NONLINEAR SCHRÖDINGER EQUATION

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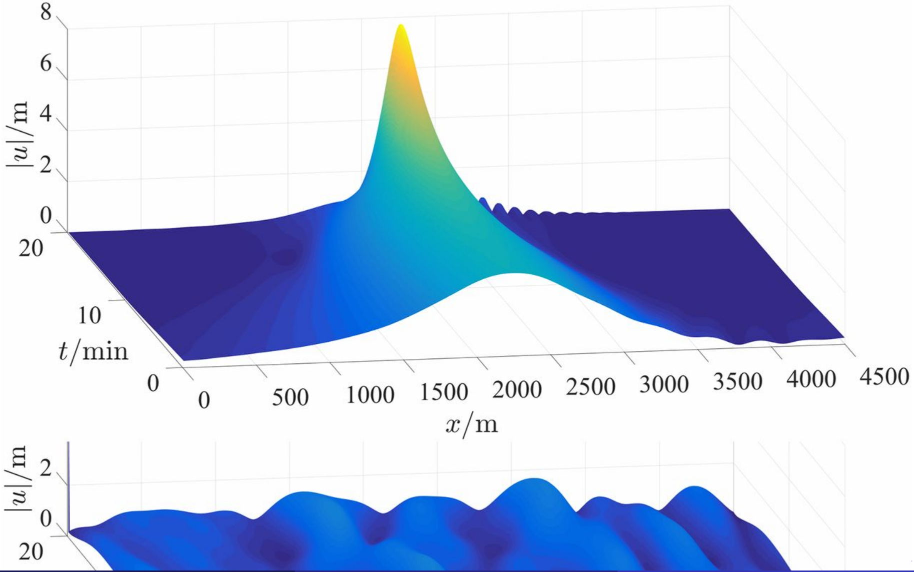
Overview

Rogue waves

Rogue waves are fascinating destructive phenomena in nature that have not been fully explained so far [1-3]. Oceanographers commonly agree that linear theories cannot provide explanations for their existence[6,7]. Only nonlinear theories can explain the dramatic concentration of energy into a single "wall of water" well above the average height of the surrounding waves[3,8,9]. Among nonlinear theories the most fundamental is based on the nonlinear Schrödinger equation (NLSE)[6]. If the fundamental approach allows us to give a basic explanation, then it can be extended to more general ones which take into account the two-dimensional nature of the problem. Which is our main goal.

Monster waves

- Waves in the ocean are chaotic



Generalization of NLSE

$$i\frac{\partial\psi}{\partial t} + \eta\left(\frac{\partial^2\psi}{\partial x^2} + 2|\psi|^2\psi\right) = 0, \tag{1}$$

- x is the propagation variable
- t is the transverse variable
- ψ is the envelope of a physical solution,complex-valued function of two variables x and t
- η is real number.
- classical field equation - propagation of light in nonlinear optical fibers, planar waveguides, Bose-Einstein condensates
- integrable(solvable) model
- find exact soliton solution

The first integral method

Consider the nonlinear partial differential equation in the form

$$F(u, u_x, u_t, u_{xx}, u_{xt}, ...) = 0, \qquad (2)$$

where $u = u(x, t)$ is the solution of the nonlinear partial differential equation (2). We use the transformations,

$$u(x, t) = f(\xi),$$

where $\xi = x + \lambda t$. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = \lambda \frac{\partial}{\partial \xi}(.), \quad \frac{\partial}{\partial x}(.) = \frac{\partial}{\partial \xi}(.), \quad \frac{\partial^2}{\partial x^2}(.) = \frac{\partial^2}{\partial \xi^2}(.), ... \qquad (3)$$

The first integral method

Use eq.(3) to transfer the NPDE (2) NODE

$$G(f(\xi), \frac{\partial f(\xi)}{\partial \xi}, \frac{\partial^2 f(\xi)}{\partial \xi^2}, ...) = 0. \tag{4}$$

Now introduce new independent variable

$$X(\xi) = f(\xi), \quad Y = \frac{\partial f(\xi)}{\partial \xi} \tag{5}$$

$$\frac{\partial X(\xi)}{\partial \xi} = Y(\xi), \tag{6}$$

$$\frac{\partial Y(\xi)}{\partial \xi} = F_1(X(\xi), Y(\xi)). \tag{7}$$

Theorem

Theorem (Division theorem.)

Suppose that P(w,z) and Q(w,z) are polynomials in C[w,z]; and P(w,z) is irreducible in C[w,z]. If Q(w,z) vanishes at all zero points of P(w,z), then there exists a polynomial G(w,z) in C[w,z] such that

$$Q(w,z) = P(w,z)G(w,z)$$

Nonlinear Schrödinger equation

$$iu_t + pu_{xx} + q|u|^2 u = 0. \tag{8}$$

Use the transformation

$$u(t,x) = e^{i\theta} f(\xi), \quad \theta = \alpha x + \beta t, \quad \xi = x - 2p\alpha t, \tag{9}$$

where α and β are constants and $f(\xi)$ is a real function. Plugging (9) into (8), we obtain ODE

$$-(\beta + p\alpha^2) f(\xi) + p\frac{\partial^2 f(\xi)}{\partial \xi^2} + q(f(\xi))^3 = 0.$$

Using (5) we get

$$\dot{X(\xi)} = Y(\xi), \tag{10}$$

$$\dot{Y(\xi)} = (\frac{\beta + p\alpha^2}{p}) X(\xi) - \frac{q}{p}(X(\xi))^3. \tag{11}$$

Frame Title

According to the first integral method, we suppose the $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (10) and (11) and

$$Q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i = 0$$

is an irreducible polynomial in the complex domain C[X,Y] such that

$$Q(X(\xi),Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi))Y^i(\xi) = 0. \qquad (12)$$

where $a_i(X)$, $(i = 0, ..., m)$, are polynomials of X and $a_m(X) \neq 0$.Eq (12) is called the first integral to (10) and (11).

Frame Title

Due to the Division theorem, there exist a polynomial $g(X) + h(X)Y$, in the complex domain C[X,Y] such that

$$\frac{dQ}{d\xi} = \frac{dQ}{dX}\frac{dX}{d\xi} + \frac{dQ}{dY}\frac{dY}{d\xi} = g((X) + h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i$$

Conclusion

The first integral method was applied successfully for solving the nonlinear Schrödinger equation. Thus we conclude the proposed method can be extended to solve the nonlinear problems which arise in the theory of solitons and other areas.

References

John Smith (2012)
Title of the publication
Journal Name 12(3), 45 – 678.

The End