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Synchronous Optimal Modulation: Frequency or Time Domain, Global or Constrained Optimization? 40 Years to the First Formulation

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***Abstract–*Alternative approaches to Total Harmonic Distortion (THD) minimization comprise frequency or time domain THD calculations and global or constrained optimization problem formulation. Time domain constrained optimization formulation suggested in 1977 that seems most adequate from computational efficiency and robustness does not attract sufficient attention it deserves from the modern researchers. Time domain current THD / harmonic loss constrained optimization is illustrated by single-phase inverter examples. New results on multilevel THD optimization for variable voltage levels are presented.**

***Index Terms*–DC-AC Power Conversion, Pulse Width Modulation, Harmonic Distortion**

I. Introduction

Over the past years, power electronics research community has shown significant interest in low frequency modulation and THD analysis ([1], [2]). For pure inductive load, current THD actually becomes voltage frequency Weighted THD (WTHD). This approximation is accurate for inductance dominated RL-loads with relatively large time constants ([1], [2]).

Low frequency synchronous modulation comprises Selective Harmonic Elimination (SHE) and THD (harmonic loss) minimization ([1]-[3]). THD minimization approaches may be classified as: 1) Frequency domain or time domain dependent on the way THD is calculated; 2) Global or constrained optimization dependent on minimization problem formulation.

Power electronics engineers use to account for a limited harmonic count of 51 as recommended by IEEE Standard 519. This causes voltage THD underestimation (for current THD, due to filtering effect, harmonic count may be even less).

Recent papers [4], [5] took into account infinite harmonic number for voltage THD by calculating voltage waveform mean square. In [6], the time domain approach was applied to current THD (WTHD) of a single-phase multilevel inverter with a staircase modulation. The required current waveform was obtained by time integration of respective voltage one.

In many papers, minimal THD problem is formulated as a global optimization one with equality constraints (e.g., desired fundamental harmonic) being heuristically included in the cost function. Different “Artificial Intelligence” optimization techniques like Genetic Algorithms, Simulated Annealing, Artificial Neutral Networks, Particle Swarm Optimization, Bee Algorithm and other are then used to solve the problem ([7]).

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This approach may suffer from computational inefficiency (need for parallel calculations ([7])) and solutions sensitivity.

Time domain constrained optimization problem formulation seems to be the most adequate one.

The constrained optimization problem formulation is used by Prof. J. Holtz and his apprentices ([1], [2]). However, they employ frequency domain harmonic loss factor calculations.

THD optimization results deliver a theoretical lower bound and may serve as a benchmark for other methods, e.g., model predictive control [8] that has excellent transient and other performances but may compromise current quality.

Time domain constrained optimization is illustrated below for single-phase two- and multilevel inverters. Finally, it is necessary to acknowledge that this approach was first suggested for two-level inverters 40 years ago independently by Prof. G. Buja ([9]) and Prof. S. Halasz ([10]).

II. Current THD Time Domain Constrained Optimization for a Single-Phase Inverter

For a 2-level single-phase inverter with 3 switching angles, voltage and current normalized positive half-waves with a quarter-wave symmetry are shown in Fig.1,a.

Modulation index m by Fourier series fundamental term

$$m = (4/\pi)[\cos\alpha_1 - \cos\alpha_2 + \cos\alpha_3]. \quad (1)$$

A closed-form expression for current ripple Normalized Mean Square (NMS) is obtained by calculating squared piece-wise linear current waveform average on a quarter of a period

$$NMS_I(m) = (2/\pi)\int_0^{\pi/2} i^2(\tau)d\tau - 0.5m^2 . \quad (2)$$

As a load inductance equally impacts all current harmonics, in Fig.1 it is assumed unity and normalized fundamental current harmonic equals to voltage modulation index m.

Based on Fig.1,a graph, current ripple NMS expression

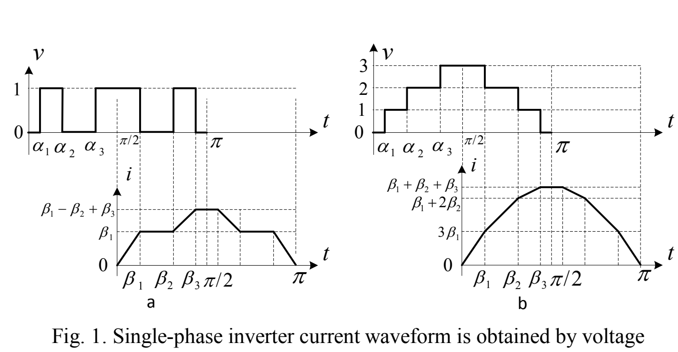

Fig. 1. Single-phase inverter current waveform is obtained by voltage waveform integration: a) 2-level 3 angles; b) 4-level 3 angles.

$$NMS_I(m) = (\beta_1 - \beta_2 + \beta_3)^2 - \frac{4\beta_3^3 + 2\beta_2^3 + 4\beta_1^3}{3\pi} + $$
$$+ 2\frac{\beta_1\beta_2^2 - \beta_1\beta_3^2 + \beta_2\beta_3^2}{\pi} - 0.5m^2; \quad (3)$$
$$\beta_1 = \pi/2 - \alpha_3, \quad \beta_2 = \pi/2 - \alpha_2, \quad \beta_3 = \pi/2 - \alpha_1.$$

Additionally, there are switching angles limitations

$$0 < \alpha_1 < \alpha_2 < \alpha_3 < \pi/2. \quad (4)$$

Formulas (1), (3), (4) can be extended for any angle count.

By frequency domain definition, current THD as WTHD

$$THD_I(m),\% = \frac{\sqrt{\sum_{n\neq 1}^{\infty} I_n^2(m)}}{I_1(m)} = \frac{\sqrt{\sum_{n\neq 1}^{\infty}\left[\frac{V_n(m)}{n}\right]^2}}{V_1(m)}. \quad (5)$$

By Parseval theorem, for current harmonic effective values

$$\frac{2}{\pi}\int_0^{\pi/2} i^2(\tau)d\tau = \sum_{n=1}^{\infty} I_n^2 \quad (6)$$

and from (2), (5), (6) current THD becomes

$$THD_I(m),\% = \frac{\sqrt{2NMS_I(m)}}{m}\cdot 100. \quad (7)$$

Using (7) instead of (5) is computationally beneficial because multiple Fourier harmonic calculations are eliminated.

Current THD minimization problem is formulated as a constrained optimization one: for a given m, $0 < m < 4/\pi$, find switching angles that satisfy equality constraint (1), inequality constraints (4) and minimize the cost function (7).

This kind of constrained optimization problems can be efficiently solved, for example, by Matlab *fmincon* function.

Fig.2 presents optimal switching angles trajectories for a 6-angle strategy. Fig. 3 demonstrates improvement in optimal THD for switching angles amount increase.

For a 3-phase inverter, additional equality constraints on line voltage switching angles arise due to the 3-phase symmetry. For a single-phase inverter, the maximum modulation index is achieved for a square-wave while for a 3-phase – for a six-step line voltage waveform. Hence maximum 3-phase modulation index becomes $2\sqrt{3}/\pi \cong 1.10$.

Positive voltage and current half-waves of a 4-level 3-angle single-phase inverter with a staircase modulation are shown in Fig.1,b. Modulation index m by Fourier fundamental term

$$m = (4/\pi)[\cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3]. \quad (8)$$

For Fig.1,b current waveform, current ripple NMS becomes

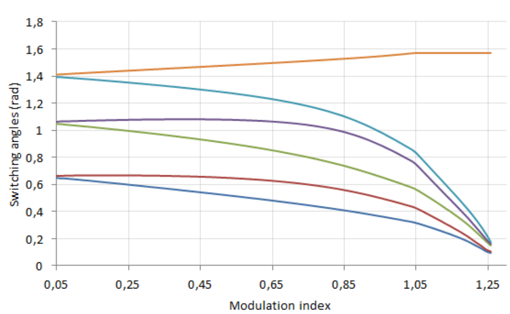

Fig. 2. Optimal angles for a 2-level 6-angle modulation.

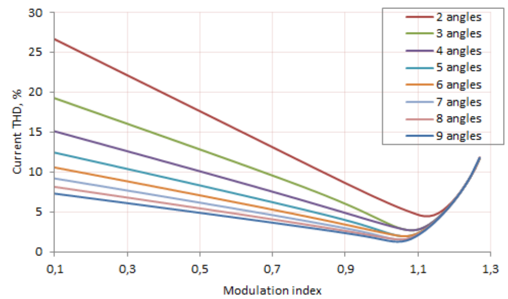

Fig. 3. Minimal THD comparison for 2-level modulation with 2-9 angles.

$$NMS_I(m) = (\beta_1 + \beta_2 + \beta_3)^2 - \frac{4\beta_3^3 + 6\beta_2^3 + 8\beta_1^3}{3\pi} - $$
$$- 2\frac{\beta_1\beta_2^2 + \beta_1\beta_3^2 + \beta_2\beta_3^2}{\pi} - 0.5m^2; \quad (9)$$
$$\beta_1 = \pi/2 - \alpha_3, \quad \beta_2 = \pi/2 - \alpha_2, \quad \beta_3 = \pi/2 - \alpha_1,$$

and there are switching angle limitations (4).

Formulas (8) and (9) can be extended for any angle count ([6]) and optimal modulation problem is formulated as a constrained optimization one similar to the previous case.

Fig. 4 and 5 present optimal angle trajectories and minimal current THD for an 8-angle 9-level inverter. Obviously, they are relevant for switching angles count less than 8 as well.

For a 3-phase cascade H-bridge inverter, voltage and current THD time domain constrained optimization is presented in [11]. For voltage THD, this was addressed in [12] where the authors used frequency domain approach and had to spend all their effort on finding closed-form expressions for infinite trigonometric sums to account for infinite harmonic count.

If the modulation index constraint is excluded from the problem formulation, a globally minimal THD with implicit modulation index will be found. For that, modulation index in (7) has to be expressed by switching angles using (1), (8).

For a given non-uniform level distribution, the problem

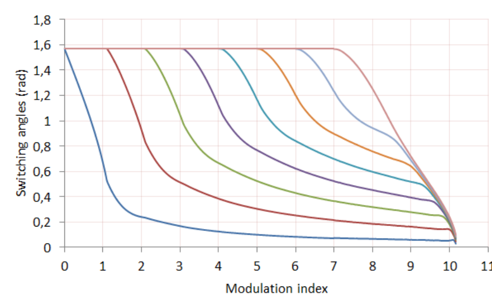

Fig. 4. Current optimal angles for an 8-angle staircase modulation.

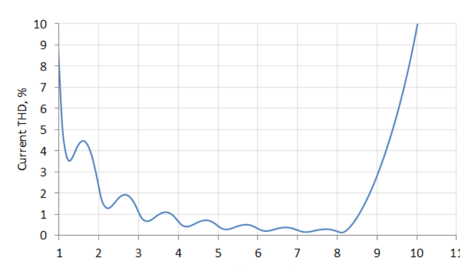

Fig. 5. Minimal current THD for an 8-angle 9-level staircase modulation.

formulation is similar to that for a uniform case. For variable voltage levels, there is a globally minimal THD. The variable levels have to be treated as optimization parameters along with the switching angles. If maximum level is fixed to unity, additional inequality constraints will be $0 < l_1 < l_2 < ... < 1$.

Current NMS formulas for variable levels become a bit cumbersome. However, a lack of compact NMS formula is not an obstacle for applying time domain approach. Current NMS algorithmic calculations are even more efficient in terms of operations count compared to analytical formulas.

Voltage THD minimization for non-uniform levels in [13] employed time domain global optimization. This probably explains the solution sensitivity - for a given level distribution, voltage THD dependence on *m* is a non-smooth "noisy" one.

The results of voltage and current THD global time domain optimization for uniform and variable levels performed using Matlab *fmincon* function are presented in Tables I (THD), II (levels) and Table III (angles in degrees and implicit *m* values).

Voltage THD values for uniform levels practically coincide with [5], [6]. From Table I, there is no much improvement in the voltage THD by variable levels. From current THD perspective, variable levels are beneficial and the improvement progressively increases with the level count.

III. Conclusion

The foundations of synchronous optimal modulation theory were laid 40 years ago. However, that time microcomputer capabilities did not to allow making full use of synchronous optimal modulation theory benefits in industrial products. Modern cheap and powerful industrial microcontrollers stimulate the interest towards optimal modulation for THD minimization in grid-connected and loss minimization in motor control applications. Optimization results may serve as a benchmark for other methods like model predictive control.

This letter is aimed to suggest that time domain constrained optimization formulation is most adequate for synchronous optimal modulation problem with the major benefits being computational effectiveness and robustness. It is notable that such kind of formulation was first suggested 40 years ago back in 1977 by Prof. G. Buja ([9]) and Prof. S. Halasz ([10]).

TABLE I
GLOBALLY MINIMAL THD (%) FOR UNIFORM AND VARIABLE LEVELS

Levels	THDv, Uni	THDv, Non-Uni	THDi, Uni	THDi, Non-Uni
3 (2)	16.42	16.38	1.50	1.41
4 (3)	11.53	11.47	0.769	0.683
5 (4)	8.90	8.83	0.474	0.402
6 (5)	7.26	7.18	0.324	0.264
7 (6)	6.13	6.06	0.238	0.187
8 (7)	5.31	5.23	0.183	0.139
9 (8)	4.68	4.61	0.144	0.108

TABLE II
OPTIMAL VOLTAGE LEVEL DISTRIBUTION FOR VARIABLE LEVELS

Levels	Voltage Optimal Levels	Current Optimal Levels
3	0, 0.523, 1	0, 0.560, 1
4	0, 0.355, 0.696, 1	0, 0.387, 0.738, 1
5	0, 0.269, 0.532, 0.780, 1	0, 0.296, 0.576, 0.821, 1
6	0, 0.216, 0.430, 0.636, 0.830, 1	0, 0.239, 0.470, 0.684, 0.867, 1
7	0, 0.181, 0.361, 0.536, 0.704, 0.862, 1	0, 0.2, 0.397, 0.582, 0.752, 0.896, 1
8	0, 0.156, 0.311, 0.463, 0.611, 0.752, 0.885, 1	0, 0.173, 0.342, 0.506, 0.659, 0.798, 0.916, 1
9	0, 0.137, 0.273, 0.407, 0.538, 0.666, 0.788, 0.901, 1	0, 0.152, 0.301, 0.447, 0.586, 0.715, 0.832, 0.930, 1

TABLE III
OPTIMAL ANGLES DISTRIBUTION FOR VARIABLE LEVELS

Level Count	Voltage Optimal Angles	Current Optimal Angles
3	13.5, 42.7 (m=1.09)	15.7, 49.3 (m=1.05)
4	9.47, 29.2, 51.9 (m=1.06)	10.9, 33.4, 58.7 (m=1.03)
5	7.31, 22.3, 38.4, 57.5 (m=1.05)	8.37, 25.4, 43.5, 64.1 (m=1.02)
6	5.96, 18.0, 30.7, 44.6, 61.3 (m=1.04)	6.79, 20.5, 34.8, 50.1, 67.7 (m=1.02)
7	5.02, 15.2, 25.7, 36.8, 49.2, 64.1 (m=1.03)	5.70, 17.2, 29.0, 41.4, 54.8, 70.3 (m=1.01)
8	4.34, 13.1, 22.1, 31.4, 41.5, 52.7, 66.3 (m=1.03)	4.91, 14.8, 24.9, 35.3, 46.3, 58.3, 72.3 (m=1.01)
9	3.83, 11.5, 19.4, 27.5, 36.0, 45.2, 55.5, 68.0 (m=1.02)	4.32, 13.0, 21.8, 30.9, 40.3, 50.2, 61.2, 73.9 (m=1.01)

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