitz.

Let us briefly explain the key ideas of the proof. As it was mentioned before, we can consider well-posedness of the integral equation. Our strategy is to use Banach fixed-point theorem. In order to satisfy all the conditions, we need to have a complete metric space X and a contraction mapping Φ.

For further discussion, for fixed $T > 0$, we define the following norm

$$|||u|||_X := \sup_{0\le t\le T} \|u(\cdot,t)\|_{L^2(\mathbb{R})} + \left(\int_0^T \|u(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8},$$

and the space X of solutions, to be

$$X := \{u \in C([0,T); L^2(\mathbb{R})) \cap L^8([0,T); L^4(\mathbb{R})) \text{ s.t. } |||u|||_X \le a\},$$

for some fixed positive constant a. It can be checked that X, defined as above, is a complete metric space.

In order to prove that the integral equation has a unique solution we need to show that the mapping Φ is a contraction mapping on X. Then by the theorem 2.5, there should exist a unique solution $u \in X$ such that $\Phi(u) = u$ or

$$\Phi_f(u)(x,t) = \Phi(u)(x,t) = e^{it\Delta}f(x) + \lambda i\int_0^t e^{i(t-\tau)\Delta}(|u|^2u)(x,\tau)d\tau. \tag{40}$$

Proof of Theorem 4.2. Since the proof is quite technical, we decompose it in a few steps:

(1) Φ is well defined on X
(2) Φ is a contraction mapping
(3) Stability of the solutions

Also, since there are two components in the norm defined above, we will divide it into "integral" and "supremum" parts in order to treat them separately. Let us proceed to the first part of the proof where we show that Φ is well-defined.

Step 1.1 (treating integral part)

$$\left(\int_0^T \|\Phi(u)(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8} = \Big\|\|\Phi(u)\|_{L^4(\mathbb{R})}\Big\|_{L^8(0,T)} = \|\Phi(u)\|_{L^8\left((0,T);L^4(\mathbb{R})\right)}$$
$$= \left\|e^{it\Delta}f + \lambda i\int_0^t e^{i(t-\tau)\Delta}(|u|^2u)(x,\tau)d\tau\right\|_{L^8\left((0,T);L^4(\mathbb{R})\right)}$$
$$\le \left\|e^{it\Delta}f\right\|_{L^8\left((0,T);L^4(\mathbb{R})\right)} + \left\|\lambda i\int_0^t e^{i(t-\tau)\Delta}(|u|^2u)(x,\tau)d\tau\right\|_{L^8\left((0,T);L^4(\mathbb{R})\right)} \tag{41}$$

In order to find the norms in (41), we use the properties (23) and (27). Remember that $\frac{1}{q} + \frac{1}{q'} = 1$, and $\frac{1}{p} + \frac{1}{p'} = 1$. In our case, $q = 8$ and $p = 4$, thus, q' and p' are calculated to be $q' = \frac{8}{7}$ and $p' = \frac{4}{3}$. Therefore, it follows that

$$\left(\int_{-\infty}^{\infty} \left\|e^{it\Delta}f\right\|^8_{L^4(\mathbb{R})}dt\right)^{1/8} \le c\|f\|_{L^2(\mathbb{R})} \tag{42}$$

$$\left(\int_{-\infty}^{\infty} \left\|\lambda i\int_0^t e^{i(t-\tau)\Delta}(|u^2|u)(\cdot,\tau)d\tau\right\|^8_{L^4(\mathbb{R})} dt\right)^{1/8} \le c|\lambda|\left(\int_0^T \||u(\cdot,t)|^3\|^{8/7}_{L^{4/3}(\mathbb{R})}dt\right)^{7/8}$$

Consequently, the inequality (41) becomes

$$\left(\int_0^T \|\Phi(u)(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8} \le c\|f\|_{L^2(\mathbb{R})} + c|\lambda|\left(\int_0^T \||u(\cdot,t)|^3\|^{8/7}_{L^{4/3}(\mathbb{R})}dt\right)^{7/8}$$
$$\le c\|f\|_{L^2(\mathbb{R})} + c|\lambda|\left(\int_0^T \|u(\cdot,t)\|^{3*(8/7)}_{L^4(\mathbb{R})}dt\right)^{7/8}. \tag{43}$$

Let us clarify that

$$\||u(\cdot,t)|^3\|_{L^{4/3}(\mathbb{R})} \le \|u(\cdot,t)\|^3_{L^4(\mathbb{R})}.$$

Specifically, by definition of the norm in space, we can write that

$$\||u(\cdot,t)|^3\|_{L^{4/3}(\mathbb{R})} = \left[\int_{\mathbb{R}} |u(x,t)|^{3*(4/3)}dx\right]^{3/4} = \left[\left(\int_{\mathbb{R}} |u(x,t)|^4 dx\right)^{1/4}\right]^3$$
$$= \|u(\cdot,t)\|^3_{L^4(\mathbb{R})}.$$

Next, we need to illustrate that

$$\left(\int_0^T \|u(\cdot,t)\|^{3*(8/7)}_{L^4(\mathbb{R})}dt\right)^{7/8} \le T^{1/2}\left(\int_0^T \|u(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{3/8}. \tag{44}$$

Take $\|u(\cdot,t)\|_{L^4(\mathbb{R})} = |w(t)|$. Therefore, the inequality 44 is equivalent to

$$\left(\int_0^T |w(t)|^{3*(8/7)}dt\right)^{7/8} \le T^{1/2}\left(\int_0^T |w(t)|^8 dt\right)^{3/8}. \tag{45}$$

Using Hölder's inequality theorem 2.8,

$$\left(\int_0^T |w(t)|^{3*(8/7)}dt\right)^{7/8} \le \left(\int_0^T \left[|w(t)|^{3*(8/7)}\right]^p dt\right)^{7/8p}\left(\int_0^T 1^{p'}dt\right)^{7/8p'}.$$

Choose p to be equal to $p = \dfrac{7}{3}$. Remembering that $p' = \dfrac{p}{p-1}$, we have $p' = \dfrac{7}{4}$. Thus,

$$\left(\int_0^T |w(t)|^{3*(8/7)}dt\right)^{7/8} \le \left(\int_0^T \left[|w(t)|^{3*(8/7)}\right]^{7/3} dt\right)^{3/8}\left(\int_0^T 1dt\right)^{1/2}$$
$$\le T^{1/2}\left(\int_0^T |w(t)|^8 dt\right)^{3/8}.$$

Replace $w(t) = \|u(\cdot,t)\|_{L^4(\mathbb{R})}$. As a result, the inequality (44) does hold. Given the latest, from (41) we have that

$$\left(\int_0^T \|\Phi(u)(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8} \le c\|f\|_{L^2(\mathbb{R})} + c|\lambda|T^{1/2}\left(\int_0^T \|u(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{3/8}.$$

If $u \in E(T,a)$, then the norm $|||u|||_T$ must be less than a. Thus,

$$\left(\int_0^T \|\Phi(u)(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8} \le c\|f\|_{L^2(\mathbb{R})} + c|\lambda|T^{1/2}a^3.$$

Step 1.2 (treating supremum part)

Now, let us move on to the supremum part of the norm. We are going to illustrate the following inequality

$$\sup_{[0,T]}\|\Phi(u)(\cdot,t)\|_{L^2(\mathbb{R})} \le c\|f\|_{L^2(\mathbb{R})} + c|\lambda|\left(\int_0^T \||u(\cdot,t)|^3\|^{8/7}_{L^{4/3}(\mathbb{R})}dt\right)^{7/8}. \tag{46}$$

Let us remember that

$$\|e^{it\Delta}f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.$$

Moreover, using the definition (40) of Φ, triangle inequality and the inequality (25), we rewrite

$$\begin{aligned}\sup_{[0,T]}\|\Phi(u)(t)\|_{L^2(\mathbb{R})} &\leq \sup_{[0,T]}\Big\|e^{it\Delta}f + \lambda i\int_0^t e^{i(t-\tau)\Delta}(|u|^2u)(\cdot,\tau)d\tau\Big\|_{L^2(\mathbb{R})}\\ &\leq \sup_{[0,T]}\big\|e^{it\Delta}f\big\|_{L^2(\mathbb{R})} + |\lambda|\sup_{[0,T]}\Big\|\int_0^t e^{i(t-\tau)\Delta}(|u|^2u)(\cdot,\tau)d\tau\Big\|_{L^2(\mathbb{R})}\\ &\leq \sup_{[0,T]}\|f\|_{L^2(\mathbb{R})} + c|\lambda|\sup_{[0,T]}\Big(\int_0^t \||u(\cdot,\tau)|^3\|^{8/7}_{L^{4/3}(\mathbb{R})}d\tau\Big)^{7/8}\\ &\leq \|f\|_{L^2(\mathbb{R})} + c|\lambda|\Big(\int_0^T \||u(\cdot,\tau)|^3\|^{8/7}_{L^{4/3}(\mathbb{R})}d\tau\Big)^{7/8}.\end{aligned}$$

It is important to note that we have checked if all the restrictions on p and q were satisfied, considering that $p' = \dfrac{4}{3}$ and $q' = \dfrac{8}{7}$. Then using the same logic as in treating the integral part, particularly (43), (44), and (45), we have

$$\begin{aligned}\sup_{[0,T]}\|\Phi(u)(\cdot,t)\|_{L^2(\mathbb{R})} &\leq c\|f\|_{L^2(\mathbb{R})} + c|\lambda|\left(\int_0^T \||u(\cdot,t)|^3\|^{8/7}_{L^{4/3}(\mathbb{R})}dt\right)^{7/8}\\ &\leq c\|f\|_{L^2(\mathbb{R})} + c|\lambda|T^{1/2}a^3.\end{aligned} \tag{47}$$

Since both integral and supremum part of the norm were proved to be bounded, we come to the conclusion that

$$|||\Phi(u)|||_T \leq c\|f\|_{L^2(\mathbb{R})} + c|\lambda|T^{1/2}a^3.$$

Moreover, we can take $a = 2c\|f\|_{L^2(\mathbb{R})}$ and $T > 0$, so that

$$8c^3|\lambda|T^{1/2}\|f\|^2_{L^2(\mathbb{R})} < 1.$$

Consequently,

$$\begin{aligned}|||\Phi(u)|||_T &\leq c\|f\|_{L^2(\mathbb{R})} + c|\lambda|T^{1/2}(2c\|f\|_{L^2(\mathbb{R})})^3\\ &= c\|f\|_{L^2(\mathbb{R})} + c\|f\|_{L^2(\mathbb{R})}\Big[8c^3|\lambda|T^{1/2}\|f\|^2_{L^2(\mathbb{R})}\Big]\\ &< 2c\|f\|_{L^2(\mathbb{R})}.\end{aligned}$$

We conclude that the mapping Φ is well-defined on our function space X. Next step is to prove that Φ is actually a contraction mapping. It is a necessary condition to be satisfied for the use of Banach fixed-point theorem. Thus, we have to show the following

$$|||\Phi(u) - \Phi(v)|||_X \leq k|||u - v|||_X$$

with $0 \leq k < 1$.

If $u, v \in X$

$$(\Phi(v) - \Phi(u))(x,t) = i\lambda\int_0^t e^{i(t-\tau)\Delta}\big((|v|^2v)(x,\tau) - (|u|^2u)(x,\tau)\big)d\tau.$$

Step 2.1 (treating integral part)

We know that u and v solve NLSE (38), but we do not know are they real or complex functions. First, assume they are real. But in the equation $i\partial_t u = -\Delta u - \lambda|u|^2u$ left side is complex, while right side will give only real value. Thus, u and v cannot be real functions. It follows that (38) can have complex functions as a solution. Considering this fact, we will use property of complex numbers that is $|z| = |\bar{z}|$.

(48)
$$\left(\int_0^T \|(\Phi(v)-\Phi(u))(\cdot,t)\|^8_{L^2(\mathbb{R})}dt\right)^{1/8} \leq c|\lambda|\left(\int_0^T \|\big(|v|^2v\big)(\cdot,t)-\big(|u|^2u\big)(\cdot,t)\|^{8/7}_{L^{4/3}(\mathbb{R})}dt\right)^{7/8}$$
$$= c|\lambda|\left(\int_0^T \|(v\bar{v}v)(\cdot,t)-(u\bar{u}u)(\cdot,t)\|^{8/7}_{L^{4/3}(\mathbb{R})}dt\right)^{7/8}$$
$$= c|\lambda|\left(\int_0^T (\|v^2\bar{v})(\cdot,t)-(u^2\bar{u})(\cdot,t)\|^{8/7}_{L^{4/3}(\mathbb{R})}dt\right)^{7/8}.$$

We divide $(v^2\bar{v}-u^2\bar{u})$ by term $(v-u)$, so we obtain
$$\begin{aligned} v^2\bar{v}-u^2\bar{u} &= (v-u)(v\bar{v}+u\bar{v})+u^2\bar{v}-u^2\bar{u} \\ &= (v-u)(|v|^2+u\bar{v})+u^2\overline{(v-u)} \end{aligned}$$

By taking norms of both sides we obtain
$$\begin{aligned} \Big||v|^2v-|u|^2u\Big| &\leq |v-u|\big(|v|^2+|v||u|\big)+|u|^2|v-u| \\ &= |v-u|\big(|v|^2+|v||u|+|u|^2\big) \\ &\leq k|v-u|\big(|v|^2+|u|^2\big) \end{aligned}$$

for some positive constant k. Turning back to the inequality (48) and continuing

$$(49)\quad c|\lambda|\left(\int_0^T \|(v^2\bar{v})(\cdot,t)-(u^2\bar{u})(\cdot,t)\|^{8/7}_{L^{4/3}(\mathbb{R})}dt\right)^{7/8}$$
$$(50)\quad \leq c|\lambda|\left(\int_0^T \Big\||v(\cdot,t)-u(\cdot,t)|\Big(|v(\cdot,t)|^2+|u(\cdot,t)|^2\Big)\Big\|^{8/7}_{L^{4/3}(\mathbb{R})}dt\right)^{7/8}$$
$$= c|\lambda|\left(\int_0^T \left(\int_{\mathbb{R}} |v(x,t)-u(x,t)|^{4/3}\big(|v(x,t)|^2+|u(x,t)|^2\big)^{4/3}dx\right)^{(3/4)(8/7)}dt\right)^{7/8}.$$

Now we can apply Hölders Inequality (2.8) where p is chosen to be equal to 3. Hence, $p'=\frac{3}{2}$.

(51)
$$c|\lambda|\left(\int_0^T \left(\int_{\mathbb{R}} |v(x,t)-u(x,t)|^{4/3}\big(|v(x,t)|^2+|u(x,t)|^2\big)^{4/3}dx\right)^{(3/4)(8/7)}dt\right)^{7/8}$$
$$\leq c|\lambda|\left[\int_0^T \left[\left(\int_{\mathbb{R}} |v(x,t)-u(x,t)|^4dx\right)^{1/3}\left(\int_{\mathbb{R}} \big(|v(x,t)|^2+|u(x,t)|^2\big)^2dx\right)^{2/3}\right]^{(3/4)(8/7)}dt\right]^{7/8}$$
$$= c|\lambda|\left[\int_0^T \left(\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}\Big(\int_{\mathbb{R}} \big(|v(x,t)|^2+|u(x,t)|^2\big)^2dx\Big)^{1/2}\right)^{8/7}dt\right]^{7/8}$$
$$= c|\lambda|\left[\int_0^T \left(\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}\Big\||v(\cdot,t)|^2+|u(\cdot,t)|^2\Big\|_{L^2(\mathbb{R})}\right)^{8/7}dt\right]^{7/8}$$
$$\leq c_1|\lambda|\left(\int_0^T \left(\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}\big(\big\||v(\cdot,t)|^2\big\|_{L^2(\mathbb{R})}+\big\||u(\cdot,t)|^2\big\|_{L^2(\mathbb{R})}\big)\right)^{8/7}dt\right)^{7/8}$$

$$\leq c_1|\lambda|\left(\int_0^T\left(\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}(\|v(\cdot,t)\|_{L^4(\mathbb{R})}^2+\|u(\cdot,t)\|_{L^4(\mathbb{R})}^2)\right)^{8/7}dt\right)^{7/8}$$

$$=c_1|\lambda|\left(\int_0^T\left(\|v(\cdot,t)\|_{L^4(\mathbb{R})}^2\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}+\|u(\cdot,t)\|_{L^4(\mathbb{R})}^2\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}\right)^{8/7}dt\right)^{7/8}$$

(52)

$$\leq c_1|\lambda|\left(\int_0^T\|v(\cdot,t)\|_{L^4(\mathbb{R})}^{2(8/7)}\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}^{8/7}dt\right)^{7/8}$$

$$+c_1|\lambda|\left(\int_0^T\|u(\cdot,t)\|_{L^4(\mathbb{R})}^{2(8/7)}\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}^{8/7}dt\right)^{7/8}.$$

For each of the above integrals from (52) we use the Hölder's Inequality again. Remembering that $r=8, r'=\dfrac{8}{7}$, and taking $p=\dfrac{7}{6}, p'=7$, the first integral splits as following

(53)

$$\left(\int_0^T\|v(\cdot,t)\|_{L^4(\mathbb{R})}^{2(8/7)}\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}^{8/7}dt\right)^{7/8}$$

$$\leq\left(\int_0^T\|v(\cdot,t)\|_{L^4(\mathbb{R})}^{2(8/7)(7/6)}dt\right)^{(7/8)(6/7)}\left(\int_0^T\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}^{(8/7)7}dt\right)^{(7/8)(1/7)}$$

$$=\left(\int_0^T\|v(\cdot,t)\|_{L^4(\mathbb{R})}^{8/3}dt\right)^{(3/4)}\left(\int_0^T\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}^{8}dt\right)^{1/8}$$

$$\leq\left(\int_0^T\|v(\cdot,t)\|_{L^4(\mathbb{R})}^{3(8/3)}dt\right)^{(3/4)(1/3)}\left(\int_0^T 1dt\right)^{(3/4)(2/3)}\left(\int_0^T\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}^{8}dt\right)^{1/8}$$

$$=T^{1/2}\left(\int_0^T\|v(\cdot,t)\|_{L^4(\mathbb{R})}^{8}dt\right)^{1/4}\left(\int_0^T\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}^{8}dt\right)^{1/8}.$$

In order to split the second integral from (52), the same procedure as above is used. Hence,

$$\left(\int_0^T\|u(\cdot,t)\|_{L^4(\mathbb{R})}^{2(8/7)}\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}^{8/7}dt\right)^{7/8}$$
$$\leq T^{1/2}\left(\int_0^T\|u(\cdot,t)\|_{L^4(\mathbb{R})}^{8}dt\right)^{1/4}\left(\int_0^T\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}^{8}dt\right)^{1/8}. \tag{54}$$

By combining (48), (51), (53), and (54), we get that

$$\left(\int_0^T\|(\Phi(v)-\Phi(u))(\cdot,t)\|_{L^2(\mathbb{R})}^{8}dt\right)^{1/8}$$

$$\leq c_1|\lambda|T^{1/2}\left[\left(\int_0^T\|v(\cdot,t)\|_{L^4(\mathbb{R})}^{8}dt\right)^{1/4}+\left(\int_0^T\|u(\cdot,t)\|_{L^4(\mathbb{R})}^{8}dt\right)^{1/4}\right]$$

$$\times\left(\int_0^T\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}^{8}dt\right)^{1/8}\leq 2c_1|\lambda|T^{1/2}a^2\left(\int_0^T\|v(\cdot,t)-u(\cdot,t)\|_{L^4(\mathbb{R})}^{8}dt\right)^{1/8}.$$

Thus we have shown that the integral part of the norm can satisfy the contraction mapping condition for some constant $c(T,a) = 2c_1|\lambda|T^{1/2}a^2$ to be determined later.

Step 2.2 (treating supremum part)

Next, we need to show the following inequality

$$\sup_{[0,T]}\Big(\|\Phi(v)-\Phi(u)\big)(\cdot,t)\|_{L^2(\mathbb{R})} \leq 2c_1|\lambda|T^{1/2}a^2\left(\int_0^T \|v(\cdot,t)-u(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8}.$$

It can be shown n a similar way as in (46). Using (24) and the arguments used in (48), we obtain

$$\begin{aligned}
&\sup_{[0,T]}\big(\|\Phi(v)-\Phi(u)\big)(\cdot,t)\|_{L^2(\mathbb{R})} \\
&= \sup_{[0,T]}\|i\lambda\int_0^t e^{i(t-\tau)\Delta}(|v|^2v-|u|^2u)(\cdot,\tau)d\tau\|_{L^2(\mathbb{R})} \\
&= |\lambda|\sup_{[0,T]}\Big\|\int_0^t e^{i(t-\tau)\Delta}(|v|^2v-|u|^2u)(\cdot,\tau)d\tau\Big\|_{L^2(\mathbb{R})} \\
&\leq c|\lambda|\sup_{[0,T]}\Big(\int_0^t \|(|v|^2v)(\cdot,\tau)-(|u|^2u)(\cdot,\tau)\|^{8/7}_{L^{4/3}(\mathbb{R})}d\tau\Big)^{7/8} \\
&\leq c|\lambda|\Big(\int_0^T \|(|v|^2v)(\cdot,t)-(|u|^2u)(\cdot,t)\|^{8/7}_{L^{4/3}(\mathbb{R})}dt\Big)^{7/8} \\
&\leq 2c_1|\lambda|T^{1/2}a^2\left(\int_0^T \|v(\cdot,t)-u(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8}.
\end{aligned} \tag{55}$$

Combining the results obtained in *step 2.1* and *step 2.2*, we get that

$$\begin{aligned}
\|\!|\Phi(u)-\Phi(v)\|\!|_X &= \left(\int_0^T \|(\Phi(v)-\Phi(u))(\cdot,t)\|^8_{L^2(\mathbb{R})}dt\right)^{1/8} + \sup_{[0,T]}\|\big(\Phi(v)-\Phi(u)\big)(\cdot,t)\|_{L^2(\mathbb{R})} \\
&\leq 2c_1|\lambda|T^{1/2}a^2\left(\int_0^T \|v(\cdot,t)-u(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8}.
\end{aligned}$$

By choosing appropriate a, $a \leq 2c\|f\|_{L^2(\mathbb{R})}$, it follows that

$$2c_1|\lambda|T^{1/2}a^2 \leq 8c^3|\lambda|T^{1/2}\|f\|^2_{L^2(\mathbb{R})} < 1. \tag{56}$$

In order the inequality (56) to be true, we take

$$T \simeq \|f\|^{\beta}_{L^2(\mathbb{R})}, \quad \text{where } \beta = -4.$$

In conclusion to the second step of the proof, we have shown that our mapping Φ is indeed a contraction mapping. Consequently, since we have the complete metric space X and the contraction mapping Φ, we can apply the Banach fixed-point theorem. According to the theorem, there exists a unique fixed-point u such that $\Phi(u) = u$. It means that

$$\Phi_f(u)(x,t) = \Phi(u)(x,t) = e^{it\Delta}f(x) + \lambda i\int_0^t e^{i(t-\tau)\Delta}(|u|^2u)(x,\tau)d\tau. \tag{57}$$

Next we need to prove the continuous dependence of the solutions to the given initial conditions. In other words, we need to show that if the initial conditions are changed insignificantly, then the new solution should not differ form the previous solution considerably. If u and v are solutions of integral equation (39) having initial data f and g

$$u(x,t)-v(x,t) = e^{it\Delta}(f-g)(x) + i\lambda\int_0^t e^{i(t-\tau)\Delta}(|u|^2u-|v|^2v)(x,\tau)d\tau.$$

Now, we will show that our mapping from the initial conditions to the solutions is Lipschitz continuous. In other words, we should conclude that

$$\left(\int_0^T \|u(\cdot,t)-v(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8} + \sup_{[0,T]}\|u(\cdot,t)-v(\cdot,t)\|_{L^2(\mathbb{R})} \le \tilde{K}\|f-g\|_{L^2(\mathbb{R})}. \tag{58}$$

Step 3.1 (treating integral part)
For the first part of the (58) we deduce the following

$$\left(\int_0^T \|u(\cdot,t)-v(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8} = \Big\|\|u-v\|_{L^4(\mathbb{R})}\Big\|_{L^8(\mathbb{R})} = \|u-v\|_{L^8\left((0,T);L^4(\mathbb{R})\right)}$$

$$= \left\|e^{it\Delta}(f-g) + i\lambda\int_0^t e^{i(t-\tau)\Delta}(|u|^2u-|v|^2v)(\cdot,\tau)d\tau\right\|_{L^8\left((0,T);L^4(\mathbb{R})\right)}$$

$$\le \left\|e^{it\Delta}(f-g)\right\|_{L^8\left((0,T);L^4(\mathbb{R})\right)} + \left\|i\lambda\int_0^t e^{i(t-\tau)\Delta}(|u|^2u-|v|^2v)(\cdot,\tau)d\tau\right\|_{L^8\left((0,T);L^4(\mathbb{R})\right)}. \tag{59}$$

Now, using the argument (42) from the previous results, the first part in (59) becomes

$$\left(\int_{-\infty}^{\infty} \left\|e^{it\Delta}(f-g)(\cdot,t)\right\|^8_{L^4(\mathbb{R})}dt\right)^{1/8} \le c\|f-g\|_{L^2(\mathbb{R})}. \tag{60}$$

For the second part of (59), the similar argument as in (42) implies that

$$\left(\int_{-\infty}^{\infty} \left\|i\lambda\int_0^t e^{i(t-\tau)\Delta}\big((|u|^2u)(\cdot,\tau)-(|v|^2v)(\cdot,\tau)\big)d\tau\right\|^8_{L^4(\mathbb{R})}dt\right)^{1/8}$$

$$\le c|\lambda|\left(\int_0^T \left\|(|u|^2u)(\cdot,t)-(|v|^2v)(\cdot,t)\right\|^{8/7}_{L^{4/3}(\mathbb{R})}dt\right)^{7/8}.$$

Moreover,at each following step using the same reasoning as in (48), (49), (51), and (54), (**??**), we deduce that

$$c|\lambda|\left(\int_0^T \left\|(|u|^2u)(\cdot,t)-(|v|^2v)(\cdot,t)\right\|^{8/7}_{L^{4/3}(\mathbb{R})}dt\right)^{7/8} = c|\lambda|\left(\int_0^T \left\|(v^2\bar{v})(\cdot,t)-(u^2\bar{u})(\cdot,t)\right\|^{8/7}_{L^{4/3}(\mathbb{R})}dt\right)$$

$$\le c|\lambda|\left(\int_0^T \left(\int_{\mathbb{R}} |v(x,t)-u(x,t)|^{4/3}\Big(|v(x,t)|^2+|u(x,t)|^2\Big)^{4/3}dx\right)^{(3/4)(8/7)}dt\right)^{7/8}$$

$$\le c_1|\lambda|\left(\int_0^T \|v(\cdot,t)\|^{2*(8/7)}_{L^4(\mathbb{R})}\|v(\cdot,t)-u(\cdot,t)\|^{8/7}_{L^4(\mathbb{R})}dt\right)^{7/8}$$

$$+ c_2|\lambda|\left(\int_0^T \|u(\cdot,t)\|^{2*(8/7)}_{L^4(\mathbb{R})}\|v(\cdot,t)-u(\cdot,t)\|^{8/7}_{L^4(\mathbb{R})}dt\right)^{7/8}$$

$$\le C|\lambda|T^{1/2}\left[\left(\int_0^T \|v(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/4} + \left(\int_0^T \|u(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/4}\right]$$

$$\times\left(\int_0^T \|v(\cdot,t)-u(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8}$$

$$\leq 2C|\lambda|T^{1/2}a^2\left(\int_0^T \|v(\cdot,t)-u(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8}.$$

Therefore, the expression (59) is rewritten to be

$$\left(\int_0^T \|u(\cdot,t)-v(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8} \leq c\|f-g\|_{L^2(\mathbb{R})}$$

$$+K_\alpha|\lambda|T^\theta(\|f\|_{L^2(\mathbb{R})}+\|g\|_{L^2(\mathbb{R})})\left(\int_0^T \|u(\cdot,t)-v(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8}.$$

Consequently,

$$\left(\int_0^T \|u(\cdot,t)-v(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8} \leq \tilde{K}\|f-g\|_{L^2(\mathbb{R})}. \tag{61}$$

Step 3.2 (treating supremum part)
Proceeding to the second part of (58), we should be able to conclude that

$$\sup_{[0,T]}\|u(\cdot,t)-v(\cdot,t)\|_{L^2(\mathbb{R})} \leq \tilde{K}\|f-g\|_{L^2(\mathbb{R})}.$$

To begin with, rewrite

$$\sup_{[0,T]}\|u(\cdot,t)-v(\cdot,t)\|_{L^2(\mathbb{R})}$$

$$= \sup_{[0,T]}\left\|e^{it\Delta}(f-g)+i\lambda\int_0^t e^{i(t-\tau)\Delta}((|u|^2u)(\cdot,\tau)-(|v|^2v)(\cdot,\tau))d\tau\right\|_{L^2(\mathbb{R})}$$

$$\leq \sup_{[0,T]}\|e^{it\Delta}(f-g)\|_{L^2(\mathbb{R})}+\sup_{[0,T]}\left\|i\lambda\int_0^t e^{i(t-\tau)\Delta}((|u|^2u)(\cdot,\tau)-(|v|^2v)(\cdot,\tau))d\tau\right\|_{L^2(\mathbb{R})}. \tag{62}$$

By lemma 3.3, the first part of (62) is reduced such that

$$\sup_{[0,T]}\|e^{it\Delta}(f-g)\|_{L^2(\mathbb{R})} \leq c\|f-g\|_{L^2(\mathbb{R})}. \tag{63}$$

For the second part of (62), recall the results of (55) and (61)

$$\sup_{[0,T]}\left\|i\lambda\int_0^t e^{i(t-\tau)\Delta}(|u|^2u-|v|^2v)(\cdot,\tau)d\tau\right\|_{L^2(\mathbb{R})} \leq 2c_1|\lambda|T^{1/2}a^2\left(\int_0^T \|v(\cdot,t)-u(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8}$$

$$\leq K\|f-g\|_{L^2(\mathbb{R})}.$$

Now, combining the inequality obtained above and (63), we get that

$$\sup_{[0,T]}\|u(\cdot,t)-v(\cdot,t)\|_{L^2(\mathbb{R})} \leq \tilde{K}\|f-g\|_{L^2(\mathbb{R})}$$

which leads us to the final conclusion about the stability of the solution. That is the mapping from the initial data to the solution is Lipschitz continuous since

$$\left(\int_0^T \|u(\cdot,t)-v(\cdot,t)\|^8_{L^4(\mathbb{R})}dt\right)^{1/8} + \sup_{[0,T]}\|u(\cdot,t)-v(\cdot,t)\|_{L^2(\mathbb{R})} \leq \tilde{K}\|f-g\|_{L^2(\mathbb{R})}.$$

$\square$

4.2. **Global well-posedness in $L^2(\mathbb{R})$.**

We have determined that the NLSE is locally well-posed. In other words, we proved that a unique solution to the equation exists in a particular interval of time T. Now we would like to study the global well-posedness of the equation.

Definition 4.3. *An equation is defined to be globally well-posed in X if T can be taken arbitrarily large.*

The following theorem states that the NLSE is also globally well-posed in $L^2(\mathbb{R})$.

Theorem 4.4. *For any $f \in L^2(\mathbb{R})$ the local solution u(x,t) of the initial value problem 4.2 extends globally with*

$$u \in C([0,\infty) : L^2(\mathbb{R})) \cap L^q_{loc}([0,\infty) : L^p(\mathbb{R}))$$

where p and q should satisfy the following condition

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{p}, \quad 2 \le p \le \infty.$$

Our aim is to illustrate that

$$\|u(\cdot,t)\|^2_{L^2(\mathbb{R})} = \|f\|^2_{L^2(\mathbb{R})}.$$

This is the same as to show the following mass conservation

$$M(t) := \|u(\cdot,t)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \|u(x,t)\|^2 dx, \quad t > 0.$$

In other words, the conservation law applies that the L^2 norm of the solution $u(x,t)$ remains the same as for initial data f. If so, the solution $u(x,t)$ can be taken as a new initial data in order to find a new solution $u(x,t')$. Then $t' = t'(\|u(\cdot,t)\|_{L^2(\mathbb{R})})$ according to the theorem for local L^2-solution 4.2. Since $\|u(\cdot,t)\|^2_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$, the length of the time intervals is preserved such that $t' = t$. Therefore, we can apply the theorem 4.2 infinitely many times, extending to a global solution.

Proof. Saying the mass is conserved means the function $M(t)$ should be constant. Therefore,

$$\partial_t M(t) = 0$$

$$\partial_t \Big(\int_{\mathbb{R}} u(x,t)\bar{u}(x,t)dx \Big) = 0$$

$$\int_{\mathbb{R}} \Big(\partial_t u(x,t)\bar{u}(x,t) + \partial_t \bar{u}(x,t)u(x,t) \Big) dx = 0$$

Take $u(x,t)$ as a solution of NLSE,

$$i\,\partial_t u(x,t) + \Delta u(x,t) + \lambda\,|u(x,t)|^2\,u(x,t) = 0 \tag{64}$$

Hence,

$$-\,i\,\partial_t \bar{u}(x,t) + \Delta \bar{u}(x,t) + \lambda\,|u(x,t)|^2\,\bar{u}(x,t) = 0 \tag{65}$$

Now, multiply (64) by $\bar{u}(x,t)$ and then integrate in x to obtain

$$i \int_{\mathbb{R}} \partial_t u(x,t)\bar{u}(x,t)dx + \int_{\mathbb{R}} \Delta u(x,t)\bar{u}(x,t)dx + \lambda \int_{\mathbb{R}} |u(x,t)|^4 dx = 0 \tag{66}$$

Use integration by parts for the middle integral above, so that

$$\begin{aligned}\int_{\mathbb{R}} \Delta u(x,t)\bar{u}(x,t)dx &= \int_{\mathbb{R}} \Big(\partial_x\partial_x u(x,t) \Big) \bar{u}(x,t)dx \\ &= -\int_{\mathbb{R}} \partial_x u(x,t)\partial_x \bar{u}(x,t)dx + \partial_x u(x,t)\bar{u}(x,t)\Big]_{x=-\infty}^{x=\infty}\end{aligned}$$

$$= \int_{\mathbb{R}} u(x,t)\partial_x^2\bar{u}(x,t)dx - u(x,t)\partial_x\bar{u}(x,t)\Big]_{x=-\infty}^{x=\infty}$$
$$= \int_{\mathbb{R}} u(x,t)\Delta\bar{u}(x,t)dx.$$

By using (65),

$$\int_{\mathbb{R}} u(x,t)\Delta\bar{u}(x,t)dx = \int_{\mathbb{R}} u(x,t)\Big(i\partial_t\bar{u}(x,t) - \lambda|u(x,t)|^2\bar{u}(x,t)\Big)dx.$$

After combining with (66), we get the following result

$$i\int_{\mathbb{R}}\Big(\partial_t u(x,t)\bar{u}(x,t) + u(x,t)\partial_t\bar{u}(x,t)\Big)dx = 0$$

which is what we wanted to prove. □

5. EXTENSION OF THE RESULTS

First of all, we can extend our results in terms of taking higher dimension in the space variable, that is taking $x \in \mathbb{R}^n$. The equation remains to be both locally and globally well-posed in $L^2(\mathbb{R}^n)$ according to Y. Tsutsumi [18]. Moreover, the author considers generalized power of the non-linearity term. Precisely, for the following IVP for NLSE

$$\begin{cases} i\,\partial_t u(x,t) + \Delta u(x,t) + \lambda\,|u(x,t)|^{\alpha-1}u(x,t) = 0, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases} \tag{67}$$

with $t \in \mathbb{R}, x \in \mathbb{R}^n$, real constant λ and for some α.

Furthermore, let us give some references to the results obtained for $H^s(\mathbb{R})$ spaces. The problem is proved to be well-posed in $H^1(\mathbb{R}^n)$ for some α by J. Ginibre and G. Velo in [9]. Also the $H^2(\mathbb{R}^n)$ theory can be found in [11]. The ill-posedness of the equation for $s < 0$ was shown by M. Christ, J. Colliander, and T. Tao in [4] as well as by C. Kenig, G. Ponce, and L. Vega in [13]. The reason for being ill-posed for $s < 0$ is because the solution does not uniformly continuously depend on the initial data in H^s norm.

Another way of extending results is changing the form of the nonlinearity. As an example, consider the following initial value problem for Logarithmic Schrödinger Equation

$$\begin{cases} i\,\partial_t u(x,t) + \Delta u(x,t) + Vu(x,t) + u(x,t)\log(|u(x,t)^2|) = 0, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases} \tag{68}$$

with V being some real-valued potential. Using a compactness method, local and global well-posedness of the equation was shown in [3, Theorem 9.3.4.].

Moreover, consider the following p-Laplace operator

$$\Delta_p u := \nabla\cdot(||^{p-2}\nabla u)$$

where $1 < p < \infty$. When $p = 2$ we have the usual Laplacian operator. If we insert the p-Laplacian into the NLSE instead of the usual Laplacian, then the problem becomes more complicated since we will add more nonlinearity. Moreover, there is no research work that concerns about such type of NLSE. However, we can give a reference to a paper where time-independent version of the Schrödinger equation was considered. The authors Wei Han and Jiangyan in their paper [10] have examined the existence and uniqueness of the positive solution for the p-Laplacian Kirchhoff- Schrödinger type equation. Particularly, the following problem is the subject of discussion in [10].

$$\begin{cases} -\Big(a + b\int_\Omega |\nabla u|^p\Big)\Delta_p u + \lambda v(x)|u|^{p-2}u = hf(u) - \mu g(u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{69}$$

where $\Omega \in \mathbb{R}^N (N \geq 3)$, $\lambda, \mu \geq 0$, $p \geq 2$, and $a, b \geq 0$ with $a + b > 0$.

In addition, let us talk about other partial differential equations that are somehow connected to NLSE. The first example is the following modified Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u \pm |u|^2 \partial_x u = 0.$$

As it can be seen, the equation is to some extent similar to NLSE. Notably, the well-posedness of the problem can be proved using the same technique as for NLSE. The local result in $H^s(\mathbb{R})$, for $s \geq 1/4$ was obtained by C. E. Kenig, G. Ponce, and L. Vega in [12]. The global well-posedness in $H^s(\mathbb{R})$, for $s \geq 1/4$ was shown in [6] by J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Next example is a combination of the NLSE and modified Korteweg-de Vries equation. It is called Nonlinear Schrödinger Airy equation

$$\partial_t u + i\, a\, \partial_x^2 u + b\, \partial_x^3 u + i\, c\, |u|^2 u + d|u|^2 \partial_x u + e\, u^2 \partial_x \bar{u} = 0.$$

The initial value problem for the equation was proved to be locally and globally well-posed in $H^s(\mathbb{R})$for $s \geq 1/4$ by G. Staffilani in [16] and by X. Carvajal in [2] respectively. In order to prove the local result, author used the Banach fixed-point theorem in [16].

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