

A Short Note On Solving 1-D Porous Medium Equation by Finite Element Methods

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Abstract

Porous Medium Equation (PME) is one of the simplest types of nonlinear evolution equation of parabolic type. It emerges in the description of different natural phenomena, and its theory and properties depart strongly from the heat equation, $u_t = \Delta u$, its most famous relative. Hence the interest of its study, both for the pure mathematician and the applied scientist (Vazquez, 2006). The aim of this paper is to study the Porous Medium Equation in one dimension.

1 Introduction

The nonlinear equation of the form

$$(1.1) \quad u_t = \Delta(u^m), \quad m > 1,$$

is usually called the Porous Medium Equation (PME), where $u = u(x, t)$ is nonnegative scalar function of space $x \in \mathbb{R}^d$ and time $t \in \mathbb{R}$, the space dimension is $d \geq 1$ and m is a constant larger than 1. $\Delta = \Delta_x$ denotes the Laplace operator acting on the space variables. The theory and properties of 1.1 strongly depend on heat equation $u_t = \Delta u$ [8]

The PME is one of the important special cases of a partial differential equation-called Nonlinear Diffusion.

Another well-known nonlinear degenerate parabolic equation

$$(1.2) \quad u_t = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$$

is called *p-Laplacian diffusion equation*, which has its own characteristics. We will only deal with the PME throughout this paper.

Derivation of PME

As it was said before, the PME departs strongly from the linear *heat equation*. The main difference is that the linear diffusion term Δu in the heat equation is replaced by $\Delta(u^m)$, for some $m > 1$ in the PME.

$$(1.3) \quad \boxed{u_t = \Delta(|u|^{m-1}u)}$$

Analogously, 1.3 could be written as a heat equation with a nonlinear diffusion constant $m|u|^{m-1}$:

$$(1.4) \quad u_t = \nabla \cdot (mu^{m-1}\nabla u)$$

Both 1.3 and 1.4 demonstrate that the diffusivity $m|u|^{m-1}$ "switches off" as $|u| \rightarrow 0$ for $m < 1$. So the question is where such equation might arise, in which the diffusion constant alters proportionally to the power of diffused amount? There are applications in biology, particularly in models of animal and insect dispersal, also in plasma physics. Moreover, as suggested by the name, there is an application in the study of the flow of a gas in a porous medium. Porous Medium Equation (1.3) can be derived in uncomplicated way in the following manner [3]. Neglecting certain constants, the gas flow is governed by the following three equations:

$$\begin{aligned} \rho_t &= -\nabla \cdot (\rho v) && \text{(conservation of mass),} \\ v &= -\nabla p && \text{(Darcy's Law (Grindrod,1991)),} \\ \rho &= p^\gamma && \text{(equation of state).} \end{aligned}$$

where ρ is the density, p is the pressure, v is the velocity, and γ is the (constant) ratio of specific heats. Substituting for p and v , we get

$$(1.5) \quad \rho_t = \nabla \cdot (\rho \nabla (\rho^{\gamma-1})).$$

Since $\rho \nabla (\rho^{\gamma-1}) = \gamma^{-1} \rho^{\gamma-1} \nabla \rho$, and $\nabla (\rho^{1+\gamma-1}) = (1 + \gamma^{-1}) \rho^{\gamma-1} \nabla \rho$, it is possible to write 1.5 as

$$\rho_t = \frac{\gamma^{-1}}{1 + \gamma^{-1}} \Delta(\rho^{1+\gamma-1}) = \frac{1}{1 + \gamma^{-1}} \Delta(\rho^{1+\gamma-1}).$$

Finally, setting $u = \rho$ and rescaling t by $1/(1 + \gamma)$ gives us 1.3.

PME is just one of many examples of PDE of Nonlinear Diffusion type. A quite general form of the nonlinear diffusion equation is

$$(1.6) \quad \partial_t H(x, t, u) = \sum_{i=1}^d \partial_{x_i} (A_i(x, t, u, Du)).$$

Conditions that are to be imposed on the functions H and A_i are as follows: $\partial_t H(x, t, u) \geq 0$, and the matrix $(a_{ij}) = (\partial_{u_j} A_i(x, t, u, Du))$ should be positive semidefinite. The so-called *p-Laplacian evolution* (PLE)

$$(1.7) \quad \partial_t u = \operatorname{div}(|\nabla u|^{p-2} \nabla u),$$

has got much attention from researchers, but we will not cover it here. It is part of a general theory of diffusion with diffusivity depending on the gradient of the main unknown. It has a parallel, sometimes divergent, sometimes convergent theory [8].

2 Introduction to Discontinuous Galerkin Method

In this section a short introduction to the discontinuous Galerkin (dG) method will be given. The theory of dG is taken from the book of Claes Johnson called “Numerical solution of partial differential equations by the finite element method” [4]. The method can be formulated as follows.

Let V_h be a finite-dimensional subspace of V with basis $\{\phi_1, \dots, \phi_M\}$. For definiteness we shall assume that Ω is a polygonal convex domain and that V_h consists of piecewise linear functions on a quasi-uniform triangulation of Ω with mesh size h and satisfying the minimum angle condition. Replacing V by the finite-dimensional subspace V_h we get the following problem: Find $u_h(t) \in V_h$, $t \in I$, such that

$$(2.1) \quad (\dot{u}_h(t), v) + a(u_h(t), v) = (f(t), v) \quad \forall v \in V_h, t \in I,$$

$$(2.2) \quad (u_h(0), v) = (u^0, v) \quad \forall v \in V_h.$$

Now let us consider the discontinuous Galerkin method for 2.1 based on using a finite element formulation to discretize in the time variable. To formulate this method we need to introduce, for a given integer $q > 0$, the space

$$W_{hk} = \{v : I \rightarrow V_h : v|_{I_n} \in \mathbf{P}_q(I_n), n = 1, \dots, N\},$$

where

$$\mathbf{P}_q(I_n) = \{v : I_n \rightarrow V_h : v(t) = \sum_{i=0}^q v_i t^i \quad \text{with} \quad v_i \in V_h\},$$

that is, W_{hk} is the space of functions on I with values in V_h that on each time interval I_n vary as polynomials of degree at most k . The functions v in W_{hk} may be discontinuous in time at the discrete time levels t_n . To account for this let us introduce the notation

$$v_+^n = \lim_{s \rightarrow 0^+} v(t_n + s), \quad v_-^n = \lim_{s \rightarrow 0^-} v(t_n + s) \\ [v^n] = v_+^n - v_-^n,$$

where $[v_n]$ is the jump of v at t_n .

Now, the discontinuous Galerkin method for 2.1 is defined as follows: Find $U \in W_{hk}$ such that

$$(2.3) \quad A(U, v) = L(v) \quad \forall v \in W_{hk},$$

where

$$A(w, v) := \sum_{n=1}^N \int_{I_n} ((w, v) + a(w, v)) dt \\ + \sum_{n=2}^N ([w^{n-1}], v_+^{n-1}) + (w_+^0, v_+^0), \\ L(v) = \int_I (f, v) dt + (u^0, v_+^0)$$

Since $v \in W_{hk}$ varies independently on each subinterval I_n , it is possible to formulate 2.3 in the alternative way: For $n = 1, 2, \dots, N$, given U_-^{n-1} , find $U \equiv U|_{I_n} \in \mathbf{P}_q(I_n)$, such that

$$(2.4) \quad \int_{I_n} ((\dot{U}, v) + a(U, v)) dt + (U_+^{n-1}, v_+^{n-1}) = \int_{I_n} (f, v) dt + (U_-^{n-1}, v_+^{n-1}), \quad \forall v \in \mathbf{P}_q(I_n),$$

where $U^0 = u^0$

For $q = 0$, if we use a notation $U^n \equiv U_-^n \equiv U_+^{n-1}$, 2.4 simplifies to the following problem: For $n = 1, \dots, N$, find $U^n \in V_h$, such that

$$(2.5) \quad (U^n - U^{n-1}, v) + k_n a(U_n, v) = \int_{I_n} (f, v) dt \quad \forall v \in V_h, n = 1, \dots, N$$

where $U_-^n = u^0$.

For $q = 1$, 2.4 is equivalent to the following system with $U(t) = U_0 + \frac{t-t_{n-1}}{k_n} U_1$, $t \in I_n, U_i \in V_h$,

$$(2.6) \quad \begin{aligned} (U_0, v) + k_n a(U_0, v) + (U_1, v) + \frac{1}{2} k_n a(U_1, v) \\ = (U_-^{n-1}, v) + \int_{I_n} (f(s), v) ds, \quad \forall v \in V_h \end{aligned}$$

$$(2.7) \quad \begin{aligned} \frac{1}{2} k_n a(U_0, v) + \frac{1}{2} (U_1, v) + \frac{1}{3} k_n a(U_1, v) \\ = \frac{1}{k_n} \int_{I_n} (s - t_{n-1})(f(s), v) ds, \quad \forall v \in V_h. \end{aligned}$$

3 The Two Finite Element Solution

In order to have a feeling how the finite element method works on a larger scope (for considerable amount of nodes), it would be beneficial to see how it operates on two nodes. Let us see it on the following type of PME, with $m = 2$:

$$(3.1) \quad \begin{aligned} u_t - (2uu_x)_x &= 0, \quad x \in [-12; 12] \\ u(-12, t) = u(12, t) &= 0 \quad u(x, 0) = 3 - \frac{x^2}{12}, \quad |x| < 6 \end{aligned}$$

To get to the weak form we need to multiply both sides of the question by the test function:

$$\int_{-12}^{12} u_t \phi(x) dx - \int_{-12}^{12} (2uu_x)_x \phi(x) dx = 0,$$

which, after integrating by parts the second integral, gives us

$$(3.2) \quad \int_{-12}^{12} u_t \phi(x) dx + \int_{-12}^{12} 2uu_x \phi(x) dx = 0.$$

Since our solution is the function of time and space, we can represent it in the following way:

$$(3.3) \quad u(x, t) = \sum_{i=1}^{GN} u_i(t) \phi_i(x) = u_1(t) \phi_1(x) + u_2(t) \phi_2(x) + u_3(t) \phi_3(x) = u_2(t) \phi_2(x),$$

where $\phi_M(x) = \chi^{(M)}(x) N_2^{M-1}(x) + \chi^{(M)}(x) N^{(M)}(x)$,

$$\text{and } N_2^{(1)} = \frac{x - x_1^{(e)}}{x_2^{(e)} - x_1^{(e)}}, \quad N_1^{(1)} = \frac{x_2^{(e)} - x}{x_2^{(e)} - x_1^{(e)}},$$

due to the boundary conditions, i.e. $u_1(t)\phi_1(x) = u_3(t)\phi_3(x) = 0$.

In our case, $N_2^{(1)} = \frac{x+12}{12}$, $N_1^{(1)} = \frac{12-x}{12}$, which is the part of global solution. Now we need to look for the local solution of 3.2

$$\int_{x_1^{(e)}}^{x_2^{(e)}} u_t \phi(x) dx = \int_{x_1^{(e)}}^{x_2^{(e)}} (N_1^{(e)} \dot{u}_1^{(e)}(t) + N_2^{(e)} \dot{u}_2) \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \end{bmatrix} dx.$$

Let us change coordinate to make computation easier:

$$dx = \frac{l}{2} d\xi, \quad N_1(\xi) = \frac{1-\xi}{2}, \quad N_2(\xi) = \frac{1+\xi}{2},$$

where l is the length of the node and is equal to 12. Thus, changing coordinates and evaluating an integral give us $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

Doing the same operations on coordinates for the second integral, we get

$$\begin{aligned} 2 \int_{x_1^{(e)}}^{x_2^{(e)}} uu_x \phi_x dx &= 2 \int_{x_1^{(e)}}^{x_2^{(e)}} u \frac{du}{d\xi} \frac{d\xi}{dx} \frac{d\phi}{d\xi} \frac{d\xi}{dx} dx \\ &= \frac{1}{l} (u_1 + u_2) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

Finally, our matrix will look like following:

$$\begin{bmatrix} \frac{u_1+u_2}{12} & \frac{-u_1-u_2}{12} & 0 \\ \frac{-u_1-u_2}{12} & \frac{u_1+u_2}{12} + \frac{u_2+u_3}{12} & \frac{-u_2-u_3}{12} \\ 0 & \frac{-u_2-u_3}{12} & \frac{u_2+u_3}{12} \end{bmatrix}$$

Now, summing two functions and equating to zero, we get:

$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} \frac{u_1+u_2}{12} & \frac{-u_1-u_2}{12} & 0 \\ \frac{-u_1-u_2}{12} & \frac{u_1+u_2}{12} + \frac{u_2+u_3}{12} & \frac{-u_2-u_3}{12} \\ 0 & \frac{-u_2-u_3}{12} & \frac{u_2+u_3}{12} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0$$

After all, we come up with the following ODE:

$$(3.4) \quad 8u' + \frac{1}{6}u_2^2 = 0.$$

Solution to 3.4 is $u = 48/(t + C)$. From Vidden (2012) [9] it is known that the exact solution for 4.1 is of the form

$$U(x, t) = \begin{cases} (t+1)^{-1/3} \left(3 - \frac{x^2}{12(t+1)^{2/3}} \right), & |x| < 6(t+1)^{1/3} \\ 0, & |x| \geq 6(t+1)^{1/3}. \end{cases}$$

It is fair enough to claim that our two-element approximation is relatively fine since it is of the form C_1/t_1 . Of course it is far from being close to exact solution of 4.1. The reason why it is so is the least possible amount of nodes that we have used in our approximation. The more we have nodes, the better the approximation to the exact solution.

4 Numerical Examples

In this section we will consider two numerical examples. In producing the results, the finite element package "COMSOL Multiphysics" computer software was used. COMSOL is a leader among software aimed for conducting a finite element analysis for different physics and engineering applications. This is the reason why COMSOL was chosen to produce a numerical approximations of two equations being considered. These two examples were taken from [9]

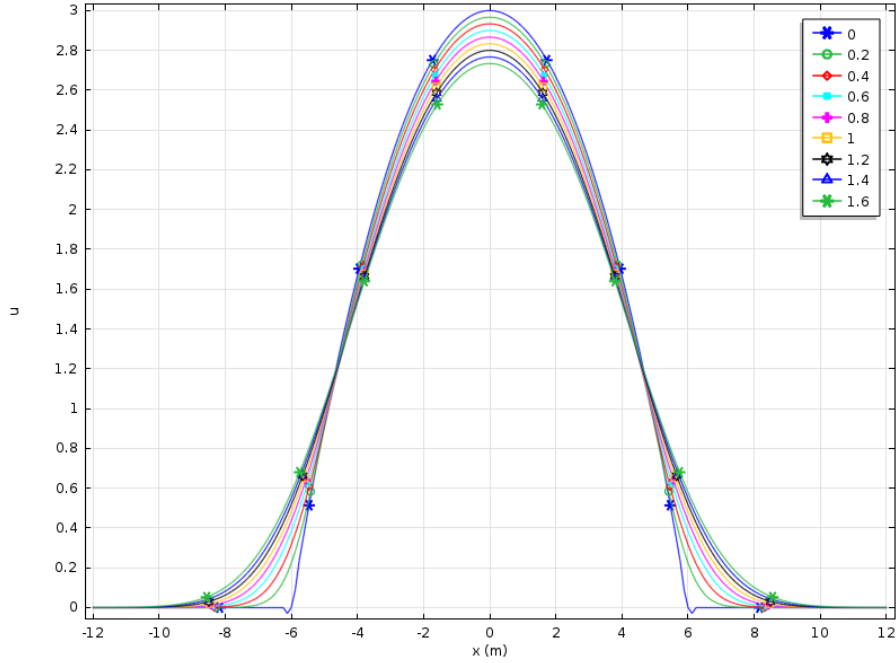


Figure 1: Evolution of numerical example 1

Example 1. The first PDE is the one that we have looked for in the last section:

$$(4.1) \quad \begin{aligned} u_t - (2uu_x)_x &= 0, \quad x \in [-12; 12] \\ u(-12, t) = u(12, t) &= 0 \quad \begin{cases} u(x, 0) = 3 - \frac{x^2}{12}, & |x| < 6. \\ u(x, 0) = 0, & |x| \geq 6 \end{cases} \end{aligned}$$

In figure 1 we illustrate the evolution of the symmetric DG solution at times $t = 0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6$. We have an oscillation for $t = 1.4$. This oscillations of the numerical solution occur because of our using of the continuous finite elements in COMSOL. As we can observe, the compact support gets larger for the numerical solution with every time step.

Example 2. The next example is an attempt to analyze the General Porous Medium Equation

$$(4.2) \quad u_t - (u^m)_{xx} = 0, \quad x \in [-6, 6].$$

For the zero boundary condition, the Barenblatt solution [9] is given by

$$U(x, t) = \begin{cases} (t+1)^{\frac{-1}{m+1}} \left(3 - \frac{m-1}{2m(m+1)} \frac{x^2}{(t+1)^{\frac{2}{m+1}}} \right), & |x| \leq \sqrt{\frac{6m(m+1)}{m-1}} (t+1)^{\frac{1}{m+1}} \\ 0, & |x| \geq \sqrt{\frac{6m(m+1)}{m-1}} (t+1)^{\frac{1}{m+1}}. \end{cases}$$

For this example, the solution was computed until time $t = 1.6$ with the 0.2 interval step. Barenblatt's

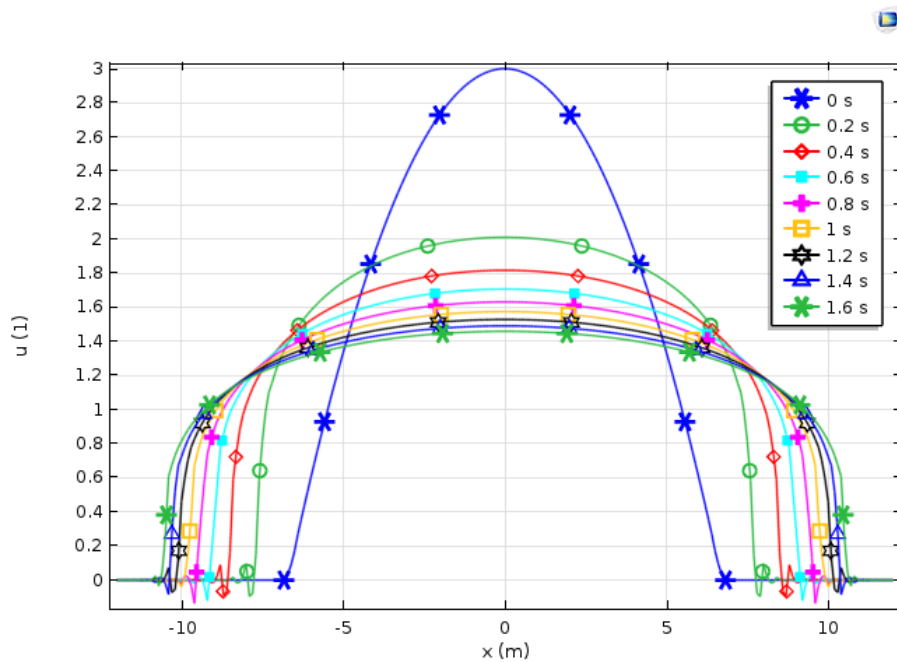


Figure 2: Evolution of numerical example 2

solution was used as an initial condition. Figure 2 shows us the run of equation 4.2 for $m=5$. There are oscillations in each of the time steps. Again, oscillations occur because of continuous finite elements in COMSOL. Also, the compact support gets larger in the finite element solution with every time step. The dG method is designed to overcome these oscillations which the continuous finite elements can not remove, however it is not yet available in the software. It can be useful to use dG on these two examples to see how much improvement dG can provide over the continuous finite element method. The mathematical analysis of PME and the associated numerical analysis is beyond a undergraduate research project and therefore we stop here.

5 Conclusion and Future Work

In this paper Porous Medium Equation was studied and analyzed. Also, the work provided some theory about PME itself and an introduction to the dG method. For the sake of understanding the basic operation of finite elements method, two element solution was considered. The result has shown that two-element solution is certainly far from being fine. Also, we have considered two numerical examples that illustrated the diffusion type of propagation of the solution on the uniform mesh. However, oscillations of the numerical solution's propagation associated with the time steps took its place. Certainly, to fix this oscillations problem it is the part of the work that remains be done. Also, examination of admissibility analysis results are still need to take place in the future research.

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