Solution a Problem of Nonlinear Elasticity Using Power Series in Complex Time

Akyl R. Shakir

Supervisor: Dr. Anastasios Bountis , Second Reader: Dr. Dongming Wei

Abstract

We apply singularity analysis in complex time to investigate the solutions of a dynamical system of one degree of freedom related to the oscillations of a Micro-Electro-Mechanical System (MEMS) of nonlinear elasticity. This problem is expressed mathematically by a second order differential equation for the position variable $x(t)$ of the oscillator. Using the fact that this equation is connected with an energy integral (kinetic plus potential energy), we first study its solutions by plotting the graph of its potential function for different values of an important parameter of the problem K>0. Then, we analyze these solutions by expanding $x(t$ using power series in complex time. Our solutions are expanded about a singularity of the equation of motion at $x = 1$, which constitutes a very important point in the analysis, and complements solutions found by other researchers using real time. As a first application, our complex time expansions can be used to estimate locations in x space where the solutions are bounded and periodic and regions where the solutions are unbounded (and hence non-physical). We also examine our approach as an extension of the well–known Frobenius theory for linear second order ODEs. Finally, we investigate the locations of the singularities near the periodic solutions using a Newton-Raphson method near the singularity of the equation at $x = 1$.

1 Introduction

An important subject in the study of Nonlinear Ordinary Differential Equations (NODEs) is to approximate their solutions by power series in the complex time plane. This approach generalizes the polynomial approximation these solutions by Taylor series expansions in powers of $t \in \Re$ around some point $t_0 \in \Re$, precisely because it converges exactly where Taylor series cease to be valid, i.e. near singularities t_* in the complex domain. One first advantage of this analysis is that it can be used to identify the domains of convergence of the classical Taylor series expansions. Secondly, by being valid in regimes where Taylor series fail, singularity analysis in complex time can be used to estimate regimes in real space where solutions are bounded (i.e periodic in many physical applications) or unbounded and hence non–physical. In this Project, our interest is focused on the solution of a so–called Micro-Electro-Mechanical System (MEMS) problem, used to model the oscillations of a thin metallic plate in a capacitor.

It is well–known that MEMS oscillators of the type studied here, involve in their potential besides the harmonic term (proportional to x^2) polynomial terms of higher nonlinearity. In this study, we have decided to ignore them and have considered only the harmonic part of the potential. However, as will be come clear below, our analysis proceeds in very much the same way, even if these higher order nonlinearities are kept, and thus we believe that the results of our analysis can be extended to also apply to such more realistic MEMS oscillators.

In Section 2 of the Project we carry out such a singularity analysis by expanding the solution of the MEMS oscillator in complex time around the value $x = 1$ which represents a real singularity of the equations of motion. In this way we find a remarkably elegant series solution, with two free constants, which can be used to satisfy any pair of initial conditions $x(0)$ and $x'(0)$ that one may choose. This solution happens to be *free of logarithms* and hence can be used to solve for the locations of the singularities in the complex t-plane to arbitrary accuracy. Thus, it can be used as an alternative solution of the problem, which is valid even in regimes of periodic solutions with very long periods, where even Fourier series expansions may fail to converge.

In Section 3, we examine the connection of our series solutions with solutions in real time, by plotting the MEMS potential energy function and identifying regimes where bounded (periodic) and unbounded solutions exist. Section 4 is devoted to our conclusions and a discussion of how this research can be extended in the future to provide useful solutions of the MEMS problem in the more complicated and realistic case where the potential includes polynomial terms in x.

2 Power series solutions of MEMS in complex time

As is well known, a simplified version of the equation of a MEMS oscillator in one dimension is given by

$$x(t)'' + x(t) = \frac{k}{(1 - x(t))^2} \tag{1}$$

where $k > 0$ is a constant parameter, $x(t)$ describes the oscillator's displacement from equilibrium and the second derivative is taken with respect to the time variable t. In the above force equation (2) the mass of our oscillator is unity and we have neglected on the left hand side all terms of order higher than linear. Let us now define by t_* a singular point of the solutions anywhere in complex t-plane, and let us rewrite this MEMS variable as $x(t - t_*)$ satisfying the equation for:

$$x(t - t_*)'' + x(t - t_*) = \frac{k}{(1 - x(t - t_*))^2} \tag{2}$$

Let consider t_* as the location of a possible singularity of our solutions in the complex time plane. Such singularities would appear positioned symmetrically with respect to the real t axis, as shown in the figure below. They correspond to time values where the solutions are expected to diverge as $t \to t^*$. For simplification, let us denote from here on the independent variable of our solution by $\tau = t - t^*$.

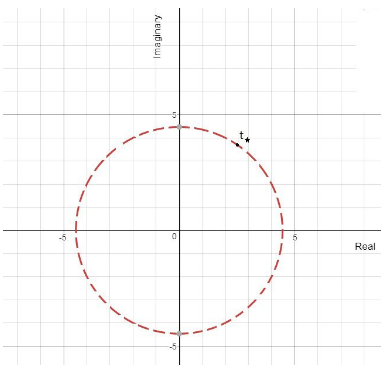


Figure 1: A singularity t_* in the complex time plane and its complex conjugate

To find out how divergence occurs, we first need to find the leading term of a series expansion of our solution by writing it in the form $x \sim c\tau^p$. One way that such a divergence may arise is if the power in this leading term is $p < 0$. However, substituting such a term in the equation produces $p = 2/3$ and hence we must seek this divergence elsewhere. This is provided by the equation of motion itself (2) and it clearly occurs at the point $x = 1$! Therefore, we proceed to expand the solution of x near this singularity as a power series in τ whose first terms are:

$$x = 1 + a_1\tau^p + ..., p > 0 \tag{3}$$

where a_1, the leading term coefficient, and $p > 0$ can be determined by substituting in (2) and equating terms of leading order. This yields

$$p(p-1)a_1\tau^{p-2} = \frac{k\tau^{-2p}}{a_1^2} \tag{4a}$$
$$p - 2 = -2p \to p = 2/3 \tag{4b}$$
$$a_1 = -(\frac{9k}{2})^{1/3} \tag{4c}$$

Let us now solve Eq. (2) for x as a function of τ, by extending the above solution (3) to higher order terms multiplied by the coefficients $a_2, a_3, a_4, ...$. To achieve this, we need to expand each side of the equation (2) separately and equate coefficients of terms with the same power of τ.
RHS:

$$\frac{k}{(a_1\tau^{2/3} + a_2\tau + a_3\tau^{4/3})^2} = \frac{k}{a_1^2\tau^{4/3}(1 + a_2/a_1\tau^{1/3} + a_3/a_1\tau^{2/3})^2} \tag{5a}$$
$$= \frac{k}{a_1^2\tau^{4/3}}(1 - 2a_2/a_1\tau^{1/3} - 2a_3/a_1\tau^{2/3} + 3a_2^2/a_1^2\tau^{2/3} + 3a_3^2/a_1^2\tau^{4/3}) \tag{5b}$$
$$= -\frac{2a_1}{9}\left(\tau^{-4/3} - 2a_2/a_1\tau^{-1} - 2a_3/a_1\tau^{-2/3} + 3a_2^2/a_1^2\tau^{-2/3} + 3a_3^2/a_1^2\right) \tag{5c}$$
$$= -2a_1/9\tau^{-4/3} + 4a_2/9\tau^{-1} + 4a_3/9\tau^{-2/3} - 6a_2^2/9a_1\tau^{-2/3} - 6a_3^2/9a_1 \tag{5d}$$

LHS:

$$x = 1 + a_1\tau^{2/3} + a_2\tau + a_3\tau^{4/3} + a_4\tau^{5/3} + a_5\tau^2 + a_6\tau^{7/3} + a_7\tau^{8/3} + a_8\tau^3 \tag{6a}$$
$$x' = 2/3a_1\tau^{-1/3} + a_2 + 4/3a_3\tau^{1/3} + 5/3a_4\tau^{2/3} + 2a_5\tau + 7/3a_6\tau^{4/3} + 8/3a_7\tau^{5/3} + ... \tag{6b}$$
$$x'' = -2/9a_1\tau^{-4/3} + 4/9a_3\tau^{-2/3} + 10/9a_4\tau^{-1/3} + 2a_5 + 28/9a_6\tau^{1/3} + 40/9a_7\tau^{2/3} \tag{6c}$$
$$-2/9a_1\tau^{-4/3} + 4/9a_3\tau^{-2/3} + 10/9a_4\tau^{-1/3} + 2a_5 + 28/9a_6\tau^{1/3} + 1 + a_1\tau^{2/3} + \tag{6d}$$
$$+a_2\tau + a_3\tau^{3/4} + a_4\tau^{5/3} + a_5\tau^2 + a_6\tau^{7/3} = \tag{6e}$$
$$= 2a_1/9\tau^{-3/4} + 4a_2/9\tau^{-1} + 2a_3/9\tau^{-2/3} - 6a_2^2/9a_1\tau^{-2/3} - 6a_3^2/9a_1 \tag{6f}$$

Assuming that the series is developed in powers of $\tau^{1/3}$ and equating coefficients of the nest higher order terms gives:

$$\tau^{-1} : (4a_2/9) = 0, \quad \text{and} \quad \tau^{-1/3} : 10/9a_4 = 0;$$

from which we conclude $a_2 = 0$ and $a_4 = 0$. So the first few terms in the series expansion of $x(t)$ now take the form:

$$x = 1 + a_1\tau^{2/3} + a_3\tau^{3/4} + a_5\tau^2 + a_6\tau^{7/3} + ... \tag{7}$$

To find higher order terms, we extend the above series to higher powers of τ and rewrite the RHS expression as:

$$\frac{k}{(a_1\tau^{2/3}+a_3\tau^{4/3}+a_5\tau^2+a_6\tau^{7/3})^2} = \frac{k}{a_1^2\tau^{4/3}(1+\left(a_3/a_1\tau^{2/3}+a_5/a_1\tau^{4/3}+a_6/a_1\tau^{5/3}\right))^2} \tag{8a}$$

$$= \frac{k}{a_1^2\tau^{4/3}}(1-2a_3/a_1\tau^{2/3}-2a_5/a_1\tau^{4/3}-2a_6/a_1\tau^{5/3}+3(a_3/a_1\tau^{2/3}+a_5/a_1\tau^{4/3}+a_6/a_1\tau^{5/3})^2) \tag{8b}$$

$$= -\frac{2a_1}{9}\left(\tau^{-3/4}-2a_3/a_1\tau^{-3/2}-2a_5/a_1-2a_6/a_1\tau^{1/3}+3a_3^2/a_1^2+6a_3a_5^2/a_1^2\tau^{2/3}+6a_3a_6/a_1^2\tau\right) \tag{8c}$$

$$= -2a_1/9\tau^{-4/3}+4a_3/9\tau^{-2/3}+4a_5/9+4a_6/9\tau^{1/3}-6a_3^2/9a_1-12a_3a_5/9a_1\tau^{2/3}-12a_3a_6/9a_1\tau \tag{8d}$$

The same approach as above now yields for the next coefficients:

$$\tau^{1/3}: 24a_6/9=0 \quad \text{and} \quad \tau^{-2/3}: 4a_3/9=4a_3/9$$

Following this procedure, we are now able to show that a_3 is the second free term of this equation, while a_6=0, and the updates series expansion for the solution $x(t)$, now reads:

$$x = 1+a_1\tau^{2/3}+a_3\tau^{4/3}+a_5\tau^2+a_7\tau^{8/3}+.... \tag{9}$$

3 A POWER SERIES IN POWERS OF $\tau^{2/3}$

Based on the result (2) of the previous section we now make the crucial observation that we can substitute $u=\tau^{2/3}$ in the above series, thus obtaining the much simpler expansion:

$$x = 1+a_1u+a_3u^2+a_5u^3+a_7u^4+... \tag{10}$$

Setting $u=(t-t_*)^{2/3}$ into the Equation (2) and transforming the derivatives in terms of the new variable we find

$$\frac{du}{dt} = \frac{2}{3}u^{-1/2} \tag{11a}$$

$$\frac{d}{dt} = \frac{d}{du}\frac{du}{dt} = \frac{2(t-t_*)^{-1/3}}{3}\frac{d}{du} = \frac{2}{3}u^{-1/2}\frac{d}{du} \tag{11b}$$

$$\frac{d^2}{dt^2} = \left(\frac{d(\frac{2}{3}u^{-1/2}\frac{d}{du})}{du}\frac{du}{dt}\right) = \frac{-u^{3/2}}{3}\frac{dx}{du} + \frac{2u^{-1/2}}{3}\frac{d^2x}{du^2}\frac{2}{3}u^{-1/2} \tag{11c}$$

$$-\frac{2}{9}u^{-2}\frac{dx}{du}+\frac{4}{9}u^{-1}\frac{d^2x}{du^2}+x = \frac{k}{(1-x)^2} = \frac{-2a_1^3u^2}{a_1^2u^2(1+a_3/a_1u+a_5/a_1u^2+a_7/a_1u^3)^2} \tag{11d}$$

Thus the equation we get after the substitution $u=(t-t_*)^{2/3}$:

$$4ux''-2x'+9u^2x = \frac{-2a_1}{(1+\frac{a_3}{a_1}u+\frac{a_5}{a_1}u^2+...)^2} \tag{12}$$

It is interesting to note from the above eq. (3), that if the right hand side were zero, $x=0$ would be a regular singular point of the equation

$$4ux'' - 2x' + 9u^2 x = 0$$

as we know from Frobenius theory of linear second order ODEs.

The starting terms and the subsequent series expansions of the two linearly independent solutions are:

$$x_1(u) = \sum_{n=0}^{\infty} a_n u^n \tag{13a}$$

$$x_2(u) = u^{3/2} \sum_{n=0}^{\infty} b_n u^n \tag{13b}$$

Substituting the first solution in the homogenous equation yields:

$$4\sum_{n=0}^{\infty} a_n n(n-1)u^{n-1} - 2\sum_{n=0}^{\infty} a_n n u^{n-1} + 9\sum_{n=0}^{\infty} a_n u^{n+2} = 0 \tag{14}$$

$$4\sum_{n=2}^{\infty} a_n n(n-1)u^{n-1} - 2\sum_{n=1}^{\infty} a_n n u^{n-1} + 9\sum_{n=0}^{\infty} a_n u^{n+2} = 0 \tag{15}$$

Now, put $n- > n+2$ in the above equation to get:

$$4\sum_{n=0}^{\infty} a_n + 2(n+2)(n+1)u^{n+1} - 2\sum_{n=0}^{\infty} a_n + 1(n+1)u^n + 9\sum_{n=0}^{\infty} a_n u^{n+2} = 0 \tag{16}$$

Let's write out the first few terms in all the sums of the above equation:
$n = 0 : a_1 = 0$
$n = 1 : 8a_2 - 4a_2 = 0, a_2 = 0$
$n = 2 : 24a_3 - 6a_3 + 9a_0 = 0, a_3 = \frac{a_0}{2}$, and a_0 is free
$n = 3 : 48a_4 - 8a_4 + 9a_1 = 0, a_4 = 0$
$n = 4 : 80a_5 - 10a_5 + 9a_2 = 0, a_5 = 0$
$n = 5 : 120a_6 - 12a_6 + 9a_3 = 0, a_6 = \frac{a_0}{24}$
Let's examine now the $x_2(u)$ solution:

$$x_2(u) = \sum_{n=0}^{\infty} b_n u^{n+3/2} \tag{17a}$$

$$x_2'(u) = \sum_{n=0}^{\infty} b_n (n+\frac{3}{2})u^{n+1/2} \tag{17b}$$

$$x_2''(u) = \sum_{n=0}^{\infty} b_n (n+\frac{3}{2})(n+\frac{1}{2})u^{n-1/2} \tag{17c}$$

After substituting into homogenous equation we obtain:

$$4u^{1/2}\sum_{n=0}^{\infty} b_n (n+\frac{3}{2})(n+\frac{1}{2})u^n - 2u^{1/2}\sum_{n=0}^{\infty} b_n (n+\frac{3}{2})u^n + 9u^{7/2}\sum_{n=0}^{\infty} b_n u^n = 0 \tag{18}$$

$$4\sum_{n=0}^{\infty} b_n (n+\frac{3}{2})(n+\frac{1}{2})u^n - 2\sum_{n=0}^{\infty} b_n (n+\frac{3}{2})u^n + 9u^3\sum_{n=0}^{\infty} b_n u^n = 0 \tag{19}$$

$n = 0 : 3b_0 = 3b_0, b_0$ is free
$n = 1 : 4b_1(\frac{15}{4})u - 2b_1(\frac{5}{2})u = 0, b_1 = 0$
$n = 24b_2(\frac{7}{2})(\frac{3}{2})u^2 - 2b_2(\frac{7}{2})u^2 = 0, b_2 = 0$
$n = 34b_3(\frac{9}{2})(\frac{7}{2})u^3 - 2b_3(\frac{9}{2})u^3 + 9b_0 u^3 = 0, b_3 = -\frac{b_0}{6}$

Can we use these solutions to solve the non-homogeneous equation? Not really, only if we are very close to the singularity where $u \to 0$.

4 Higher order expansion of the u-power series

In this section, we carry out the same calculations as in Sections 2 and 3 to get the higher order a_i coefficients as a function of the free term a_3 and $a_1 = -(\frac{9k}{2})^{1/3}$. We will proceed by expanding $x = 1 + a_1 u + a_3 u^2 + a_5 u^3 + a_7 u^4 + ...$ Hence, let's expand the LHS and RHS and match the coefficients for u terms.

$$\frac{dx}{du} = a_1 + 2a_3 u + 3a_5 u^2 + 4a_7 u^3 \tag{20a}$$

$$\frac{d^2x}{du^2} = 2a_3 + 6a_5 u + 12a_7 u^2 \tag{20b}$$

$$-2\frac{dx}{du} + 4u\frac{d^2x}{du^2} + 9u^2 x = \frac{9ku^2}{(a_1 u + a_3 u^2 + a_5 u^3 + a_7 u^4)^2} \tag{20c}$$

$$-2\left(a_1 + 2a_3 u + 3a_5 u^2 + 4a_7 u^3\right) + 4u\left(2a_3 + 6a_5 u + 12a_7 u^2\right) + \tag{20d}$$

$$+9u^2\left(1 + a_1 u + a_3 u^2 + a_5 u^3 + a_7 u^4\right) = \tag{20e}$$

$$= \frac{-2a_1^3 u^2}{a_1^2 u^2 (1 + a_3/a_1 u + a_5/a_1 u^2 + a_7/a_1 u^3)^2} \tag{20f}$$

LHS:
$-2\left(a_1 + 2a_3 u + 3a_5 u^2 + 4a_7 u^3\right) + 4u\left(2a_3 + 6a_5 u + 12a_7 u^2\right) + 9u^2\left(1 + a_1 u + a_3 u^2 + a_5 u^3 + a_7 u^4\right) =$
$-2a_1 - 4a_3 u - 6a_5 u^2 - 8a_7 u^3 + 8a_3 u + 24a_5 u^2 + 48a_7 u^3 + 9u^2 + 9a_1 u^3 + 9a_3 u^4 + 9a_5 u^5 + 9a_7 u^6$
RHS:
$-2a_1\left(1 - 2[a_3/a_1 u + a_5/a_1 u^2 + a_7/a_1 u^3] + 3[a_3/a_1 u + a_5/a_1 u^2 + a_7/a_1 u^3]^2\right) = -2a_1 + 4a_3 u + 4a_5 u^2 + 4a_7 u^3 - 6a_3^2/a_1 u^2 - 12a_1(a_3 a_5/a_1^2 u^3 + a_3 a_7/a_1^2 u^4) - 6a_1(a_5^2/a_1^2 u^4 + 2a_5 a_7/a_1^2 u^5 + a_7^2/a_1^2 u^6)$

$$u^2 : 4a_5 - 6a_3^2/a_1 = 18a_5 + 9$$
$$a_5 = -3a_3^2/7a_1 - 9/14$$
$$u^3 : 4a_7 - 12a_3 a_5/a_1 = -8a_7 + 48a_7 + 9a_1$$
$$a_7 = -a_3 a_5/3a_1 - a_1/4$$

$$x = 1 + a_1 u + a_3 u^2 + (-3a_3^2/7a_1 - 9/14)u^3 + (-a_3 a_5/3a_1 - a_1/4)u^4 + ... \tag{21}$$

Section 5: The location of the singularity

Case 1: The location near the $x = 1$

$$x(t) = 1 + a_1(t - t_*)^{2/3} + a_3(t - t_*)^4/3 + ...$$

$$x(0) = 1 + a_1(-t_*)^{2/3} + a_3(-t_*)^4/3 + ...$$

$$x'(0) = \tfrac{2}{3}a_1(-t_*)^{-1/3} + \tfrac{4}{3}a_3(-t_*)^{1/3} + ...$$

Take, $x'(0) = 0$ and call $z = (-t_*)^{1/3}$:

$$0 = \tfrac{2}{3}a_1 z^{-1} + \tfrac{4}{3}a_3 z$$

$$z^2 = -\tfrac{a_1}{2a_3}$$

$$x(0) = 1 + a_1 z^2 + a_3 z^4$$

$$x(0) = 1 + a_1(-\tfrac{a_1}{2a_3}) + a_3\tfrac{a_1^2}{4a_3^2} = 1 - \tfrac{a_1^2}{4a_3}$$

$$a_3 = \tfrac{a_1^2}{4(1-x(0))}$$

$$z^2 = -\tfrac{2a_1(1-x(0))}{a_1^2} = \tfrac{2}{(9k/2)^{1/3}}(1-x_0)$$

$$z^2 = (-t_*)^{2/3}, t_*^{2/3} = \tfrac{2}{(9k/2)^{1/3}}(1-x_0)$$

$$x_0 < 1, t_* \subseteq \Re$$

Note that for $x(0) > 1$ we have periodic solutions. Of course for $x(0) < 1$, we also have at some $x'(0) = 0$ periodic solution. But we would need better approximations to see them (i.e include more terms in the series expansions of x beyond the $a_3 u^2$ term, $a_5 u^3$, $a_7 u^7$, etc.)

Case 2: Location of the singularity near the periodic solution

Define $z = (-t_*)^{1/3}$, with initial conditions $x(0) = 1$ and $x'(0) = 0$:

$$x(t) = 1 + a_1(t-t_*)^{2/3} + a_3(t-t_*)^4/3 + a_5(t-t_*)^2 + a_7(t-t_*)^{8/3} + \ldots \tag{22a}$$

$$x(0) = 1 + a_1(-t_*)^{2/3} + a_3(-t_*)^4/3 + a_5(-t_*)^2 + a_7(-t_*)^{8/3+\ldots} \tag{22b}$$

$$x'(0) = \frac{2}{3}a_1(-t_*)^{-1/3} + \frac{4}{3}a_3(-t_*)^{1/3} + 2a_5(-t_*) + \frac{8}{3}a_7(-t_*)^{5/3} + \ldots \tag{22c}$$

$$x'(0) = \frac{2}{3}a_1(z)^{-1} + \frac{4}{3}a_3(z) + 2a_5(z)^3 + \frac{8}{3}a_7(z)^5 + \ldots \tag{22d}$$

$$\frac{2}{3}a_1(z)^2 + \frac{4}{3}a_3(z)^4 + 2a_5(z)^6 + \frac{8}{3}a_7(z)^8 = 0 \tag{23}$$

$$\frac{4}{3} + \frac{4}{3}a_1 z^2 + \frac{4}{3}a_3 z^4 + \frac{4}{3}a_5 z^6 + \frac{4}{3}a_7 z^8 = 1 \tag{24}$$

By adding these two equations above, we get:

$$f(x) = \frac{1}{4} - \frac{2}{3}a_1 z^2 + \frac{2}{3}a_5 z^6 + \frac{4}{3}a_7 z^8 \tag{25}$$

By the method of Newton-Raphson, we find the complex singularities near the periodic solutions of potential in bounded region for $K = \frac{3}{27}$.

Algorithm 1. Matlab script for Newton-Raphson method: singularity analysis

```
a1=(1/2)^(1/3);
a5=(-9/14);
a7=a1/4;

f=inline('1/4␣+2/3*a1*x^2+2/3*a5*x^6+4/3*a7*x^8','x');
df=inline('+2*2/3*a1*x+6*2/3*a5*x^5+8*4/3*a7*x^7','x');

x1=-0.1 +0.6i
x2=x1-f(x1)/df(x1)
x3=x2-f(x2)/df(x2)
x4=x3-f(x3)/df(x3)
x5=x4-f(x4)/df(x4)
x6=x5-f(x5)/df(x5)
x7=x6-f(x6)/df(x6)
x8=x7-f(x7)/df(x7)
x9=x8-f(x8)/df(x8)
x10=x9-f(x9)/df(x9)

p1=[4/3*a7, 0 , 2/3*a5, 0 , 0, 0, 2/3*a1, 0, 1/4]
roots(p1)
x1 = -0.10000 + 0.60000i
x2 = -0.055472 + 0.778807i
x3 = -0.14178 + 0.69515i
x4 = -0.12552 + 0.76228i
x5 = -0.14192 + 0.75677i
x6 = -0.14145 + 0.75799i
x7 = -0.14146 + 0.75799i
x8 = -0.14146 + 0.75799i
x9 = -0.14146 + 0.75799i
x10 = -0.14146 + 0.75799i
p1 =

Columns 1 through 8:

0.26457   0.00000   -0.42857   0.00000   0.00000   0.00000   0.52913   0.00000

Column 9:

0.25000

ans =

-1.22463 + 0.36775i
-1.22463 - 0.36775i
1.22463 + 0.36775i
1.22463 - 0.36775i
-0.14146 + 0.75799i
```

```
−0.14146 − 0.75799 i
0.14146 + 0.75799 i
0.14146 − 0.75799 i
```

5 SOLUTIONS OF MEMS IN REAL SPACE

Let us now plot the potential of the MEMS problem

$$V(x) = \frac{x^2}{2} - \frac{k}{(1-x)} \tag{26}$$

as a function of x for different values of the parameter $k > 0$ to understand the different types of solutions in connection with the results of our power series in complex time.

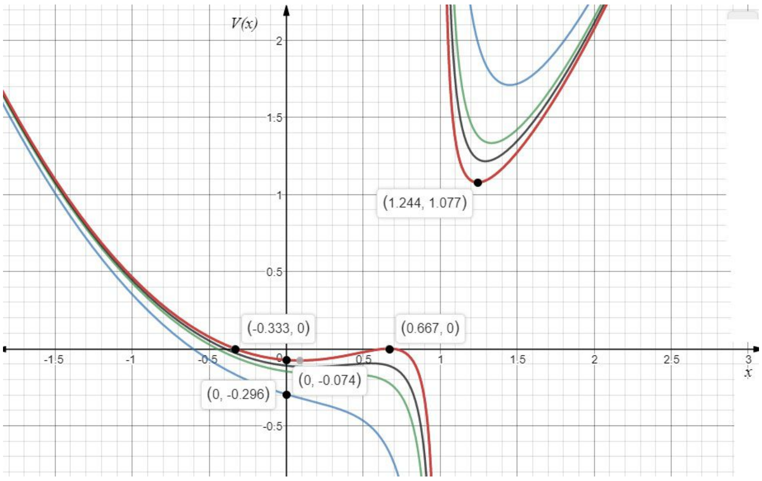
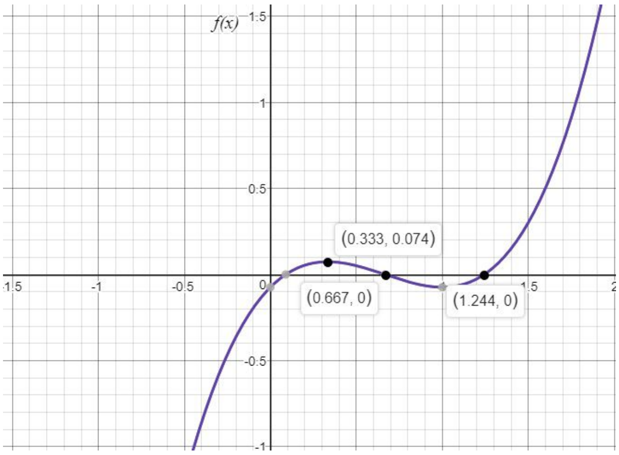


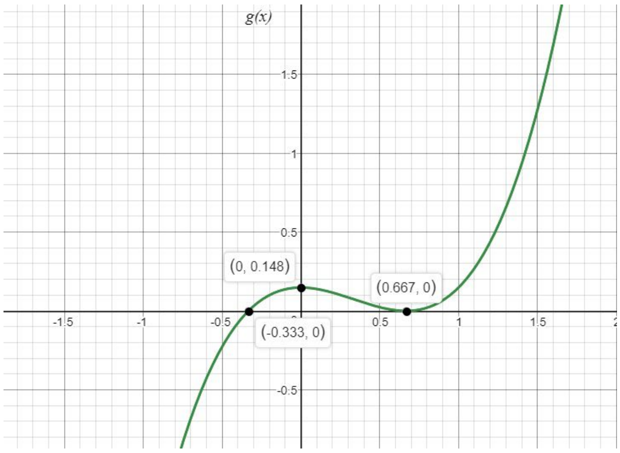
Figure 2: The potential of the function for $\frac{2}{27} < k < \frac{4}{27}$

Figure 2 above represents the potential function of the MEMS problem, where the solution at $x = 1$ goes to infinity, so $x = 1$ is a vertical asymptote. By computing critical points, i.e. maxima and minima of the potential function, we have found conditions for the k parameter so that periodic solutions exist. Note that such solutions can be found in the domain $0 < x < 1$ provided k satisfies the inequalities $\frac{2}{27} < k < \frac{4}{27}$. These are the physically meaningful solutions of the MEMS problem. However, observe from Figure 2 that non-physical periodic solutions also exist in the domain $x > 1$. As we demonstrate below these regions are also found by our singularity analysis, since the correspond to cases where t^* is complex.

On the other hand, unbounded solutions shown in Figure 2, can also be determined by our singularity analysis. As our calculations below demonstrate these can be found by requiring that our singularities t^* are real.



(a) The graph of the function $f(x)$ for $K < \frac{4}{27}$



(b) The graph of the function for $K > \frac{2}{27}$

$$\ddot{x} = -x + \frac{k}{(1-x)^2} = -V'(x) \tag{27a}$$
$$V(x) = \frac{x^2}{2} + \frac{k}{x-1} \tag{27b}$$
$$V'(x) = x - \frac{k}{(1-x)^2} = 0 \tag{27c}$$
$$\frac{x(1-x)^2 - k}{(1-x)^2} = 0 \tag{27d}$$
$$f(x) = x^3 - 2x^2 + x - k = 0 \tag{27e}$$
$$f'(x) = 3x^2 - 4x + 1 = 0 \tag{27f}$$
$$x_1 = \frac{1}{3}, x_2 = 1 \tag{27g}$$
$$f(1/3) = 1/27 - 6/27 + 9/27 - k > 0 \tag{27h}$$
$$k < \frac{4}{27} \tag{27i}$$

$$V(x) = 0 : \frac{x^2}{2} = \frac{k}{1-x} \tag{28a}$$
$$x^2(1-x) = 2k \tag{28b}$$
$$g(x) = x^3 - x^2 + 2k \tag{28c}$$
$$g'(x) = 3x^2 - 2x = 0 \tag{28d}$$
$$x_1 = 0, x_2 = \frac{2}{3} \tag{28e}$$
$$g(2/3) = (2/3)^3 - (2/3)^2 + 2k > 0 \tag{28f}$$
$$k > \frac{2}{27} \tag{28g}$$

6 Conclusion and topics for future research

7 References

Reference

[1] Weideman J. A. 2003. "Computing the Dynamics of Complex Singularities of Nonlinear PDEs". Vol. 2, No. 2, pp. 171–186. Society for Industrial and Applied Mathematics

[2]Pauls W, Frisch U. 2007. "A Borel Transform Method for Locating Singularities of Taylor and Fourier Series". Vol. 127, No. 6. Journal of Statistical Physics.

[3] Baker G., Xie C. 2010. "Singularities in the complex physical plane for deep water waves". J. Fluid Mech.

[4] Skrzypacz, Piotr, S. Kadyrov, D. Nurakhmetov, and D. Wei. “Analysis of Dynamic Pull-in Voltage of a Nonlinear Material NEMS Model.” accepted for publication in Materials Today: Proceedings.