Applied Mathematical Sciences, Vol. 7, 2013, no. 66, 3271 - 3280
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2013.33195

Existence of Equilibrium States of Hollow Hollomon Cylinders Submerged in a Fluid

M. B. M. Elgindi

Taxes A & M University in Qatar
P.O. Box 23874, Doha, Qatar,
mohamed.elgindi@qatar.tamu.edu

D. Wei

University of New Orleans, 2000 Lakeshore Drive
New Orleans, LA 70148, USA
dwei@uno.edu

Abstract

This paper is concerned with the existence of equilibrium states of a thin-walled hollow elasto-plastic cylinders fully or partially submerged in a fluid. This problem serves as a model for many problems with engineering importance. Previous studies on the deformation of such a shell assumed that the material is linear elastic. This paper takes into consideration the nonlinear Hollomon materials that (are plastic) can deform plastically. The effect of gravity on pressure is also taken into account.

Mathematics Subject Classification: 49G99, 73H05, 73K15

Keywords: Existence of equilibrium states, Hollomon power-law material, hollow plastic cylinders, nonlinear eigenvalue problem, nonlinear integro-differential equation, minimization of functional with constraints, Browder Theorem.

1. INTRODUCTION

In this paper, we consider the existence of equilibrium states of hollow plastic cylinders fully or partially submerged in a fluid. The materials considered represent the Hollomon power-law plastics. We treat the fluid pressure as non-uniform by taking gravity into account. To balance the buoyancy force due to the fluid an external line load is applied at the bottom of the cylinder. By treating the problem as independent of the variable along the axis of the cylinder we arrive at a similar system of nonlinear ordinary differential equations governing the equilibrium to that of an inextensible Hollomon elasto-plastica for a circular ring presented in [9]. Our work generalizes that of [7], [8] and [10], where linear elastic cylinders are considered.

In Section 2 we give several mathematical formulations for the equilibrium equations. These are obtained by generalizing the formulations given in [8] for linear (elastica) elastic cylinders to the case when the cylinder is made of Hollomon plastic material. In Section 3, we use the Implicit Function Theorem to give a proof of the existence of positive bifurcation solutions of small norms merging off the trivial solution for the case when the pressure is uniform. In Section 4, we present a proof of the existence of solutions of arbitrary large norms. In section 5, we prove existence of equilibrium states treating the pressure parameters as given constants. In Section 6 we give some concluding remarks. The main mathematical tools used in the proofs of Sections 4 and 5 come from variational methods for nonlinear monotone elliptic eigenvalue problems and the Browder theory for monotone operators on Sobolev spaces (see [1]-[6] and [11].)

2. THE MATHEMATICAL FORMULATION

We consider a typical cross section of a hollow cylinder as shown in Figure 1.

FIGURE 1

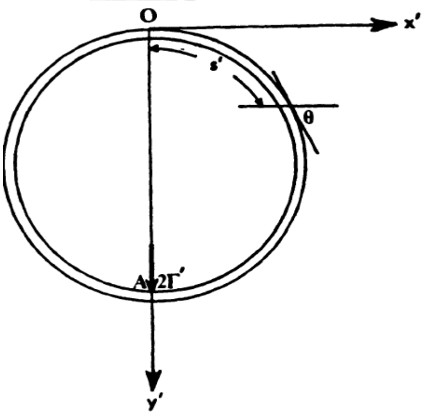
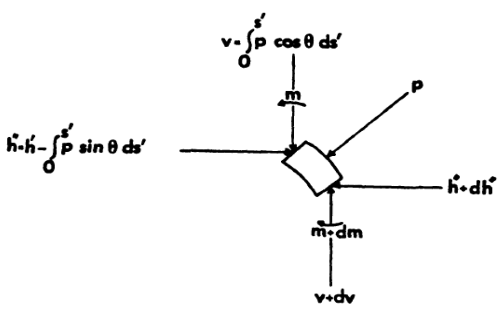


FIGURE 2



The coordinates x' and y' of a point on the ring are related to the local angle θ by the relations:

$$\frac{dx'}{ds'} = \cos\theta, \ \frac{dy'}{ds'} = \sin\theta. \tag{2.1}$$

The hydrostatic pressure at a point (x', y') (per unit length along the cylinder) is

$$p(s') = p_0 + \rho g y' \tag{2.2}$$

where p_0 is the external pressure at $s' = 0$, ρ the fluid density, and g the gravitational acceleration.

For Hollomon material, since $\sigma = K\,|\varepsilon|^{n-1}\,\varepsilon$, the local bending moment is given by:

$$m = KI_n \left|\frac{d\theta}{ds'}\right|^{n-1} \frac{d\theta}{ds'}, \tag{2.3}$$

where $I_n = \int_A r^{n+1} dA$ represents the cylinder's flexural rigidity, see [9] for more details.

Balancing the moments acting on an element of length ds' (Figure 2) yields to:

$$\frac{dm}{ds'} = (-h' + \int_0^{s'} p(t)\sin\theta(t)dt)\sin\theta + (\int_0^{s'} p(t)\cos\theta(t)dt)\cos\theta, \tag{2.4}$$

where h' represents the horizontal component of the internal force at $s' = 0$.

Using the non-dimensional quantities:

$$s = \frac{s'}{L},\ x = \frac{x'}{L},\ y = \frac{y'}{L},\ \lambda = \frac{p_0 L^{n+2}}{KI_n},\ \tau = \frac{\rho g L^{n+3}}{KI_n},\ h = \frac{h' L^{n+1}}{KI_n},\ \Gamma = \frac{\Gamma' L^{n+1}}{KI_n},$$

equations (2.1)-(2.4) lead to the θ- formulation:

$$x_s = \cos\theta, y_s = \sin\theta, \tag{2.5}$$

$$(|\theta_s|^{n-1}\theta_s)_s = -h\sin\theta + \lambda(x\cos\theta + y\sin\theta) + \tau(u\cos\theta + \frac{1}{2}y^2\sin\theta), \tag{2.6}$$

where:

$$u = \int_0^s y(t)\cos\theta(t)dt, \text{ and } \quad x\cos\theta + y\sin\theta = \int_0^s \cos(\theta(s)-\theta(t))dt \tag{2.7}$$

Let φ(s) represents the (deviation from the circular case) function defined by:

φ(s) =θ(s)-πs. (2.8)

Using $\varphi(s)$ equations (2.5)-(2.8) lead to the φ - formulation:

$$(|\varphi_s+\pi|^{n-1}(\varphi_s+\pi))_s = -h\sin(\varphi+\pi s)+\lambda\int_0^s \cos(\varphi(s)-\varphi(t)+\pi(s-t))dt +$$

$$\tau[u\cos(\varphi+\pi s)+\frac{1}{2}y^2\sin(\varphi+\pi s)], \tag{2.9}$$

where

$$x(s)=\int_0^s \cos(\varphi(t)+\pi t)dt,\ y(s)=\int_0^s \sin(\varphi(t)+\pi t)dt,\ \ u(s)=\int_0^s y(t)\cos(\varphi(t)+\pi t)dt. \tag{2.10}$$

We seek equilibrium states φ that are symmetric about the y-axis and therefore, confine ourselves to the interval $0<s<1$. Since $\theta(0)=0$ and $\theta(1)=\pi$, $\varphi(s)$ must satisfy the boundary conditions:

$$\varphi(0)=\varphi(1)=0, \tag{2.11}$$

and the nonlinear constraints:

$$x(1)=0 \text{ and } \varphi_{ss}(1^-)=\Gamma. \tag{2.12}$$

Using $w=\varphi_s+\pi$ in equations (2.9)-(2.12) lead to the w -formulation:

$$\left(|w|^{n-1}w\right)_{ss} + \nu w = \delta - f(w) - \tau\int_0^s \sin(\pi t + \int_0^t w(\xi)d\xi)dt, \tag{2.13}$$

where

$$c=\frac{1}{2}\theta_s^2(0)-h=\frac{1}{2}[\varphi_s^2(0)+\pi]^2-h,$$

and (2.14)

$$\nu=\frac{3}{2}\pi^2-c,\quad \delta=\lambda+c\pi-\frac{\pi^3}{2},\quad f(w)=\frac{w^3}{2}+\frac{3\pi w^2}{2},$$

$$c=\frac{1}{2}\theta_s^2(0)-h=\frac{1}{2}[\varphi_s(0)+\pi]^2-h$$

with the following boundary conditions and constraints:

$$w_s(0)=0,\ \ w_s(1)=\Gamma,\ \int_0^1 w(s)ds=0,\ \int_0^1 \cos(\pi s+\int_0^s w(\xi)d\xi)ds=0,\ x(1)=0, \tag{2.15}$$

where c is an arbitrary constant of integration to be determined along with the solutions, and $w_s(0) = \theta_s(0) = 0$ due to symmetry.

Finally, with $u = |w|^{n-1} w$, we get $w = |u|^{\frac{1}{n}-1} u$ and equation (2.13) takes the form:

$$u_{ss} + \nu |u|^{\frac{1}{n}-1} u = \delta - f(|u|^{\frac{1}{n}-1} u) - \tau \int_0^s \sin(\pi t + \int_0^t |u(\xi)|^{\frac{1}{n}-1} u(\xi) d\xi) dt, \qquad (2.16)$$

where

$$\nu = \frac{3}{2}\pi^2 - c, \delta = \lambda + c\pi - \frac{\pi^2}{2}, f(u) = \frac{|u|^{\frac{3}{n}-1} u}{2} + \frac{3\pi |u|^{\frac{2}{n}-1} u}{2}.$$

3. EXISTENCE OF SOLUTIONS WITH SMALL NORMS

In this section we consider the existence of positive solutions of the boundary value problem (2.13)-(2.16) with small norms. We apply the implicit function theorem to prove the existence of positive solution in the neighborhood of some linearized solutions.

Let $\varepsilon > 0$ denotes the value of the solution at $s = 0$, that is $w(0) = \varepsilon,$ and seek positive solutions of (12) in the form:

$$w(s) = \varepsilon v(s), \qquad (3.1)$$

and therefore, $v(s)$ satisfies the (initial) conditions:

$$v(0) = 1, v_s(0) = 0, \qquad (3.2)$$

and the constraints:

$$v_s(1) - \frac{\Gamma}{\varepsilon} = 0, \quad \int_0^1 v(s) ds = 0, \text{ and } \int_0^1 \cos(\pi s + \varepsilon \int_0^s v(\xi) d\xi) ds = 0. \qquad (3.3)$$

The differential equation $v(s)$ need to satisfy is given by:

$$(v^n(s))_{ss} + \mu' v(s) = \delta' - \varepsilon^{2-n}\left(\frac{v^3(s)}{2} + \frac{3\pi\varepsilon v^2(s)}{2}\right) - \tau' \int_0^s \sin(\pi t + \varepsilon \int_0^t v(\xi) d\xi) dt, \qquad (3.4)$$

where: $\mu' = \varepsilon^{1-n}\nu,\ \delta' = \varepsilon^{-n}\delta,\ \tau' = \varepsilon^{1-n}\tau.$

We observe that the differential equation (3.4) has a unique solution $v^*(s, \varepsilon, \mu', \delta', \tau')$ which satisfies (3.2), exists for all $s \in [0,1]$, is bounded and is differentiable with respect to the parameters $\varepsilon, \mu', \delta',$ and τ'. (In fact, (3.4) can be transformed into an equation similar to (2.16) with differentiable and Lipchitz functions on the right hand side, therefore standard existence and uniqueness and smoothness of solutions apply). The existence of smooth functions $\mu'(\varepsilon), \delta'(\varepsilon),$ and $\tau'(\varepsilon)$ defined in a neighborhood of $\varepsilon = 0$ satisfying the set of the three constraints (3.3), and $\mu'(0) = n(N^2-1)\pi^3, \delta'(0) = 0,$ and $\tau'(0) = 0,$ N $\geq 2,$ follow from the Implicit Function Theorem by observing that the Jacobean: $J = \dfrac{\partial(A,B,C)}{\partial(\mu',\delta',\tau')} \neq 0$ at $\varepsilon = 0$, where:

$$A(\mu',\delta',\tau') = v_s(1) - \frac{\Gamma}{\varepsilon},\ B(\mu',\delta',\tau') = \int_0^1 v(s)ds, \text{ and } C(\mu',\delta',\tau') = \int_0^1 \cos(\pi s + \varepsilon\int_0^s v(\xi)d\xi)ds.$$

The perturbation expansions of the functions $w(s,\varepsilon), v(\varepsilon), \delta(\varepsilon),$ and $\tau(\varepsilon)$ for the case of uniform pressure: $\tau = \Gamma = 0,$ are given in [9].

4. EXISTENCE OF SOLUTIONS WITH ARBITRARY LARGE NORMS

In this section we consider the existence of solutions of the w-problem described by equations (2.13)-(2.15) for a given value of K, where: $\int_0^1 |u(s)|^2 ds = \int_0^1 |w(s)|^{2n} ds = K.$

Since the function w is a measure of the departure of the curvature from the circular state, K is a measure the bending energy level of the deformation.

THEOREM 4.1 For given values of $\Gamma > 0$ and $K > 0$, there are v, δ, τ in R and w in $W^{1,2n}(0,1)$ such that (w, v, δ, τ) is a weak solution of the nonlinear eigenvalue problem (2.13) satisfying the following boundary conditions and constraints:

$$w_s(0) = 0, \quad w_s(1) = \Gamma,$$

$$\int_0^1 |w(s)|^{2n} ds = K, \quad \int_0^1 w(s)ds = 0, \quad \int_0^1 \cos(\pi s + \int_0^s w(\xi)d\xi)ds = 0.$$

Equivalently, there are v, δ, τ in R and u in the Hilbert space $W^{1,2}(0,1)$ such that (u, v, δ, τ) is a weak solution of the nonlinear eigenvalue problem (2.16) satisfying the following boundary conditions and constraints:

$$u_s(0) = 0, \quad u_s(1) = n\Gamma\,|u(1)|^{\frac{n-1}{n}} u(1),$$

$$\int_0^1 |u(s)|^2 ds = K, \quad \int_0^1 |u(s)|^{\frac{1}{n}-1} u(s)ds = 0, \quad \int_0^1 \cos(\pi s + \int_0^s |u(\xi)|^{\frac{1}{n}-1} u(\xi)d\xi)ds = 0.$$

To prove Theorem 4.1 we need to verify some properties of the following functional on $W^{1,2}(0,1) \times W^{1,2}(0,1)$:

$$\Phi(v,u) = \frac{1}{2}\int_0^1 v_s^2 ds - \int_0^1 [\frac{n}{2(3+n)}[u(s)]^{\frac{3+n}{n}} + \frac{n}{2+n}\pi[u(s)]^{\frac{2+n}{n}}]ds - n\Gamma\,|v(1)|^{\frac{1}{n}-1} v(1)\ ,$$

and the following functionals on $W^{1,2}(0,1)$:

$$g_1(u) = \int_0^1 |u(s)|^{\frac{1}{n}-1} u(s)ds,$$

$$g_2(u) = \int_0^1 |u(s)|^2 ds, \text{ and}$$

$$g_3(u) = \int_0^1 \cos(\pi s + \int_0^s |u(\xi)|^{\frac{1}{n}-1} u(\xi)d\xi)ds.$$

We state the properties of the above functionals needed for the proof of Theorem 4.1 in the following lemma.

Lemma 4.1 The functionals defined above satisfy the following properties:
(i) Φ is differentiable semi-convex on $W^{1,2}(0,1)\times W^{1,2}(0,1)$;
(ii) Each of $g_i (i=1,2,3)$, is differentiable and weakly continuous on $W^{1,2}(0,1)$;
(iii) The set $C \equiv \{w \in W^{1,2}(0,1) : g_i(w) = c_i, i = 1,2,3, \text{are real constants}\}$ is non-empty;
(vi) $g_i', i=1,2,3,$ are linearly independent in $W^{1,2}(0,1)$; and
(v) $J(u) \equiv \Phi(u,u) \to \infty$ as $\| u \|_{1,2} \to \infty$ on the set C.

The proof s of properties (i)-(iii) of Lemma 3.1 are similar to those given in [8], and therefore omitted here.

The proof of Theorem 4.1follows by applying Lemma 3.2 of [8] and observing that the weak formulation of the nonlinear eigenvalue problem (2.13)-(2.15) requires us to solve for $(u,\lambda) \in W^{1,2}(0,1)\times R^3$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ such that:

$$J'(u) - \sum_{i=1}^{3} \lambda_i g_i'(u) = 0 \text{, where } u \in W^{1,2}(0,1).$$

That is

$$\int_0^1 u_s(s)v_s(s)ds + \int_0^1 [\frac{1}{2}u(s)^{\frac{3}{n}} + \frac{3}{2}\pi u(s)^{\frac{2}{n}}]v(s)ds + \frac{\nu}{n}\int_0^1 |u(s)|^{\frac{1}{n}} v(s)ds$$

$$+\tau \int_0^1 [\int_0^s \sin(\pi t + \int_0^t |u(\xi)|^{\frac{1}{n}-1} u(\xi)d\xi)dt]v(s)ds = \delta \int_0^1 v(s)ds$$

where: $\nu = -2\lambda_2, \tau = -\lambda_3, \delta = \lambda_1 - \lambda_3 \int_0^1 \sin(\pi t + \int_0^t u(\xi)|^{\frac{1}{n}-1} u(\xi)d\xi)dt.$

This completes the proof for Theorem 4.1.

5. EXISTENCE OF SOLUTIONS FOR GIVEN PRESSURE GRADIENTS

In this section we establish two existence theorems with τ being given but with K unspecified. In the first theorem both τ and Γ are given. It turns out that they must satisfy the inequality $\Gamma < \frac{\tau}{2\pi}$ which has a simple geometric interpretation.

THEOREM 5.1 For given values of $\tau > 0$ and $\Gamma > 0$ such that $\Gamma < \frac{\tau}{2\pi}$ there exists a weak solution of the boundary value problem described by (2.9) through (2.12).

Proof:

We let $H \equiv W^{1,1+n}(0,1)$, and define for $\psi, \varphi \in H$ the following three functionals:

$$\Phi(\psi,\varphi) = \frac{1}{n+1}\int_0^1 (|\psi_s + \pi|^n (\psi_s + \pi))_s \, ds + \tau \int_0^1 [u(\varphi)\sin(\varphi + \pi t) - \frac{1}{2} y^2(\varphi)\cos(\varphi + \pi t)]dt,$$

$$g_1(\varphi) = \int_0^1 \cos(\varphi + \pi t)dt,$$

and

$$g_2(\varphi) = \int_0^1 y(\varphi)\cos(\varphi + \pi t)dt.$$

We can verify (see [8] for details) the following properties:

(i) Φ is differentiable, semi-convex on $H \times H$,

(ii) The functionals g_i, $i = 1, 2,$ are differentiable and weakly continuous on H,

(iii) The functionals $g_i^{'}$, $i = 1, 2,$ are linearly independent,

(iv) The set: $C \equiv \left\{ \varphi \in H : g_1(\varphi) = 0, g_2(\varphi) = -\frac{\Gamma}{\tau} \right\}$ is non-empty, and

(v) $\Phi(\varphi,\varphi) \to \infty$ as $\|\varphi\| \to \infty$ on C.

Properties (i)-(v) above allow us to apply Theorem 5 of [2] (which holds for any reflexive Banach space and in particular for H) to conclude the existence of φ in H and constants λ_1 and λ_2 such that:

$$\Phi^{'}(\varphi,\varphi) = \lambda_1 g_1^{'}(\varphi) + \lambda_2 g_2^{'}(\varphi).$$

Taking $h = \lambda_1$ and $\lambda = \lambda_2$, it follows that (φ, h, λ) is a weak solution of the boundary value problem (2.9)-(2.12).

THEOREM 5.2 For given positive values of the parameters λ and τ, there exists a weak solution (φ, h), $\varphi \in H$, of the boundary value problem (2.9)-(2.12).

The proof of this theorem goes along the same lines as that of Theorem 5.1, and is, therefore, omitted.

6. CONCLUDING REMARKS

(1) In this paper, we have been concerned with the existence of equilibrium states of hollow elasto-plastic cylinders fully submerged in a fluid. By considering "planar" deformations of the cylinders we have generalized the results obtained for an inextensible elastic model in [8].

(2) A similar existence of solutions theorem to Theorem 4.3 of [8] can be verified for case of a hollow Hollomon cylindrical tube partially submerged in a fluid.
(3) The buoyancy force Γ in Theorem 5.2 is to be determined along the solution by the formula: $\Gamma = -\tau \int_0^1 y(t)\cos(\varphi(t) + \pi t)dt.$

Acknowledgement: The research in this paper is a part of a research project funded by the Research office, Texas A & M University at Qatar.

REFERENCES

[1] Adams, R. A., Sobolev Spaces, Academic Press, San Diego, 1978.

[2] ANTMANN, S.S., A note on a paper of Tadjbakhsh, J. Math. An. 21 (1968), 132-135.

[3] ANTMANN, S.S., General solutions for plane extensible elastica having nonlinear stress-strain laws, Quart. Appl. Math. 26. (1968), 35-47.

[4] ANTMANN, S.S., The shape of buckled nonlinearly elastic rings, Z.A.M.P. 21 (1970), 422-438.

[5] ANTMANN, S.S., Existence of solutions of equilibrium equations for nonlinearly elastic rings and arches, Indiana University Math. J. 20 No. 3 (1970), 281-302.

[6] BROWDER, F., Variational methods for non-linear elliptic eigenvalue problems, Bull. Amer. Math. Soc. 71 (1965), 176-183.

[7] Djondjorov P.A., Vassil M Vassilev, Ivaïlo M Mladenov, Analytic Description of the Equilibrium Shapes of Elastic Rings Under Uniform Hydrostatic Pressure, AIP Conference Proceedings 01/2011; 1340:189-202. DOI:10.1063/1.3567137.

[8] Elgindi, M.B.M. and Yen, D.H.Y., Existence of Equilibrium States of Hollow Elastic Cylinders Submerged in a Fluid, Int. J. Math. & Math. Sci. Vol. 15- 2 (1992) 385-396.

[9] Kamran, K, Elgindi, M. B. M., Wei, D. and Liu, Y., Buckling and Deformation of Hollomon's Power-Law Tubes, (in preparation) preprint.

[10] Tadjbakhsk, I. and Odeh, F., Equilibrium states of elastic rings, J. Math. Anal. Appl,18 (1967), 59- 74.

[11] Zeidler E., Nonlinear Functional Analysis and its Applications III, Variational Methods and Optimization, Springer-Verlag, Berlin 1985.

Received: April 29, 2013