# ON THE SOLVABILITY OF THE BRINKMAN-FORCHHEIMER-EXTENDED DARCY EQUATION 

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#### Abstract

The nonlinear Brinkman-Forchheimer-extended Darcy equation is used to model some porous medium flow in chemical reactors of packed bed type. The results concerning the existence and uniqueness of a weak solution are presented for nonlinear convective flows in medium with nonconstant porosity and for small data. Furthermore, the finite element approximations to the flow profiles in the fixed bed reactor are presented for several Reynolds numbers at the non-Darcy's range.


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## 1 Introduction

In this section we introduce the mathematical model describing incompressible isothermal flow in porous medium without reaction. The considered equations for the velocity and pressure fields are for flows in fluid saturated porous media. Most of research results for flows in porous media are based on the Darcy equation which is considered to be a suitable model at a small range of Reynolds numbers. However, there are restrictions of Darcy equation for modeling some porous medium flows, e.g. in closely packed medium, saturated fluid flows at slow velocity but with relatively large Reynolds numbers. The flows in such closely packed medium behave nonlinearly and can not be modelled accurately by the Darcy equation which is linear. The deficiency can be circumvented with the Brinkman-Forchheimer-extended Darcy law for flows in closely packed media, which leads to the following model: Let $\Omega \subset \mathbb{R}^{n}$, $n=2,3$, represent the reactor channel. We denote its boundary by $\Gamma=\partial \Omega$. The conservation of volume-averaged values of momentum and mass in the packed reactor reads as follows

$$
\begin{align*}
-\operatorname{div}(\varepsilon \nu \nabla \boldsymbol{u}-\varepsilon \boldsymbol{u} \otimes \boldsymbol{u})+\frac{\varepsilon}{\varrho} \nabla p+\sigma(\boldsymbol{u})=\boldsymbol{f} & \text { in } \Omega,  \tag{1}\\
\operatorname{div}(\varepsilon \boldsymbol{u})=0 & \text { in } \Omega,
\end{align*}
$$

[^0]where $\boldsymbol{u}: \Omega \rightarrow \mathbb{R}^{n}, p: \Omega \rightarrow \mathbb{R}$ denote the unknown velocity and pressure, respectively. The positive quantity $\varepsilon=\varepsilon(\boldsymbol{x})$ stands for porosity which describes the proportion of the non-solid volume to the total volume of material and varies spatially in general. The expression $\sigma(\boldsymbol{u})$ represents the friction forces caused by the packing and will be specified later on. The righthand side $f$ represents an outer force (e.g. gravitation), $\varrho$ the constant fluid density and $\nu$ the constant kinematic viscosity of the fluid, respectively. The expression $\boldsymbol{u} \otimes \boldsymbol{u}$ symbolizes the dyadic product of $\boldsymbol{u}$ with itself.

The formula given by Ergun [3] will be used to model the influence of the packing on the flow inertia effects

$$
\begin{equation*}
\sigma(\boldsymbol{u})=150 \nu \frac{(1-\varepsilon)^{2}}{\varepsilon^{2} d_{p}^{2}} \boldsymbol{u}+1.75 \frac{1-\varepsilon}{\varepsilon d_{p}} \boldsymbol{u}|\boldsymbol{u}| . \tag{2}
\end{equation*}
$$

Thereby $d_{p}$ stands for the diameter of pellets and $|\cdot|$ denotes the Euclidean vector norm. The linear term in (2) accounts for the head loss according to Darcy and the quadratic term according to Forchheimer law, respectively. For the derivation of the equations, modelling and homogenization questions in porous media we refer to e.g. [2, 4. . To close the system (1) we prescribe Dirichlet boundary condition

$$
\begin{equation*}
\left.\boldsymbol{u}\right|_{\Gamma}=\boldsymbol{g} \tag{3}
\end{equation*}
$$

whereby

$$
\begin{equation*}
\int_{\Gamma_{i}} \varepsilon \boldsymbol{g} \cdot \boldsymbol{n} d s=0 \tag{4}
\end{equation*}
$$

has to be fulfilled on each connected component $\Gamma_{i}$ of the boundary $\Gamma$. We remark that in the case of polygonally bounded domain the outer normal vector $\boldsymbol{n}$ has jumps and thus the above integral should be replaced by a sum of integrals over each side of $\Gamma$. The distribution of porosity $\varepsilon$ is assumed to satisfy the following bounds

$$
\begin{equation*}
0<\varepsilon_{0} \leq \varepsilon(\boldsymbol{x}) \leq \varepsilon_{1} \leq 1 \quad \forall \boldsymbol{x} \in \Omega \tag{A1}
\end{equation*}
$$

with some constants $0<\varepsilon_{0}, \varepsilon_{1} \leq 1$.
A comprehemsive account of fluid flows through porous media beyond the Darcy law's valid regimes and classified by the Reynolds number, can be found in, e.g., [10]. Also, see [11] for simulating pumped water levels in abstraction boreholes using such nonlinear DarcyForchheimer law, and [12, [13], and [14] for recent referenes on this model.

In the next section we use the porosity distribution which is estimated for packed beds consisting of spherical particles and takes the near wall channelling effect into account. This kind of porosity distribution obeys assumption A1.

Let us introduce dimensionless quantities

$$
\boldsymbol{u}^{*}=\frac{\boldsymbol{u}}{U_{0}}, \quad p^{*}=\frac{p}{\varrho U_{0}^{2}}, \quad \boldsymbol{x}^{*}=\frac{\boldsymbol{x}}{d_{p}}, \quad \boldsymbol{g}^{*}=\frac{\boldsymbol{g}}{U_{0}},
$$

whereby $U_{0}$ denotes the magnitude of some reference velocity. For simplicity of notation we omit the asterisks. Then, the reactor flow problem reads in dimensionless form as follows

$$
\left\{\begin{array}{rllll}
-\operatorname{div}\left(\frac{\varepsilon}{R e} \nabla \boldsymbol{u}-\varepsilon \boldsymbol{u} \otimes \boldsymbol{u}\right)+\varepsilon \nabla p+\frac{\alpha}{R e} \boldsymbol{u}+\beta \boldsymbol{u}|\boldsymbol{u}| & =\boldsymbol{f} & \text { in } \Omega,  \tag{5}\\
\operatorname{div}(\varepsilon \boldsymbol{u}) & =0 & \text { in } \Omega, \\
\boldsymbol{u} & =\boldsymbol{g} & \text { on } & \Gamma,
\end{array}\right.
$$

where

$$
\begin{equation*}
\alpha(\boldsymbol{x})=150 \kappa^{2}(\boldsymbol{x}), \quad \beta(\boldsymbol{x})=1.75 \kappa(\boldsymbol{x}) \tag{6}
\end{equation*}
$$

with

$$
\begin{equation*}
\kappa(\boldsymbol{x})=\frac{1-\varepsilon(\boldsymbol{x})}{\varepsilon(\boldsymbol{x})} \tag{7}
\end{equation*}
$$

and the Reynolds number is defined by

$$
R e=\frac{U_{0} d_{p}}{\nu} .
$$

The existence and uniqueness of solution of the nonlinear model (5) with constant porosity and without the convective term has been established in [5]. We will extend this result to the case when the porosity depends on the location and with the convective term in this work.

Remark 1 (5) becomes a Navier-Stokes problem if $\varepsilon \equiv 1$.
Notation Throughout the work we use the following notations for function spaces. For $m \in \mathbb{N}_{0}, p \geq 1$ and bounded subdomain $G \subset \Omega$ let $W^{m, p}(G)$ be the usual Sobolev space equipped with norm $\|\cdot\|_{m, p, G}$. If $p=2$, we denote the Sobolev space by $H^{m}(G)$ and use the standard abbreviations $\|\cdot\|_{m, G}$ and $|\cdot|_{m, G}$ for the norm and seminorm, respectively. We denote by $D(G)$ the space of $C^{\infty}(G)$ functions with compact support contained in $G$. Furthermore, $H_{0}^{m}(G)$ stands for the closure of $D(G)$ with respect to the norm $\|\cdot\|_{m, G}$. The counterparts spaces consisting of vector valued functions will be denoted by bold faced symbols like $\boldsymbol{H}^{m}(G):=\left[H^{m}(G)\right]^{n}$ or $\boldsymbol{D}(G):=[D(G)]^{n}$. The $L^{2}$ inner product over $G \subset \Omega$ and $\partial G \subset \partial \Omega$ will be denoted by $(\cdot, \cdot)_{G}$ and $\langle\cdot, \cdot\rangle_{\partial G}$, respectively. In the case $G=\Omega$ the domain index will be omitted. In the following we denote by $C$ the generic constant which is usually independent of the model parameters, otherwise dependences will be indicated.

## 2 Existence and uniqueness results

In the following the porosity $\varepsilon$ is assumed to belong to $W^{1,3}(\Omega) \cap L^{\infty}(\Omega)$. We start with the weak formulation of problem (5) and look for its solution in suitable Sobolev spaces.

### 2.1 Variational formulation

Let

$$
L_{0}^{2}(\Omega):=\left\{v \in L^{2}(\Omega):(v, 1)=0\right\}
$$

be the space consisting of $L^{2}$ functions with zero mean value. We define the spaces

$$
\boldsymbol{X}:=\boldsymbol{H}^{1}(\Omega), \quad \boldsymbol{X}_{0}:=\boldsymbol{H}_{0}^{1}(\Omega), \quad Q:=L^{2}(\Omega), \quad M:=L_{0}^{2}(\Omega),
$$

and

$$
\boldsymbol{V}:=\boldsymbol{X}_{0} \times M .
$$

Let us introduce the following bilinear forms

$$
\begin{array}{ll}
a: \boldsymbol{X} \times \boldsymbol{X} \rightarrow \mathbb{R}, & a(\boldsymbol{u}, \boldsymbol{v})=\frac{1}{R e}(\varepsilon \nabla \boldsymbol{u}, \nabla \boldsymbol{v}), \\
b: \boldsymbol{X} \times Q \rightarrow \mathbb{R}, & b(\boldsymbol{u}, q)=(\operatorname{div}(\varepsilon \boldsymbol{u}), q), \\
c: \boldsymbol{X} \times \boldsymbol{X} \rightarrow \mathbb{R}, & c(\boldsymbol{u}, \boldsymbol{v})=\frac{1}{R e}(\alpha \boldsymbol{u}, \boldsymbol{v}) .
\end{array}
$$

Furthermore, we define the semilinear form

$$
d: \boldsymbol{X} \times \boldsymbol{X} \times \boldsymbol{X} \rightarrow \mathbb{R}, \quad d(\boldsymbol{w} ; \boldsymbol{u}, \boldsymbol{v})=(\beta|\boldsymbol{w}| \boldsymbol{u}, \boldsymbol{v}),
$$

and trilinear form

$$
n: \boldsymbol{X} \times \boldsymbol{X} \times \boldsymbol{X} \rightarrow \mathbb{R}, \quad n(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v})=((\varepsilon \boldsymbol{w} \cdot \nabla) \boldsymbol{u}, \boldsymbol{v}) .
$$

We set

$$
A(\boldsymbol{w} ; \boldsymbol{u}, \boldsymbol{v}):=a(\boldsymbol{u}, \boldsymbol{v})+c(\boldsymbol{u}, \boldsymbol{v})+n(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v})+d(\boldsymbol{w} ; \boldsymbol{u}, \boldsymbol{v}) .
$$

Multiplying momentum and mass balances in (5) by test functions $\boldsymbol{v} \in \boldsymbol{X}_{0}$ and $q \in M$, respectively, and integrating by parts implies the weak formulation:

Find $(\boldsymbol{u}, p) \in \boldsymbol{X} \times M$ with $\left.\boldsymbol{u}\right|_{\Gamma}=\boldsymbol{g}$ such that

$$
\begin{equation*}
A(\boldsymbol{u} ; \boldsymbol{u}, \boldsymbol{v})-b(\boldsymbol{v}, p)+b(\boldsymbol{u}, q)=(\boldsymbol{f}, \boldsymbol{v}) \quad \forall(\boldsymbol{v}, q) \in \boldsymbol{V} \tag{8}
\end{equation*}
$$

First, we recall the following result from [6]:
Theorem 2 The mapping $u \mapsto \varepsilon u$ is an isomorphism from $H^{1}(\Omega)$ onto itself and from $H_{0}^{1}(\Omega)$ onto itself. It holds for all $u \in H^{1}(\Omega)$

$$
\|\varepsilon u\|_{1} \leq C\left\{\varepsilon_{1}+|\varepsilon|_{1,3}\right\}\|u\|_{1} \quad \text { and } \quad\left\|\frac{u}{\varepsilon}\right\|_{1} \leq C\left\{\varepsilon_{0}^{-1}+\varepsilon_{0}^{-2}|\varepsilon|_{1,3}\right\}\|u\|_{1} .
$$

In the following the closed subspace of $\boldsymbol{H}_{0}^{1}(\Omega)$ defined by

$$
\boldsymbol{W}=\left\{\boldsymbol{w} \in \boldsymbol{H}_{0}^{1}(\Omega): \quad b(\boldsymbol{w}, q)=0 \quad \forall q \in L_{0}^{2}(\Omega)\right\} .
$$

will be employed. Next, we establish and prove some properties of trilinear form $n(\cdot, \cdot, \cdot)$ and nonlinear form $d(\cdot ; \cdot, \cdot)$.

Lemma 3 Let $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}^{1}(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{H}^{1}(\Omega)$ with $\operatorname{div}(\varepsilon \boldsymbol{w})=0$ and $\left.\boldsymbol{w} \cdot \boldsymbol{n}\right|_{\Gamma}=0$. Then we have

$$
\begin{equation*}
n(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v})=-n(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}) . \tag{9}
\end{equation*}
$$

Furthermore, the trilinear form $n(\cdot, \cdot, \cdot)$ and the nonlinear form $d(\cdot ; \cdot, \cdot)$ are continuous, i.e.

$$
\begin{align*}
|n(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq C_{\varepsilon}\|\boldsymbol{u}\|_{1}\|\boldsymbol{v}\|_{1}\|\boldsymbol{w}\|_{1} & \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^{1}(\Omega),  \tag{10}\\
|d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq C_{\varepsilon}\|\boldsymbol{u}\|_{1}\|\boldsymbol{v}\|_{1}\|\boldsymbol{w}\|_{1} & \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^{1}(\Omega), \tag{11}
\end{align*}
$$

and for $\boldsymbol{u} \in \boldsymbol{W}$ and for a sequence $\boldsymbol{u}^{k} \in \boldsymbol{W}$ with $\lim _{k \rightarrow \infty}\left\|\boldsymbol{u}^{k}-\boldsymbol{u}\right\|_{0}=0$, we have also

$$
\begin{equation*}
\lim _{k \rightarrow \infty} n\left(\boldsymbol{u}^{k}, \boldsymbol{u}^{k}, \boldsymbol{v}\right)=n(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{W} \tag{12}
\end{equation*}
$$

Proof. We follow the proof of [7, Lemma 2.1, §2, Chapter IV] and adapt it to the trilinear form

$$
n(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v})=((\varepsilon \boldsymbol{w} \cdot \nabla) \boldsymbol{u}, \boldsymbol{v})=\sum_{i, j=1}^{n}\left(\varepsilon w_{j} \partial_{j} u_{i}, v_{i}\right),
$$

which has the weighting factor $\varepsilon$. Hereby, symbols with subscripts denote components of bold faced vectors, e.g. $\boldsymbol{u}=\left(u_{i}\right)_{i=1, \ldots, n}$. Let $\boldsymbol{u} \in \boldsymbol{H}^{1}, \boldsymbol{v} \in \boldsymbol{D}(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{W}$. Integrating by parts and employing density argument, we obtain immediately (9)

$$
\begin{aligned}
& \sum_{i, j=1}^{n}\left(\varepsilon w_{j} \partial_{j} u_{i}, v_{i}\right)=-\sum_{i, j=1}^{n}\left(\partial_{j}\left(\varepsilon w_{j} v_{i}\right), u_{i}\right)+\sum_{i, j=1}^{n}\left\langle\varepsilon w_{j} n_{j} u_{i}, v_{i}\right\rangle \\
& =-\sum_{i, j=1}^{n}\left(\varepsilon w_{j} \partial_{j} v_{i}, u_{i}\right)-(\operatorname{div}(\varepsilon \boldsymbol{w}) \boldsymbol{u}, \boldsymbol{v})+\langle(\varepsilon \boldsymbol{w} \cdot \boldsymbol{n}) \boldsymbol{u}, \boldsymbol{v}\rangle \\
& =-n(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}) .
\end{aligned}
$$

From Sobolev embedding $H^{1}(\Omega) \hookrightarrow L^{4}(\Omega)$ (see [1]) and Hölder inequality follows

$$
\left|\left(\varepsilon w_{j} \partial_{j} u_{i}, v_{i}\right)\right| \leq|\varepsilon|_{0, \infty}\left\|w_{j}\right\|_{0,4}\left\|\partial_{j} u_{i}\right\|_{0}\left\|v_{i}\right\|_{0,4} \leq C|\varepsilon|_{0, \infty}\left\|w_{j}\right\|_{1}\left|u_{i}\right|_{1}\left\|v_{i}\right\|_{1},
$$

and consequently the proof of (10) is completed. Since $\lim _{k \rightarrow \infty}\left\|u_{i}^{k} u_{j}^{k}-u_{i} u_{j}\right\|_{0,1}=0$ and $\varepsilon \partial_{j} v_{i} \in$ $L^{\infty}(\Omega)$, the continuity estimate 10 implies

$$
\begin{aligned}
\lim _{k \rightarrow \infty} n\left(\boldsymbol{u}^{k}, \boldsymbol{u}^{k}, \boldsymbol{v}\right) & =-\lim _{k \rightarrow \infty} n\left(\boldsymbol{u}^{k}, \boldsymbol{v}, \boldsymbol{u}^{k}\right)=-\lim _{k \rightarrow \infty} \sum_{i, j=1}^{n}\left(\varepsilon u_{j}^{k} \partial_{j} v_{i}^{k}, u_{i}^{k}\right) \\
& =-\sum_{i, j=1}^{n}\left(\varepsilon u_{j} \partial_{j} v_{i}, u_{i}\right)=-n(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u})=n(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})
\end{aligned}
$$

The continuity of $d(\cdot ; \cdot, \cdot)$ follows from Hölder inequality and Sobolev embedding $H^{1}(\Omega) \hookrightarrow$ $L^{4}(\Omega)$ (see [1])

$$
|d(\boldsymbol{u} ; \boldsymbol{v}, \boldsymbol{w})| \leq|\beta|_{\infty}\|\boldsymbol{u}\|_{0,4}\|\boldsymbol{v}\|_{0,4}\|\boldsymbol{w}\|_{0} \leq C_{\varepsilon}\|\boldsymbol{u}\|_{1}\|\boldsymbol{v}\|_{1}\|\boldsymbol{w}\|_{1} .
$$

In the next stage we consider the difficulties caused by prescribing the inhomogeneous Dirichlet boundary condition. Analogous difficulties are already encountered in the analysis of NavierStokes problem. We will carry out the study of three dimensional case. The extension in two dimensions can be constructed analogously. Since $\boldsymbol{g} \in \boldsymbol{H}^{1 / 2}(\Gamma)$, we can extend $\boldsymbol{g}$ inside of $\Omega$ in the form of

$$
\boldsymbol{g}=\varepsilon^{-1} \operatorname{curl} \boldsymbol{h}
$$

with some $\boldsymbol{h} \in \boldsymbol{H}^{2}(\Omega)$. The operator curl is defined then as

$$
\operatorname{curl} \boldsymbol{h}=\left(\partial_{2} h_{3}-\partial_{3} h_{2}, \partial_{3} h_{1}-\partial_{1} h_{3}, \partial_{1} h_{2}-\partial_{2} h_{1}\right) .
$$

We note that in the two dimensional case the vector potential $\boldsymbol{h} \in \boldsymbol{H}^{2}(\Omega)$ can be replaced by a scalar function $h \in H^{2}(\Omega)$ and the operator curl is then redefined as curl $h=\left(\partial_{2} h,-\partial_{1} h\right)$.

Our aim is to adapt the extension of Hopf (see [8]) to our model. We recall that for any parameter $\mu>0$ there exists a scalar function $\varphi_{\mu} \in C^{2}(\bar{\Omega})$ such that

- $\varphi_{\mu}=1$ in some neighborhood of $\Gamma($ depending on $\mu)$,
- $\varphi_{\mu}(\boldsymbol{x})=0$ if $d_{\Gamma}(\boldsymbol{x}) \geq 2 \exp (-1 / \mu)$, where $d_{\Gamma}(\boldsymbol{x}):=\inf _{\boldsymbol{y} \in \Gamma}|\boldsymbol{x}-\boldsymbol{y}|$ denotes the distance of $\boldsymbol{x}$ to $\Gamma$,
- $\left|\partial_{j} \varphi_{\mu}(\boldsymbol{x})\right| \leq \mu / d_{\Gamma}(\boldsymbol{x})$ if $d_{\Gamma}(\boldsymbol{x})<2 \exp (-1 / \mu), j=1, \ldots, n$.

For the construction of $\varphi_{\mu}$ see also [7, Lemma 2.4, §2, Chapter IV].
Let us define

$$
\begin{equation*}
\boldsymbol{g}_{\mu}:=\varepsilon^{-1} \operatorname{curl}\left(\varphi_{\mu} \boldsymbol{h}\right) . \tag{13}
\end{equation*}
$$

In the following lemma we establish bounds which are crucial for proving existence of velocity.

Lemma 4 The function $\boldsymbol{g}_{\mu}$ satisfies the following conditions

$$
\begin{equation*}
\operatorname{div}\left(\varepsilon \boldsymbol{g}_{\mu}\right)=0,\left.\quad \boldsymbol{g}_{\mu}\right|_{\Gamma}=\boldsymbol{g} \quad \forall \mu>0, \tag{14}
\end{equation*}
$$

and for any $\delta>0$ there exists sufficiently small $\mu>0$ such that

$$
\begin{align*}
&\left|d\left(\boldsymbol{u}+\boldsymbol{g}_{\mu} ; \boldsymbol{g}_{\mu}, \boldsymbol{u}\right)\right| \leq \delta\|\beta\|_{0, \infty}|\boldsymbol{u}|_{1}\left(|\boldsymbol{u}|_{1}+\left\|\boldsymbol{g}_{\mu}\right\|_{0}\right) \quad \forall \boldsymbol{u} \in \boldsymbol{X}_{0},  \tag{15}\\
&\left|n\left(\boldsymbol{u}, \boldsymbol{g}_{\mu}, \boldsymbol{u}\right)\right| \leq \delta|\boldsymbol{u}|_{1}^{2} \quad \forall \boldsymbol{u} \in \boldsymbol{W} . \tag{16}
\end{align*}
$$

Proof. The relations in (14) are obvious. We follow [5 in order to show (15). Since $\boldsymbol{h} \in \boldsymbol{H}^{2}(\Omega)$ Sobolev's embedding theorem implies $\boldsymbol{h} \in \boldsymbol{L}^{\infty}(\Omega)$, so we get according to the properties of $\varphi_{\mu}$ in (Ex) the following bound

$$
\left|\boldsymbol{g}_{\mu}\right| \leq C \varepsilon_{0}^{-1}\left\{|\nabla \boldsymbol{h}|+\frac{\mu}{d_{\Gamma}(\boldsymbol{x})}|\boldsymbol{h}|\right\} \leq C\left\{\frac{\mu}{d_{\Gamma}(\boldsymbol{x})}+|\nabla \boldsymbol{h}|\right\} .
$$

Defining

$$
\Omega_{\mu}:=\left\{\boldsymbol{x} \in \Omega: d_{\Gamma}(\boldsymbol{x})<2 \exp (-1 / \mu)\right\}
$$

we obtain from Cauchy-Schwarz and triangle inequalities

$$
\begin{align*}
& \qquad\left|\left(\beta\left|\boldsymbol{u}+\boldsymbol{g}_{\mu}\right|, \boldsymbol{g}_{\mu} \cdot \boldsymbol{u}\right)\right| \leq \begin{array}{l}
\|\beta\|_{0, \infty}\|\boldsymbol{u}\|_{0}\left\|\boldsymbol{u} \cdot \boldsymbol{g}_{\mu}\right\|_{0, \Omega_{\mu}} \\
+\|\beta\|_{0, \infty}\left\|\boldsymbol{g}_{\mu}\right\|_{0}\left\|\boldsymbol{u} \cdot \boldsymbol{g}_{\mu}\right\|_{0, \Omega_{\mu}}
\end{array}  \tag{17}\\
& \left\|\boldsymbol{u} \cdot \boldsymbol{g}_{\mu}\right\|_{0, \Omega_{\mu}}^{2} \leq \int_{\Omega_{\mu}}|\boldsymbol{u}|^{2}\left|\boldsymbol{g}_{\mu}\right|^{2} d \boldsymbol{x} \\
& \leq C \int_{\Omega_{\mu}}|\boldsymbol{u}|^{2}\left\{\left(\mu / d_{\Gamma}(\boldsymbol{x})\right)^{2}+2 \mu / d_{\Gamma}(\boldsymbol{x})|\nabla \boldsymbol{h}|+|\nabla \boldsymbol{h}|^{2}\right\} d \boldsymbol{x} \\
& \leq C\left\{\mu^{2}\left\|\boldsymbol{u} / d_{\Gamma}\right\|_{0, \Omega_{\mu}}^{2}+2 \mu\left\|\boldsymbol{u} / d_{\Gamma}\right\|_{0, \Omega_{\mu}}\|\boldsymbol{u}\|_{0,4, \Omega_{\mu}}\||\nabla \boldsymbol{h}|\|_{0,4, \Omega_{\mu}}+\|\boldsymbol{u}\|_{0,4, \Omega_{\mu}}^{2}\||\nabla \boldsymbol{h}|\|_{0,4, \Omega_{\mu}}^{2}\right\} \\
& \leq C\left\{\mu\left\|\boldsymbol{u} / d_{\Gamma}\right\|_{0, \Omega_{\mu}}+\|\boldsymbol{u}\|_{0,4}\||\nabla \boldsymbol{h}|\|_{0,4, \Omega_{\mu}}\right\}^{2}
\end{align*}
$$

and consequently

$$
\begin{equation*}
\left\|\boldsymbol{u} \cdot \boldsymbol{g}_{\mu}\right\|_{0, \Omega_{\mu}} \leq C\left\{\mu\left\|\boldsymbol{u} / d_{\Gamma}\right\|_{0, \Omega_{\mu}}+\|\boldsymbol{u}\|_{0,4}\||\nabla \boldsymbol{h}|\|_{0,4, \Omega_{\mu}}\right\} \tag{18}
\end{equation*}
$$

Applying Hardy inequality (see [1)

$$
\left\|v / d_{\Gamma}\right\|_{0} \leq C|v|_{1} \quad \forall v \in H_{0}^{1}(\Omega)
$$

and using Sobolev embedding $H^{1}(\Omega) \hookrightarrow L^{4}(\Omega)$, estimate (18) becomes

$$
\begin{equation*}
\left\|\boldsymbol{u} \cdot \boldsymbol{g}_{\mu}\right\|_{0, \Omega_{\mu}} \leq C \lambda(\mu)\|\boldsymbol{u}\|_{1} \tag{19}
\end{equation*}
$$

where

$$
\lambda(\mu):=\max \left\{\mu,\||\nabla \boldsymbol{h}|\|_{0,4, \Omega_{\mu}}\right\}
$$

From (17), (19), Poincaré inequality and from the fact that $\lim _{\mu \rightarrow 0} \lambda(\mu)=0$ we conclude that for any $\delta>0$ we can choose sufficiently small $\mu>0$ such that

$$
\left|\left(\beta\left|\boldsymbol{u}+\boldsymbol{g}_{\mu}\right| \boldsymbol{g}_{\mu}, \boldsymbol{u}\right)\right| \leq \delta\|\beta\|_{0, \infty}|\boldsymbol{u}|_{1}\left(|\boldsymbol{u}|_{1}+\left\|\boldsymbol{g}_{\mu}\right\|_{0}\right)
$$

holds. Therefore the proof of estimate (15) is completed. Now, we take a look at the trilinear convective term

$$
\begin{aligned}
n\left(\boldsymbol{u}, \boldsymbol{g}_{\mu}, \boldsymbol{u}\right) & =\left((\varepsilon \boldsymbol{u} \cdot \nabla) \boldsymbol{g}_{\mu}, \boldsymbol{u}\right)_{\Omega_{\mu}}=\left((\varepsilon \boldsymbol{u} \cdot \nabla)\left\{\varepsilon^{-1} \operatorname{curl}\left(\varphi_{\mu} \boldsymbol{h}\right)\right\}, \boldsymbol{u}\right)_{\Omega_{\mu}} \\
& =\left((\boldsymbol{u} \cdot \nabla)\left\{\operatorname{curl}\left(\varphi_{\mu} \boldsymbol{h}\right)\right\}, \boldsymbol{u}\right)_{\Omega_{\mu}}-\left((\boldsymbol{u} \cdot \nabla \varepsilon) \boldsymbol{g}_{\mu}, \boldsymbol{u}\right)_{\Omega_{\mu}}
\end{aligned}
$$

The first term of above difference becomes small due to [7, Lemma 2.3, $\S 2$, Chapter IV], and it satisfies

$$
\begin{equation*}
\left|\left((\boldsymbol{u} \cdot \nabla)\left\{\operatorname{curl}\left(\varphi_{\mu} \boldsymbol{h}\right)\right\}, \boldsymbol{u}\right)_{\Omega_{\mu}}\right|=\left|\left((\boldsymbol{u} \cdot \nabla)\left(\varepsilon \boldsymbol{g}_{\mu}\right), \boldsymbol{u}\right)_{\Omega_{\mu}}\right| \leq \delta|\boldsymbol{u}|_{1}^{2} \tag{20}
\end{equation*}
$$

as long as $\mu>0$ is chosen sufficiently small. Using Hölder inequality, Sobolev embedding $H^{1}(\Omega) \hookrightarrow L^{6}(\Omega)$ yields

$$
\left|\left((\boldsymbol{u} \cdot \nabla \varepsilon) \boldsymbol{g}_{\mu}, \boldsymbol{u}\right)_{\Omega_{\mu}}\right| \leq C\|\varepsilon\|_{1,3}\left\|\boldsymbol{g}_{\mu} \cdot \boldsymbol{u}\right\|_{0}\|\boldsymbol{u}\|_{1}
$$

which together with (19) implies for sufficiently small $\mu>0$ the bound

$$
\begin{equation*}
\left|\left((\boldsymbol{u} \cdot \nabla \varepsilon) \boldsymbol{g}_{\mu}, \boldsymbol{u}\right)_{\Omega_{\mu}}\right| \leq \delta|\boldsymbol{u}|_{1}^{2} \tag{21}
\end{equation*}
$$

From (20) and (21) follows the desired estimate (16).
While the general framework for linear and non-symmetric saddle point problems can be found in [6], our problem requires more attention due to its nonlinear character. Setting $\boldsymbol{w}:=\boldsymbol{u}-\boldsymbol{g}_{\mu}$, the weak formulation (8) is equivalent to the following problem

Find $(\boldsymbol{w}, p) \in \boldsymbol{V}$ such that

$$
\begin{equation*}
A\left(\boldsymbol{w}+\boldsymbol{g}_{\mu} ; \boldsymbol{w}+\boldsymbol{g}_{\mu}, \boldsymbol{v}\right)-b(\boldsymbol{v}, p)+b\left(\boldsymbol{w}+\boldsymbol{g}_{\mu}, q\right)=(\boldsymbol{f}, \boldsymbol{v}) \quad \forall(\boldsymbol{v}, q) \in \boldsymbol{V} \tag{22}
\end{equation*}
$$

Let us define the nonlinear mapping $G: \boldsymbol{W} \rightarrow \boldsymbol{W}$ with

$$
\begin{align*}
{[G(\boldsymbol{w}), \boldsymbol{v}]:=} & a\left(\boldsymbol{w}+\boldsymbol{g}_{\mu}, \boldsymbol{v}\right)+c\left(\boldsymbol{w}+\boldsymbol{g}_{\mu}, \boldsymbol{v}\right)-(\boldsymbol{f}, \boldsymbol{v})  \tag{23}\\
& +n\left(\boldsymbol{w}+\boldsymbol{g}_{\mu}, \boldsymbol{w}+\boldsymbol{g}_{\mu}, \boldsymbol{v}\right)+d\left(\boldsymbol{w}+\boldsymbol{g}_{\mu} ; \boldsymbol{w}+\boldsymbol{g}_{\mu}, \boldsymbol{v}\right),
\end{align*}
$$

whereby $[\cdot, \cdot]$ defines the inner product in $\boldsymbol{W}$ via $[u, v]:=(\nabla u, \nabla v)$. Then, the variational problem (22) reads in the space of $\varepsilon$-weighted divergence free functions $\boldsymbol{W}$ as follows

Find $\boldsymbol{w} \in \boldsymbol{W}$ such that

$$
\begin{equation*}
[G(\boldsymbol{w}), \boldsymbol{v}]=0 \quad \forall \boldsymbol{v} \in \boldsymbol{W} \tag{24}
\end{equation*}
$$

### 2.2 Solvability of nonlinear saddle point problem

We start our study of the nonlinear operator problem (24) with the following lemma.

Lemma 5 The mapping $G$ defined in (23) is continuous and there exists $r>0$ such that

$$
\begin{equation*}
[G(\boldsymbol{u}), \boldsymbol{u}]>0 \quad \forall \boldsymbol{u} \in \boldsymbol{W} \quad \text { with } \quad|\boldsymbol{u}|_{1}=r . \tag{25}
\end{equation*}
$$

Proof. Let $\left(\boldsymbol{u}^{k}\right)_{k \in \mathbb{N}}$ be a sequence in $\boldsymbol{W}$ with $\lim _{k \rightarrow \infty}\left\|\boldsymbol{u}^{k}-\boldsymbol{u}\right\|_{1}=0$. Then, applying CauchySchwarz inequality and (16), we obtain for any $\boldsymbol{v} \in \boldsymbol{W}$

$$
\begin{aligned}
& \left|\left[G\left(\boldsymbol{u}^{k}\right)-G(\boldsymbol{u}), \boldsymbol{v}\right]\right| \leq \frac{1}{R e}\left|\left(\varepsilon \nabla\left(\boldsymbol{u}^{k}-\boldsymbol{u}\right), \nabla \boldsymbol{v}\right)\right|+\frac{1}{R e}\left|\left(\alpha\left(\boldsymbol{u}^{k}-\boldsymbol{u}\right), \boldsymbol{v}\right)\right| \\
& \quad+\left|\left(\beta\left|\boldsymbol{u}^{k}+\boldsymbol{g}_{\mu}\right|\left(\boldsymbol{u}^{k}-\boldsymbol{u}\right), \boldsymbol{v}\right)\right|+\left|\left(\beta\left(\left|\boldsymbol{u}^{k}+\boldsymbol{g}_{\mu}\right|-\left|\boldsymbol{u}+\boldsymbol{g}_{\mu}\right|\right)\left(\boldsymbol{u}+\boldsymbol{g}_{\mu}\right), \boldsymbol{v}\right)\right| \\
& \quad+\left|n\left(\boldsymbol{u}^{k}, \boldsymbol{u}^{k}, \boldsymbol{v}\right)-n(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})\right|+\left|n\left(\boldsymbol{u}^{k}-\boldsymbol{u}, \boldsymbol{g}_{\mu}, \boldsymbol{v}\right)\right|+\left|n\left(\boldsymbol{g}_{\mu}, \boldsymbol{u}^{k}-\boldsymbol{u}, \boldsymbol{v}\right)\right| \\
& \leq \frac{\varepsilon_{1}}{R e}\left|\boldsymbol{u}^{k}-\boldsymbol{u}\right|_{1}|\boldsymbol{v}|_{1}+\frac{1}{R e}\|\alpha\|_{0, \infty}\left\|\boldsymbol{u}^{k}-\boldsymbol{u}\right\|_{0}\|\boldsymbol{v}\|_{0} \\
& \quad+\|\beta\|_{0, \infty}\left\|\boldsymbol{u}^{k}+\boldsymbol{g}_{\mu}\right\|_{0,4}\left\|\boldsymbol{u}^{k}-\boldsymbol{u}\right\|_{0}\|\boldsymbol{v}\|_{0,4}+\|\beta\|_{0, \infty}\left\|\boldsymbol{u}+\boldsymbol{g}_{\mu}\right\|_{0,4}\left\|\boldsymbol{u}^{k}-\boldsymbol{u}\right\|_{0}\|\boldsymbol{v}\|_{0,4} \\
& \quad+\left|n\left(\boldsymbol{u}^{k}, \boldsymbol{u}^{k}, \boldsymbol{v}\right)-n(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})\right|+C\left\|\boldsymbol{u}^{k}-\boldsymbol{u}\right\|_{1}\left\|\boldsymbol{g}_{\mu}\right\|_{1}\|\boldsymbol{v}\|_{1} .
\end{aligned}
$$

The boundedness of $\boldsymbol{u}^{k}$ in $\boldsymbol{W}, 12$, the Poincaré inequality, and the above inequality imply that

$$
\left|\left[G\left(\boldsymbol{u}^{k}\right)-G(\boldsymbol{u}), \boldsymbol{v}\right]\right| \rightarrow 0 \quad \text { as } \quad k \rightarrow \infty \quad \forall \boldsymbol{v} \in \boldsymbol{W}
$$

Thus, employing

$$
\left|G\left(\boldsymbol{u}^{k}\right)-G(\boldsymbol{u})\right|_{1}=\sup _{\substack{\boldsymbol{v} \in \boldsymbol{W} \\ \boldsymbol{v} \neq \boldsymbol{0}}} \frac{\left[G\left(\boldsymbol{u}^{k}\right)-G(\boldsymbol{u}), \boldsymbol{v}\right]}{|\boldsymbol{v}|_{1}},
$$

we state that $G$ is continuous. Now, we note that for any $\boldsymbol{u} \in \boldsymbol{W}$ we have

$$
\begin{align*}
& {[G(\boldsymbol{u}), \boldsymbol{u}]=\frac{1}{R e}\left(\varepsilon \nabla\left(\boldsymbol{u}+\boldsymbol{g}_{\mu}\right), \nabla \boldsymbol{u}\right)+\frac{1}{R e}\left(\alpha\left(\boldsymbol{u}+\boldsymbol{g}_{\mu}\right), \boldsymbol{u}\right)} \\
& \quad+\left(\beta\left|\boldsymbol{u}+\boldsymbol{g}_{\mu}\right|\left(\boldsymbol{u}+\boldsymbol{g}_{\mu}\right), \boldsymbol{u}\right)+n\left(\boldsymbol{u}+\boldsymbol{g}_{\mu}, \boldsymbol{u}+\boldsymbol{g}_{\mu}, \boldsymbol{u}\right)-(\boldsymbol{f}, \boldsymbol{u}) \\
& \geq \frac{\varepsilon_{0}}{R e}|\boldsymbol{u}|_{1}^{2}-\frac{\varepsilon_{1}}{R e}\left|\left(\nabla \boldsymbol{g}_{\mu}, \nabla \boldsymbol{u}\right)\right|+\frac{1}{R e}(\alpha \boldsymbol{u}, \boldsymbol{u})-\frac{1}{R e}\left|\left(\alpha \boldsymbol{g}_{\mu}, \boldsymbol{u}\right)\right| \\
& \quad+\left(\beta\left|\boldsymbol{u}+\boldsymbol{g}_{\mu}\right|,|\boldsymbol{u}|^{2}\right)-\left|\left(\beta\left|\boldsymbol{u}+\boldsymbol{g}_{\mu}\right| \boldsymbol{g}_{\mu}, \boldsymbol{u}\right)\right|  \tag{26}\\
& \quad+n\left(\boldsymbol{u}, \boldsymbol{g}_{\mu}, \boldsymbol{u}\right)+n\left(\boldsymbol{g}_{\mu}, \boldsymbol{g}_{\mu}, \boldsymbol{u}\right)-\|\boldsymbol{f}\|_{0}\|\boldsymbol{u}\|_{0} \\
& \geq \frac{\varepsilon_{0}}{R e}|\boldsymbol{u}|_{1}^{2}-\frac{\varepsilon_{1}}{R e}\left|\boldsymbol{g}_{\mu}\right|_{1}|\boldsymbol{u}|_{1} \\
& \quad-\frac{1}{R e}\|\alpha\|_{0, \infty}\left\|\boldsymbol{g}_{\mu}\right\|_{0}\|\boldsymbol{u}\|_{0}-\left|\left(\beta\left|\boldsymbol{u}+\boldsymbol{g}_{\mu}\right| \boldsymbol{g}_{\mu}, \boldsymbol{u}\right)\right| \\
& \quad-\left|n\left(\boldsymbol{u}, \boldsymbol{g}_{\mu}, \boldsymbol{u}\right)\right|-C\left\|\boldsymbol{g}_{\mu}\right\|_{1}^{2}\|\boldsymbol{u}\|_{1}-\|\boldsymbol{f}\|_{0}\|\boldsymbol{u}\|_{0} .
\end{align*}
$$

From the Poincaré inequality, we infer the estimate

$$
\|v\|_{1} \leq C|v|_{1} \quad \forall v \in H_{0}^{1}(\Omega)
$$

which together with (15), (16) and (26) results in

$$
\begin{aligned}
& {[G(\boldsymbol{u}), \boldsymbol{u}] \geq\left\{\frac{\varepsilon_{0}}{R e}-\delta\left(1+\|\beta\|_{0, \infty}\right)\right\}|\boldsymbol{u}|_{1}^{2}} \\
& \quad-\left\{\frac{\varepsilon_{1}}{R e}\left|\boldsymbol{g}_{\mu}\right|_{1}+C_{1} \frac{1}{R e}\|\alpha\|_{0, \infty}\left\|\boldsymbol{g}_{\mu}\right\|_{0}+\delta\|\beta\|_{0, \infty}\left\|\boldsymbol{g}_{\mu}\right\|_{0}+C_{2}\left\|\boldsymbol{g}_{\mu}\right\|_{1}^{2}+C_{3}\|\boldsymbol{f}\|_{0}\right\}|\boldsymbol{u}|_{1}
\end{aligned}
$$

Choosing $\delta$ such that

$$
0<\delta<\delta_{0}:=\frac{\varepsilon_{0}}{R e}\left(1+\|\beta\|_{0, \infty}\right)^{-1}
$$

and $r>r_{0}$ with

$$
\begin{equation*}
r_{0}:=\frac{\frac{\varepsilon_{1}}{R e}\left|\boldsymbol{g}_{\mu}\right|_{1}+\frac{1}{R e} C_{1}\|\alpha\|_{0, \infty}\left\|\boldsymbol{g}_{\mu}\right\|_{0}+\delta\|\beta\|_{0, \infty}\left\|\boldsymbol{g}_{\mu}\right\|_{0}+C_{2}\left\|\boldsymbol{g}_{\mu}\right\|_{1}^{2}+C_{3}\|\boldsymbol{f}\|_{0}}{\frac{\varepsilon_{0}}{R e}-\delta\left(1+\|\beta\|_{0, \infty}\right)}, \tag{27}
\end{equation*}
$$

leads to the desired assertion (25).
The following lemma plays a key role in the existence proof.
Lemma 6 Let $Y$ be finite-dimensional Hilbert space with inner product $[\cdot, \cdot]$ inducing a norm $\|\cdot\|$, and $T: Y \rightarrow Y$ be a continuous mapping such that

$$
[T(x), x]>0 \quad \text { for } \quad\|x\|=r_{0}>0
$$

Then there exists $x \in Y$, with $\|x\| \leq r_{0}$, such that

$$
T(x)=0 .
$$

Proof. See [9].
Now we are able to prove the main result concerning existence of velocity.
Theorem 7 The problem (24) has at least one solution $\boldsymbol{u} \in \boldsymbol{W}$.

Proof. We construct the approximate sequence of Galerkin solutions. Since the space $\boldsymbol{W}$ is separable, there exists a sequence of linearly independent elements $\left(\boldsymbol{w}^{i}\right)_{i \in \mathbb{N}} \subset \boldsymbol{W}$. Let $\boldsymbol{X}_{m}$ be the finite dimensional subspace of $\boldsymbol{W}$ with

$$
\boldsymbol{X}_{m}:=\operatorname{span}\left\{\boldsymbol{w}^{i}, i=1, \ldots, m\right\}
$$

and endowed with the scalar product of $\boldsymbol{W}$. Let $\boldsymbol{u}^{m}=\sum_{j=1}^{m} a_{j} \boldsymbol{w}^{j}, a_{j} \in \mathbb{R}$, be a Galerkin solution of (24) defined by

$$
\begin{equation*}
\left[G\left(\boldsymbol{u}^{m}\right), \boldsymbol{w}^{j}\right]=0, \quad \forall j=1, \ldots, m . \tag{28}
\end{equation*}
$$

From Lemma 5 and Lemma 6 we conclude that

$$
\begin{equation*}
\left[G\left(\boldsymbol{u}^{m}\right), \boldsymbol{w}\right]=0 \quad \forall \boldsymbol{w} \in \boldsymbol{X}_{m} \tag{29}
\end{equation*}
$$

has a solution $\boldsymbol{u}^{m} \in \boldsymbol{X}_{m}$. The unknown coefficients $a_{j}$ can be obtained from the algebraic system (28). On the other hand, multiplying (28) by $a_{j}$, and adding the equations for $j=$ $1, \ldots, m$ we have

$$
\begin{aligned}
0= & {\left[G\left(\boldsymbol{u}^{m}\right), \boldsymbol{u}^{m}\right] } \\
\geq & \left\{\frac{1}{R e}-\delta\left(1+\|\beta\|_{0, \infty}\right)\right\}\left|\boldsymbol{u}^{m}\right|_{1}^{2} \\
& -\left\{\frac{1}{R e}\left|\boldsymbol{g}_{\mu}\right|_{1}+C_{1} \frac{1}{R e}\|\alpha\|_{0, \infty}\left\|\boldsymbol{g}_{\mu}\right\|_{0}+\delta\|\beta\|_{0, \infty}\left\|\boldsymbol{g}_{\mu}\right\|_{0}+C_{2}\left\|\boldsymbol{g}_{\mu}\right\|_{1}^{2}+C_{3}\|\boldsymbol{f}\|_{0}\right\}\left|\boldsymbol{u}^{m}\right|_{1} .
\end{aligned}
$$

This gives together with (27) the uniform boundedness in $\boldsymbol{W}$

$$
\left|\boldsymbol{u}^{m}\right|_{1} \leq r_{0},
$$

therefore there exists $\boldsymbol{u} \in \boldsymbol{W}$ and a subsequence $m_{k} \rightarrow \infty$ ( we write for the convenience $m$ instead of $m_{k}$ ) such that

$$
\boldsymbol{u}^{m} \rightharpoonup \boldsymbol{u} \quad \text { in } \boldsymbol{W} .
$$

Furthermore, the compactness of embedding $H^{1}(\Omega) \hookrightarrow L^{4}(\Omega)$ implies

$$
\boldsymbol{u}^{m} \rightarrow \boldsymbol{u} \quad \text { in } \quad \boldsymbol{L}^{4}(\Omega) .
$$

Taking the limit in (29) with $m \rightarrow \infty$ we get

$$
\begin{equation*}
[G(\boldsymbol{u}), \boldsymbol{w}]=0 \quad \forall \boldsymbol{w} \in \boldsymbol{X}_{m} \tag{30}
\end{equation*}
$$

Finally, we apply the continuity argument and state that 30 is preserved for any $\boldsymbol{w} \in \boldsymbol{W}$, therefore $\boldsymbol{u}$ is the solution of (24).
For the reconstruction of the pressure we need inf-sup-theorem
Theorem 8 Assume that the bilinear form $b(\cdot, \cdot)$ satisfies the inf-sup condition

$$
\begin{equation*}
\inf _{q \in M} \sup _{\boldsymbol{v} \in \boldsymbol{X}_{0}} \frac{b(\boldsymbol{v}, q)}{|\boldsymbol{v}|_{1}\|q\|_{0}} \geq \gamma>0 \tag{31}
\end{equation*}
$$

Then, for each solution $\boldsymbol{u}$ of the nonlinear problem (24) there exists a unique pressure $p \in M$ such that the pair $(\boldsymbol{u}, p) \in \boldsymbol{V}$ is a solution of the homogeneous problem (22).

Proof. See [7, Theorem 1.4, §1, Chapter IV].
We end up this subsection by proving the existence of the pressure.
Theorem 9 Let $\boldsymbol{w}$ be solution of problem (24). Then, there exists unique pressure $p \in M$.
Proof. We verify the inf-sup condition (31) of Theorem 8 by employing the isomorphism of Theorem 2. From [7, Corollary 2.4, $\S 2$, Chapter I] follows that for any $q$ in $L_{0}^{2}(\Omega)$ there exists $\boldsymbol{v}$ in $\boldsymbol{H}_{0}^{1}(\Omega)$ such that

$$
(\operatorname{div} \boldsymbol{v}, q) \geq \gamma^{*}\|\boldsymbol{v}\|_{1}\|q\|_{0}
$$

with a positive constant $\gamma^{*}$. Setting $\boldsymbol{u}=\boldsymbol{v} / \varepsilon$ and applying the isomorphism in Theorem 2 , we obtain the estimate

$$
b(\boldsymbol{u}, q)=(\operatorname{div} \boldsymbol{v}, q) \geq \gamma^{*}\|\boldsymbol{v}\|_{1}\|q\|_{0} \geq \gamma_{\varepsilon}\|\boldsymbol{u}\|_{1}\|q\|_{0}
$$

where $\gamma_{\varepsilon}=\frac{\gamma^{*}}{C\left\{\varepsilon_{0}^{-1}+\varepsilon_{0}^{-2}|\varepsilon|_{1,3}\right\}}$. From the above estimate we conclude the inf-sup condition (31).

### 2.3 Uniqueness of weak solution

We exploit a priori estimates in order to prove uniqueness of weak velocity and pressure.
Theorem 10 If $\left\|\boldsymbol{g}_{\mu}\right\|_{1},\|\boldsymbol{f}\|_{-1}:=\sup _{\mathbf{0} \neq \boldsymbol{v} \in \boldsymbol{H}^{1}(\Omega)} \frac{(\boldsymbol{f}, \boldsymbol{v})}{\|\boldsymbol{v}\|_{1}}$ are sufficiently small, then the solution of (24) is unique.

Proof. Assume that $\left(\boldsymbol{u}_{1}, p_{1}\right)$ and $\left(\boldsymbol{u}_{2}, p_{2}\right)$ are two different solutions of (22). From (9) in Lemma 3 we obtain $n(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{u})=0 \forall \boldsymbol{w}, \boldsymbol{u} \in \boldsymbol{W}$. Then, we obtain

$$
\begin{align*}
0= & {\left[G\left(\boldsymbol{u}_{1}\right)-G\left(\boldsymbol{u}_{2}\right), \boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right] } \\
= & a\left(\boldsymbol{u}_{1}-\boldsymbol{u}_{2}, \boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right)+c\left(\boldsymbol{u}_{1}-\boldsymbol{u}_{2}, \boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right)-\left(\boldsymbol{f}, \boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right) \\
& +n\left(\boldsymbol{u}_{1}+\boldsymbol{g}_{\mu}, \boldsymbol{u}_{1}+\boldsymbol{g}_{\mu}, \boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right)-n\left(\boldsymbol{u}_{2}+\boldsymbol{g}_{\mu}, \boldsymbol{u}_{2}+\boldsymbol{g}_{\mu}, \boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right) \\
& +\left(\beta\left|\boldsymbol{u}_{1}+\boldsymbol{g}_{\mu}\right|\left(\boldsymbol{u}_{1}+\boldsymbol{g}_{\mu}\right), \boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right) \\
& -\left(\beta\left|\boldsymbol{u}_{2}+\boldsymbol{g}_{\mu}\right|\left(\boldsymbol{u}_{2}+\boldsymbol{g}_{\mu}\right), \boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right) \\
\geq & \frac{\varepsilon_{0}}{R e}\left|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right|_{1}^{2}-\|\boldsymbol{f}\|_{-1}\left\|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right\|_{1} \\
& +n\left(\boldsymbol{u}_{1}-\boldsymbol{u}_{2}, \boldsymbol{u}_{2}+\boldsymbol{g}_{\mu}, \boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right)  \tag{32}\\
& +\left(\beta\left|\boldsymbol{u}_{1}+\boldsymbol{g}_{\mu}\right|\left(\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right), \boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right) \\
& +\left(\beta\left(\left|\boldsymbol{u}_{1}+\boldsymbol{g}_{\mu}\right|-\left|\boldsymbol{u}_{2}+\boldsymbol{g}_{\mu}\right|\right)\left(\boldsymbol{u}_{2}+\boldsymbol{g}_{\mu}\right), \boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right) \\
\geq & \frac{\varepsilon_{0}}{R e}\left|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right|_{1}^{2}-\|\boldsymbol{f}\|_{-1}\left\|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right\|_{1} \\
& -\left|n\left(\boldsymbol{u}_{1}-\boldsymbol{u}_{2}, \boldsymbol{u}_{2}, \boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right)\right|-\left|n\left(\boldsymbol{u}_{1}-\boldsymbol{u}_{2}, \boldsymbol{g}_{\mu}, \boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right)\right| \\
& -\|\beta\|_{0, \infty}\left|\left(\left|\boldsymbol{u}_{1}+\boldsymbol{g}_{\mu}\right| \cdot\left|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right|,\left|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right|\right)\right| \\
& -\|\beta\|_{0, \infty}\left|\left(| | \boldsymbol{u}_{1}+\boldsymbol{g}_{\mu}\left|-\left|\boldsymbol{u}_{2}+\boldsymbol{g}_{\mu}\right|\right| \cdot\left|\boldsymbol{u}_{2}+\boldsymbol{g}_{\mu}\right|,\left|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right|\right)\right| .
\end{align*}
$$

From Cauchy-Schwarz inequality and Sobolev embedding $H^{1}(\Omega) \hookrightarrow L^{4}(\Omega)$ we deduce

$$
\begin{align*}
& \left|\left(\left|\boldsymbol{u}_{1}+\boldsymbol{g}_{\mu}\right| \cdot\left|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right|,\left|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right|\right)\right| \leq C\left\{\left\|\boldsymbol{u}_{1}\right\|_{0}+\left\|\boldsymbol{g}_{\mu}\right\|_{0}\right\}\left\|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right\|_{1}^{2},  \tag{33}\\
& \left|\left(\left|\left|\boldsymbol{u}_{1}+\boldsymbol{g}_{\mu}\right|-\left|\boldsymbol{u}_{2}+\boldsymbol{g}_{\mu}\right|\right| \cdot\left|\boldsymbol{u}_{2}+\boldsymbol{g}_{\mu}\right|,\left|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right|\right)\right|  \tag{34}\\
& \quad \leq C\left\{\left\|\boldsymbol{u}_{2}\right\|_{0}+\left\|\boldsymbol{g}_{\mu}\right\|_{0}\right\}\left\|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right\|_{1}^{2},
\end{align*}
$$

and according to (10) we have

$$
\begin{equation*}
\left|n\left(\boldsymbol{u}_{1}-\boldsymbol{u}_{2}, \boldsymbol{u}_{2}, \boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right)\right| \leq C\left\|\boldsymbol{u}_{2}\right\|_{1}\left\|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right\|_{1}^{2} \tag{35}
\end{equation*}
$$

and by (14) we can find $\mu$ such that

$$
\begin{equation*}
\left|n\left(\boldsymbol{u}_{1}-\boldsymbol{u}_{2}, \boldsymbol{g}_{\mu}, \boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right)\right| \leq \frac{\varepsilon_{0}}{4 R e}\left\|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right\|_{1}^{2} \tag{36}
\end{equation*}
$$

Now, we find upper bounds for $\boldsymbol{u}_{1}$ and $\boldsymbol{u}_{2}$. Testing the equation (22) with $\boldsymbol{u}$ results in

$$
\begin{aligned}
\frac{\varepsilon_{0}}{R e}\|\boldsymbol{u}\|_{1}^{2} \leq & \|\boldsymbol{f}\|_{-1}\|\boldsymbol{u}\|_{1}+\frac{\varepsilon_{0}}{R e}\left\|\boldsymbol{g}_{\mu}\right\|_{1}\|\boldsymbol{u}\|_{1}+C\left\|\boldsymbol{g}_{\mu}\right\|_{0}\|\boldsymbol{u}\|_{0} \\
& +C\left\|\boldsymbol{g}_{\mu}\right\|_{1}^{2}\|\boldsymbol{u}\|_{1}+C\|\beta\|_{0, \infty}\left\|\boldsymbol{g}_{\mu}\right\|_{0}\|\boldsymbol{u}\|_{1}^{2}+C\|\beta\|_{0, \infty}\left\|\boldsymbol{g}_{\mu}\right\|_{0,4}^{2}\|\boldsymbol{u}\|_{1} .
\end{aligned}
$$

From Sobolev embedding $H^{1}(\Omega) \hookrightarrow L^{4}(\Omega)$ we deduce for sufficiently small $\left\|\boldsymbol{g}_{\mu}\right\|_{1}$

$$
\begin{equation*}
\|\boldsymbol{u}\|_{1} \leq \frac{\|\boldsymbol{f}\|_{-1}+C_{1}\left\|\boldsymbol{g}_{\mu}\right\|_{1}+C_{2}\left\|\boldsymbol{g}_{\mu}\right\|_{1}^{2}}{\frac{\varepsilon_{0}}{R e}-C_{3}\|\beta\|_{0, \infty}\left\|\boldsymbol{g}_{\mu}\right\|_{1}}=: C\left(\left\|\boldsymbol{g}_{\mu}\right\|_{1},\|\boldsymbol{f}\|_{-1}\right) . \tag{37}
\end{equation*}
$$

Putting (33)-(37) into (32) and using the inequality

$$
\|\boldsymbol{f}\|_{-1}\left\|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right\|_{1} \leq \frac{\varepsilon_{0}}{4 R e}\left\|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right\|_{1}^{2}+\frac{2 R e}{\varepsilon_{0}}\|\boldsymbol{f}\|_{-1}^{2}
$$

we obtain

$$
\begin{align*}
0 \geq & \frac{\varepsilon_{0}}{2 R e}\left\|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right\|_{1}^{2}-\frac{2 R e}{\varepsilon_{0}}\|\boldsymbol{f}\|_{-1}^{2}-C\left(\left\|\boldsymbol{g}_{\mu}\right\|_{1},\|\boldsymbol{f}\|_{-1}\right)\|\beta\|_{0, \infty}\left\|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right\|_{1}^{2}  \tag{38}\\
& -\frac{\varepsilon_{0}}{4 R e}\left\|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right\|_{1}^{2}-C\left(\left\|\boldsymbol{g}_{\mu}\right\|_{1},\|\boldsymbol{f}\|_{-1}\right)\left\|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right\|_{1}^{2}
\end{align*}
$$

For sufficiently small $\left\|\boldsymbol{g}_{\mu}\right\|_{1},\|\boldsymbol{f}\|_{-1}$ the constant $C\left(\left\|\boldsymbol{g}_{\mu}\right\|_{1},\|\boldsymbol{f}\|_{-1}\right)$ in (37) gets small and consequently the right hand side of (38) is nonnegative. This implies $\boldsymbol{u}_{1}=\boldsymbol{u}_{2}$ and according to Theorem 9 is $p_{1}-p_{2}=0$.

## 3 A Channel Flow Problem in Packed Bed Reactors

In this section, we provide an example of the flow problem in packed bed reactors with numerical solutions at small and relatively large Reynolds numbers to show the nonlinear behavior of the velocity solutions. Let the reactor channel be represented by $\Omega=(0, L) \times$ $(-R, R)$ where $R=5$ and $L=60$. In all computations we use the porosity distribution which


Figure 1: Varying porosity.
is determined experimentally and takes into account the effect of wall channelling in packed bed reactors

$$
\begin{equation*}
\varepsilon(x, y)=\varepsilon(y)=\varepsilon_{\infty}\left\{1+\frac{1-\varepsilon_{\infty}}{\varepsilon_{\infty}} e^{-6(R-|y|)}\right\} \tag{39}
\end{equation*}
$$

where $\varepsilon_{\infty}=0.45$. The distribution of the porosity is presented in Figure 1. We distinguish between the inlet, outlet and membrane parts of domain boundary $\Gamma$, and denote them by $\Gamma_{i n}, \Gamma_{\text {out }}$ and $\Gamma_{w}$, respectively. Let

$$
\begin{aligned}
& \Gamma_{\text {in }}=\{(x, y) \in \Gamma: x=0\}, \\
& \Gamma_{o u t}=\{(x, y) \in \Gamma: x=L\}, \\
& \Gamma_{w}=\{(x, y) \in \Gamma: y=-R, y=R\} .
\end{aligned}
$$

At the inlet $\Gamma_{i n}$ and at the membrane wall we prescribe Dirichlet boundary conditions, namely the plug flow conditions

$$
\left.\boldsymbol{u}\right|_{\Gamma_{i n}}=\boldsymbol{u}_{i n}=\left(u_{i n}, 0\right)^{T},
$$

and

$$
\left.\boldsymbol{u}\right|_{\Gamma_{w}}=\boldsymbol{u}_{w}= \begin{cases}\left(0, u_{w}\right)^{T} & \text { for } \quad y=-R \\ \left(0,-u_{w}\right)^{T} & \text { for } \quad y=R\end{cases}
$$

whereby $u_{i n}>0, u_{w}>0$. At the outlet $\Gamma_{\text {out }}$ we set the following outflow boundary condition

$$
-\frac{1}{R e} \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}+p \boldsymbol{n}=\mathbf{0}
$$

where $\boldsymbol{n}$ denotes the outer normal. In order to avoid discontinuity between the inflow and wall conditions we replace constant profile by trapezoidal one with zero value at the corners. Our computations are carried out on the Cartesian mesh using biquadratic conforming and discontinuous piecewise linear finite elements for the approximation of the velocity and pressure, respectively. The finite element analysis of the Brinkman-Forchheimer-extended Darcy equation will be conducted in the forthcoming work. The plots of velocity magnitude in fixed bed reactor $\left(u_{w}=0\right)$ are presented along the vertical axis $x=50$. In the investigated reactor the inlet velocity is assumed to be normalized ( $u_{i n}=1$ ). Due to the variation of porosity we might expect higher velocity at the reactor walls $\Gamma_{w}$. This tunnelling effect can be well observed in Figure 2 which shows the velocity profiles for different Reynolds numbers. We remark that the maximum of velocity magnitude decreases with increasing Reynolds numbers.


Figure 2: Flow profiles in fixed bed reactor at $x=50$.

## 4 Conclusion

In this work, we have extended the existence and uniqueness of solution result in literature for the porous medium flow problem based on the nonlinear Brinkman-Forchheimer-extended Darcy law. The existing result is valid only for constant porosity and without the considered convection effects, and our result holds for variable porosity and it includes convective effects. We also provided a numerical solution to demonstrate the nonlinear velocity solutions at moderately large Reynolds numbers for which case the Brinkman-Forchheimer-extended

Darcy law applies.

## References

[1] R. A. Adams Sobolev Spaces Pure and applied mathematics, Academic Press, 1995.
[2] O. Bey. Strömungsverteilung und Wärmetransport in Schüttungen Nummer 570 in Fortschritt-Berichte, VDI Reihe 3, Verfahrenstechnik, Düsseldorf: VDI Verlag, 1998.
[3] S. Ergun. Fluid Flow Through Packed Columns Chemical Engineering Progress, 48(2):89-94, 1952.
[4] U. Hornung. Homogenization and Porous Media, Springer-Verlag, 1997.
[5] P. N. Kaloni and Jianlin Guo. Steady nonlinear double-diffusive convection in a porous medium based upon the Brinkman-Forchheimer model, J. Math. Anal. Appl., 204(1):138-155, 1996.
[6] Christine Bernardi, Frédéric Laval, Brigitte Métivet and Bernadette Pernaud-Thomas. Finite element approximation of viscous flows with varying density, SIAM J. Numer. Anal., 29(5):1203-1243, 1992.
[7] V. Girault and P.-A. Raviart. Finite Element Methods for Navier-Stokes Equations. Theorie and Algorithms, Springer-Verlag, 1986
[8] E. Hopf. Ein allgemeiner Endlichkeitssatz der Hydrodynamik, Math. Ann., 117:764-775, 1941.
[9] J. L. Lions. Quelques methodes de resolution des problemes aux limites non lineaires, Gauthier-Villars, Paris, 1969.
[10] T. Zhao. Investigation of Landslide-Induced Debris Flows by the DEM and CFD, Ph.D thesis, University of Oxford, 2014.
[11] K. Upton. Multi-scale modelling of borehole yields in chalk aquifers, Ph.D thesis, Imperial College London, 2015.
[12] A. Grillo, M. Carfagna, and S Federico. The Darcy-Forchheimer law for modelling fluid in biological tissues, Theoret. Appl. Mech. TEOPM7, Vol.41, No.4, 283-322, Belgrade 2014.
[13] W. Sobieski, A. Trykozko. Darcy's and Forchheimer's law in practice. Part 1. the experiment, Technical Sciences 17(4), 321335, 2014.
[14] A. S. Lal and A. C Menon. Design of a new porous medium heat exchanger for an aircraft refrigeration system, IJTEL, ISSN: 2319-2135, VOL.3, NO.4, 545-548, 2014.


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