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Stable holomorphic polynomials on the half-plane and generalizations

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1. Introduction

The study of locations of zeroes of functions became popular among mathematicians many years ago. This investigation contributes a lot to wide range of theories and topics in Mathematics and Physics.

We start now giving some definitions and notations which will be used further.

Definition 1. *Let A be a set on the complex plane, and $f(z)$ be a holomorphic function defined on A. Then $f(z)$ is said to be **stable on** A, or A**-stable**, if $f(z)$ is nonzero for all points in A.*

When $A = {H_R}^n = \{z \in \mathbb{C}^n | \Re(z_i) > 0, i = 1, 2, ..., n\}$ in Definition 1, we call $f(z)$ to be **Hurwitz stable**. When $A = {H_U}^n = \{z \in \mathbb{C}^n | \Im(z_i) > 0, i = 1, 2, ..., n\}$ in Definition 1, we call $f(z)$ to be **stable**. We say $f(z)$ has a **half-plane property** if $f(z) \neq 0$ for all $z \in H^n$, where H is any half-plane of the complex plane.

The study of stable functions can be applied to matrix theory and theory of graphs, see [1] and [5]. There are also connections with matroid theory in [4] as well as with combinatorics and probability in [8]. Some importance in physics can be noted including application to conformal field theory in [7], and the Lee-Yang theory of phase transitions, see [3] and [2].

This paper shows results on the stability of polynomials in n-variables with complex coefficients $P \in \mathbb{C}(z_1, z_2, ..., z_n)$.

Definition 2. *The set of indexes of variables $\{1, 2, ..., n\}$ is called **a ground set of** P and we denote it as $E = E(P)$. Usually, $P = \sum_{m \in S} a_m z^m$, where $a_m \in \mathbb{C}$ and S is a collection of n-tuple vectors with elements of type $m = (m_1, m_2, ..., m_n) \in (\{0\} \cup \mathbb{Z}_+)^n$ and $z^m = \prod_{q \in E(P)} {z_q}^{m_q}$. A polynomial P is called to be **multiaffine** if each z_k in $P = \sum_{m \in S} a_m z^m$ has a degree at most 1. The **support** of the polynomial P is $supp(P) = \{m | m \in S, a_m \neq 0\}$.*

We also consider polynomials stable on the unit ball. In this paper we investigate quadratic polynomials $P(z, w)$ which are nonzero for all $z, w \in \mathbb{C}$ defined by the inequality $|z|^2 + |w|^2 < 1$.

The plan of this paper is as follows: In Section 2 we give some background information related to the polynomials stable on the right and upper half-planes. In Section 3 we define matroids, delta-matroids and jump systems, and expand the proof of the theorem that the support of a stable polynomial is a jump system which was initially given by Branden, [4]. In Section 4 we discuss thre stability of quadratic polynomials and transformations done to identify connections between the stability on the unit ball and half-planes. In Section 5 we prove some properties of polynomials stable on the unit ball. In Section 6 we conclude, and give possible directions for further research.

2. Stable and Hurwitz stable polynomials

Here we will give previous results on polynomials stable on right and upper half-planes. First we will start with Hurwitz stable polynomials which are nonzero on the product of the right half-planes. Some properties of such polynomials are given in [5].

Using the following methods of constructions of new polynomials, we define:

(1) Let $e \in E(P)$ be arbitrary, we define $P^{\setminus e}$ to be a polynomial obtained from the $P = \sum_{m \in S} a_m z^m$ by taking $z_e = 0$. The new polynomial is called *the deletion of* e *from* P.
(2) Let $e \in E(P)$ be arbitrary, then $P^{/e} = \frac{\partial P}{\partial z_e}$ is called *the contraction of* e *from* P.
(3) Let A be arbitrary subset of $E(P)$, then $P^{\beta A}(z) = \sum_{m \in S: m_e \leq 1, \, all \, e \in A} a_m z^m$ is called *the multiaffine part of* P.
(4) Let A be arbitrary subset of $E(P)$, then the process of making new polynomial $P^{\# A}(z) = \sum_m a_m \prod_{e \in A} {z_e}^{m_e \mod 2} \prod_{e \in E \setminus A} {z_e}^{m_e}$ is called *folding mod* 2.

The collection of new polynomials constructed above has the following property:

Lemma 3. If polynomial P is Hurwitz stable and A is any subset of $E(P)$, then each of $P^{\setminus e}$, $P^{/e}$, $P^{\beta A}$ and $P^{\# A}$ is Hurwitz stable.

As an example we can consider a polynomial

$$P(z_1, z_2) = a_{22} z_1^2 z_2^2 + a_{21} z_1^2 z_2 + a_{12} z_1 z_2^2 + a_{20} z_1^2 + a_{02} z_2^2 + a_{11} z_1 z_2 + a_{10} z_1 + a_{01} z_2 + a_{00}.$$

If $P(z_1, z_2)$ is Hurwitz stable, then, by Lemma 3, both

$$P^{\setminus 1} = a_{02} z_2^2 + a_{01} z_2 + a_{00}$$

and

$$P^{/1} = 2a_{22} z_1 z_2^2 + 2a_{21} z_1 z_2 + a_{12} z_2^2 + 2a_{20} z_1 + a_{11} z_2 + a_{10}$$

are also Hurwitz stable.

Definition 4. *A polynomial is* ***real stable*** *if it is stable and all coefficients are real.*

Actually, we can imply the definition of real stability for any set A, just saying that a polynomial with real coefficients is nonzero on the set A. (real stable on A)
The theorems related to stable and real stable polynomials are proved in [4]:

Theorem 5. Let $P = H + iG \neq 0$, where $H, G \in \mathbb{R}[z_1, z_2, ..., z_n]$, and let z_{n+1} be a new indeterminate. Then the followings are equivalent:
(a) $P = H + iG$ is stable,
(b) $H + z_{n+1}G$ is real stable,
(c) All nonzero polynomials in the pencil $\{\alpha H + \beta G : \alpha, \beta \in \mathbb{R}\}$ are real stable and

$$\frac{\partial h}{\partial z_j}(x) \cdot g(x) - h(x) \cdot \frac{\partial g}{\partial z_j}(x) \geq 0 : \forall 1 \leq j \leq n, x \in \mathbb{R}^n.$$

Let the operation $\Delta_{ij}(f)$ be defined as:

$$\Delta_{ij}(f) = \frac{\partial f}{\partial z_i} \cdot \frac{\partial f}{\partial z_j} - \frac{\partial^2 f}{\partial z_i \partial z_j} \cdot f.$$

Theorem 6. Let $P \in \mathbb{R}[z_1 ... z_n]$ be multiaffine. Then the following are equivalent
(a) For all $x \in \mathbb{R}^n$ and $1 \leq i, j \leq n$

$$\Delta_{ij}(P)(x) \geq 0,$$

(b) P is stable.

Corollary 7. Let $P(z_1, z_2) = a_{11}z_1z_2 + a_{10}z_1 + a_{01}z_2 + a_{00} \in \mathbb{R}[z_1, z_2]$. Then $P(z_1, z_2)$ is stable if and only if

$$\Delta_{12}(f) = \frac{\partial f}{\partial z_1} \cdot \frac{\partial f}{\partial z_2} - \frac{\partial^2 f}{\partial z_1 \partial z_2} \cdot f$$

$$= (a_{11}z_2 + a_{10})(a_{11}z_1 + a_{01}) - a_{11}(a_{11}z_1z_2 + a_{10}z_1 + a_{01}z_2 + a_{00})$$

$$= a_{10}a_{01} - a_{11}a_{00} \geq 0.$$

3. Stability of polynomials and matroid theory

Definition 8. *A* ***matroid*** *is a pair (E, M), where M is a collection of subsets of a finite set E satisfying,*
(1) $M \neq \emptyset$,
(2) for any $B \in M$ and $A \subseteq B$, $A \in M$,
(3) if $A, B \in M$ and $|A| > |B|$, then there exists $x \in A \backslash B$ such that $B \cup \{x\} \in M$,
(4) let Γ be the set of maximal elements of M, then, for any $A, B \in \Gamma$ and $x \in A \backslash B$, there is $y \in B \backslash A$ such that $(A - \{x\}) \cup \{y\} \in \Gamma$.

All elements of M are called *independent sets*, and all elements of Γ are called *bases* of the matroid.

Definition 9. *A **delta matroid** is a pair (E, F), where F is a collection of subsets of a finite set E satisfying,*
(1) $F \neq \emptyset$,
(2) $\bigcup_{A \in F} A = E$,
(3) symmetric exchange axiom: if $A, B \in F$ and $x \in A \triangle B$, then there exists $y \in A \triangle B$ such that $A \triangle \{x, y\} \in F$.

Here $A \triangle B = (A \cup B) \backslash (A \cap B)$ is the symmetric difference.
Notice that the set of bases of the matroid is a delta matroid.

Definition 10. *Let $\alpha, \beta \in \mathbb{Z}^n$. Define $St(\alpha, \beta) = \{\sigma \in \mathbb{Z}^n : |\sigma| = 1, |\alpha + \sigma - \beta| = |\alpha - \beta| - 1\}$ as the set of **steps from** α **to** β, where $|\alpha| = \sum_{i=1}^{n} |\alpha_i|$.*

Two-step Axiom: Let F be a collection of points in $\mathbb{Z}^n$. If $\alpha, \beta \in F, \sigma \in St(\alpha, \beta)$ and $\alpha + \sigma \notin F$, then there is a $\tau \in St(\alpha + \sigma, \beta)$ such that $\alpha + \sigma + \tau \in F$.

Definition 11. *A collection F of points in $\mathbb{Z}^n$ is called a **jump system** if it satisfies Two-step Axiom.*

The connection between delta matroids and jump system can be defined by the following statement from [6]:
A delta matroid is a collection of subsets of $V_n = (1, 2, \ldots, n)$ whose characteristic vectors define a jump system.
Here, $x = (x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$ is the characteristic vector of the set $A \subseteq V$ if it satisfies $a \in A \iff x_a = 1$.
For example, the set $(1, 3, 5) \subset V_5$ has a characteristic vector $(1, 0, 1, 0, 1) \subset \{0, 1\}^5$.
To show one of the relations of stable polynomials to matroid theory, we introduce the theorem that the support of a stable polynomial is a jump system, see [4].

Theorem 12. If P is a polynomial with half-plane property, then the support of P is a jump system.

Proof. Given $P(z) = \sum_m a_m z^m$ is a polynomial with the half-plane property which is nonzero for some product of half-planes $H^n \subset \mathbb{C}^n$. Each half-plane can be presented as $H = e^{i\theta} z : \Re(z) > 0$, and $P(z)$ is stable on H^n if and only if $P(e^{i\theta} z)$ is Hurwitz stable. Multiplication of coefficients by $e^{i\theta}$ does not nullify them, so $supp(P(z)) = supp(P(e^{i\theta} z))$. Therefore, we can consider $P(z)$ as Hurwitz stable polynomial.
Let $\alpha, \beta \in supp(P)$ for which the Two-step Axiom does not hold. Suppose

$\alpha_i > \beta_i$ for some values $i \in S \subset E(P)$, and $\alpha_j \leq \beta_j$ for $j \in T \subset E(P)$, such that $S \neq \emptyset \neq T$ and $S \cup T = E(P)$.

We say $\alpha \leq \beta$ if $\alpha, \beta \in \mathbb{R}^n$ and $\alpha_k \leq \beta_k$ for all $k = 1, 2, \ldots, n$.

Let's introduce next function which changes the variables of polynomial P:

$$\mu(z) : z_i \to \begin{cases} z_i^{-1} & \forall i \in S \\ z_i & \forall i \in T \end{cases}$$

and substitute it in polynomial P instead of z:

$g(z) = z^\gamma P(\mu(z))$, where $\gamma \in \mathbb{Z}_+^{|E(P)|}$ is sufficiently large so that $g(z)$ is a polynomial.

Then, elements α and β of support of P considered above become α' and β' elements of support of $g(z)$ respectively, and:

$$\begin{cases} \alpha' = \begin{cases} \gamma - \alpha_i & \forall i \in S \\ \gamma + \alpha_j & \forall j \in T \end{cases} \\ \beta' = \begin{cases} \gamma - \beta_i & \forall i \in S \\ \gamma + \beta_j & \forall j \in T \end{cases} \end{cases} \to \begin{cases} \gamma - \alpha_i < \gamma - \beta_i & \forall i \in S \\ \gamma + \alpha_j < \gamma + \beta_j & \forall j \in T \end{cases} \to \alpha' \leq \beta'.$$

Now, both z_i and z_i^{-1} have real parts of equal sign, so change of variables $\mu(z)$ will preserve stability of function, thus $P(\mu(z))$ is stable if and only if $P(z)$ is stable. Multiplication of $P(\mu(z))$ by z^γ will preserve stability also, so, finally, stability of $P(z)$ leads to stability of $g(z)$ and vice versa. Therefore, without loss of generalization, using transformation of (α, β) to (α', β'), we can suppose that Two-step Axiom is invalid for some $\alpha, \beta \in supp(P)$ with $\alpha \leq \beta$.

Next, suppose that in polynomial $P(z) = \sum_m a_m z^m$ the maximal degree of each z_i is k_i, and $k = (k_1, k_2 \ldots k_n)$, thus:

$$P(\frac{1}{z}) = \sum_m a_m (\frac{1}{z})^m \to z^k P(\frac{1}{z}) = z^k \sum_m a_m (\frac{1}{z})^m = \sum_m a_m z^{k-m}.$$

We use the following operation for $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$:

$$\partial^\alpha P(z_1, z_2, \ldots, z_n) = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}} P(z_1, z_2, \ldots, z_n)$$

. Hence, for the above polynomial we have:

$$P_\beta(z) = \partial^{k-\beta}\left(z^k P(\frac{1}{z})\right) = \sum_m a_m \frac{(k-m)!}{(\beta-m)!} z^{\beta-m}$$

$$\rightarrow P_\beta(\frac{1}{z}) = \sum_m a_m \frac{(k-m)!}{(\beta-m)!} (\frac{1}{z})^{\beta-m} \rightarrow z^\beta P_\beta(\frac{1}{z}) = \sum_m a_m \frac{(k-m)!}{(\beta-m)!} z^m$$

$$\rightarrow P_{\alpha,\beta}(z) = \partial^\alpha\left(z^\beta P_\beta(\frac{1}{z})\right) = \sum_m a_m \frac{(k-m)!}{(\beta-m)!}\frac{m!}{(m-\alpha)!} z^{m-\alpha}$$

$P_(\alpha,\beta)(z)$ is polynomial because during the transformations of $P(z)$, only those m with $\alpha \le m \le \beta$ remained. The coefficient of $z^{m-\alpha}$ in the new polynomial is $a_{m-\alpha} = a_m \frac{(k-m)!}{(\beta-m)!}\frac{m!}{(m-\alpha)!}$, which transforms the elements $(\alpha,\beta) \in supp(P) \rightarrow (0, \beta-\alpha)$ in $supp(P_{\alpha,\beta}(z))$, and the last pair is considered as the boundary for support of new polynomial.

From the above, if a polynomial P and its elements of support $\alpha, \beta(\alpha \le \beta)$ constitute a counterexample then so does a polynomial $P_(\alpha,\beta)$ and its elements of support $0, \beta-\alpha$. Thus, considering the minimal counterexample with respect to $|\beta-\alpha|$, we can suppose that $\alpha = 0$ in $P_(\alpha,\beta)(z)$ and write it as $P(z) = \sum_m a_m z^m \in \mathbb{C}[z_1, z_2 \dots z_n]$ with positive maximal degree of each component z_i, $supp(P) \subseteq [0,\beta]$ and $a_0 a_\beta \neq 0$ which is the minimal counterexample with respect to $|\beta-\alpha| = |\beta|$.

Let $e_1, e_2 \dots e_n$ be the standard orthonormal basis of $\mathbb{R}^n$. Without loss of generality, we may take $\sigma = e_1$ in the Two-step Axiom. By our assumption, the axiom is not valid for the above polynomial $P(z)$, so $e_1, 2e_1, e_1+e_2 ... e_1+e_n \notin supp(P)$. If there exists $\epsilon \in (e_1,\beta) \cap supp(P)$ then $P_{0,\epsilon}(z)$ is a smaller counterexample than $P(z) = P_{0,\beta}(z)$, which contradicts to our assumption about minimal choice of $P(z)$. Therefore, for all $\gamma \in \mathbb{R}^n \vee \gamma_1 > 0$, $a_\gamma = 0$ unless $\gamma = \beta$.

Let $0 < \mu \in \mathbb{R}$ and $r = \frac{\sum_{i=2}^n \beta_i}{\beta_1}$. The considered polynomial $P(z) = P(z_1, z_2, \dots, z_n)$ is stable, and multiplication of variables by positive numbers does not affect stability, so univariate polynomial $P(\mu^{-r}z, \mu z ... \mu z)$ is also stable. From the previous paragraph, $a_\gamma \neq 0$ only for those $\gamma \in \mathbb{R}^n$ with $\gamma_1 = 0$, so all terms of $P(\mu^{-r}z, \mu z ... \mu z)$, except that with a_0 and a_β, are multiplied by a positive power of μ. Letting $\mu \rightarrow 0$, we have that:

$$\lim_{\mu \rightarrow 0} P(\mu^{-r}z, \mu z ... \mu z) = a_0 + a_\beta z^{|\beta|}.$$

We cannot have $|\beta| \leq 2$, because this means $e_1 + e_i \in supp(P)$ which contradicts our assumption, so $|\beta| \geq 3$. But, then at least one of the roots of $a_0 + a_\beta z^{|\beta|} = 0$ will be in the half-plane with positive real part, which is contradiction. ■

4. Stability on the unit ball

Let $\Omega \subset \mathbb{C}^2$ be the unit ball $|z|^2 + |w|^2 < 1$ for $z, w \in \mathbb{C}$, and let Ω_r be used to denote the collection of points (z, w) such that $|z| < r$ and $|w| < \sqrt{1-r^2}$. It is easy to see that $\Omega_r \subset \Omega$. We consider a quadratic polynomial $P(z, w) = az^2 + bzw + cw^2 + dz + fw + g$, which is Ω-stable.

Lemma 13. Stability on Ω is equivalent to stability on H_R^2 under the mapping $z = \frac{1-\delta}{1+\delta} r$ and $w = \frac{1-\tau}{1+\tau}\sqrt{1-r^2}$ for all $|z| < r$ and $|w| < \sqrt{1-r^2}$ for all $r \in (0, 1)$.

Proof. $|z| < r$ implies $\left|\frac{1-\delta}{1+\delta} r\right| < r$, i.e $\left|\frac{1-\delta}{1+\delta}\right| < 1$. $\left|\frac{1-\delta}{1+\delta}\right|^2 < 1 \Rightarrow \Re(\delta) > 0$. Similarly, for $|w| < \sqrt{1-r^2}$. ■

Lemma 14. Stability on Ω is equivalent to stability on H_U^2 under the mapping $z = \frac{i-\delta}{i+\delta} r$ and $w = \frac{i-\tau}{i+\tau}\sqrt{1-r^2}$ for all $|z| < r$ and $|w| < \sqrt{1-r^2}$ for all $r \in (0, 1)$.

Proof. $|z| < r$ implies $\left|\frac{i-\delta}{i+\delta} r\right| < r$, i.e $\left|\frac{i-\delta}{i+\delta}\right| < 1$. $\left|\frac{i-\delta}{i+\delta}\right|^2 < 1 \Rightarrow \Im(\delta) > 0$. Similarly, for $|w| < \sqrt{1-r^2}$. ■

Applying the transformations described in Lemma 13 to quadratic Ω-stable polynomial $P(z, w) = az^2 + bzw + cw^2 + dz + fw + g$ we have that it converts to

$$Q(\delta, \tau) = c(2,2)\delta^2\tau^2 + c(2,1)\delta^2\tau + c(1,2)\delta\tau^2 + c(2,0)\delta^2 + c(0,2)\tau^2 +$$

$$+c(1,1)\delta\tau + c(1,0)\delta + c(0,1)\tau + c(0,0),$$

where coefficients are

$$\begin{aligned}
c(2,2) &= ar^2 + br\sqrt{1-r^2} + c(1-r^2) - dr - f\sqrt{1-r^2} + g,\\
c(2,1) &= 2ar^2 - 2c(1-r^2) - 2dr + 2g,\\
c(1,2) &= -2ar^2 + 2c(1-r^2) - 2f\sqrt{1-r^2} + 2g,\\
c(2,0) &= ar^2 - br\sqrt{1-r^2} + c(1-r^2) - dr + f\sqrt{1-r^2} + g,\\
c(0,2) &= ar^2 - br\sqrt{1-r^2} + c(1-r^2) + dr - f\sqrt{1-r^2} + g,\\
c(1,1) &= -4ar^2 - 4c(1-r^2) + 4g,\\
c(1,0) &= -2ar^2 + 2c(1-r^2) + 2f\sqrt{1-r^2} + 2g,\\
c(0,1) &= 2ar^2 - 2c(1-r^2) + 2dr + 2g,\\
c(0,0) &= ar^2 + br\sqrt{1-r^2} + c(1-r^2) + dr + f\sqrt{1-r^2} + g.
\end{aligned}$$

Ω-stability of $P(z, w)$ is equivalent to H_R^2-stability of $Q(\delta, \tau)$. Using Lemma 3 for $Q(\delta, \tau)$, we have that the following polynomials are Hurwitz stable:

$$\frac{\partial^2 Q}{\partial \delta^2} = 2c(2,2)\tau^2 + 2c(2,1)\tau + 2c(2,0)$$

$$\frac{\partial^2 Q}{\partial \tau^2} = 2c(2,2)\delta^2 + 2c(1,2)\delta + 2c(0,2)$$

$$Q_{\delta=0} = c(0,2)\tau^2 + c(0,1)\tau + c(0,0)$$

$$Q_{\tau=0} = c(2,0)\delta^2 + c(1,0)\delta + c(0,0)$$

The above four polynomials do not have roots in the right half-plane, so if x_1, x_2 are roots of any of these polynomials then real part of $(x_1 + x_2)$ is nonpositive, i.e.:

$$\Re\left(\frac{-c(2,1)}{c(2,2)}\right) = \Re\left(-\frac{2ar^2 - 2c(1-r^2) - 2dr + 2g}{ar^2 + br\sqrt{1-r^2} + c(1-r^2) - dr - f\sqrt{1-r^2} + g}\right) \leq 0$$

,

$$\Re\left(\frac{-c(1,2)}{c(2,2)}\right) = \Re\left(-\frac{-2ar^2 + 2c(1-r^2) - 2f\sqrt{1-r^2} + 2g}{ar^2 + br\sqrt{1-r^2} + c(1-r^2) - dr - f\sqrt{1-r^2} + g}\right) \leq 0,$$

$$\Re\left(\frac{-c(0,1)}{c(0,2)}\right) = \Re\left(-\frac{2ar^2 - 2c(1-r^2) + 2dr + 2g}{ar^2 - br\sqrt{1-r^2} + c(1-r^2) + dr - f\sqrt{1-r^2} + g}\right) \leq 0$$

$$\Re\left(\frac{-c(1,0)}{c(2,0)}\right) = \Re\left(-\frac{-2ar^2 + 2c(1-r^2) + 2f\sqrt{1-r^2} + 2g}{ar^2 - br\sqrt{1-r^2} + c(1-r^2) - dr + f\sqrt{1-r^2} + g}\right) \leq 0.$$

These inequalities are consequences of transformation of stability, and become true if the initial polynomial $P(z,w)$ is stable on the unit ball. Another result could be obtained by linear transformation of (z,w) satisfying $|z|^2 + |w|^2 < 1$ on the unit ball. For mapping (z,w) to (z',w') we use matrix $\left(\begin{smallmatrix} m & n \\ -\bar{n} & \bar{m} \end{smallmatrix}\right)$ with $|n|^2 + |m|^2 = 1$ for any $n, m \in \mathbb{C}$.

$$\begin{bmatrix} m & n \\ -\bar{n} & \bar{m} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} mz + nw \\ -\bar{n}z + \bar{m}w \end{bmatrix} = \begin{bmatrix} z' \\ w' \end{bmatrix}$$

For new indeterminate (z',w') the inequality $|z'|^2 + |w'|^2 < 1$ holds.

From this point of consideration, the polynomial $P(z,w) = az^2 + bzw + cw^2 + dz + fw + g$ becomes

$$P(z',w') = (am^2 - bm\bar{n} + c\bar{n}^2)z'^2 + (2amn + bm\bar{m} - bn\bar{n} - 2c\bar{n}\bar{m})zw +$$

$$+(an^2 + bn\bar{m} + c\bar{m}^2)w^2 + (dm - f\bar{n})z + (dn + f\bar{m})w + g$$

.

Further investigation of the obtained polynomial does not give us considerable results; however, it could be regarded as one of the possible ways of research in this area.

5. Property of Ω-stability

Theorem 15. Let $P(z_1,z_2) = a_1z_1^2+bz_1z_2+a_2z_2^2+c_1z_1+c_2z_2+d$ be a polynomial with real coefficients. Then the following are equivalent for $i = 1, 2$:
(1) $\frac{\partial P}{\partial z_i}$ is real stable on Ω_r,
(2) $|c_i| \geq 2r_i|a_i| + |b|\sqrt{1-r_i^2}$, where $r_1 = r$ and $r_2 = \sqrt{1-r^2}$

Proof. Given $P(z_1,z_2) = a_1z_1^2+bz_1z_2+a_2z_2^2+c_1z_1+c_2z_2+d$ is real stable on Ω. Further, we will write i and j assuming that $(i,j) = (1,2)$ or $(i,j) = (2,1)$.

$$\frac{\partial P}{\partial z_i} = 2a_iz_i + bz_j + c_i.$$

Let $r_1 = r$ and $r_2 = \sqrt{1-r^2}$. Thus, $r_1^2+r_2^2 = 1$ and $r_j^2 = 1-r_i^2$. Suppose $|z_i| < r_i$ and $|w_j| < r_j = \sqrt{1-r_i^2}$.
Next, changing variables: $z_i = \frac{i-\delta_i}{i+\delta_i}r_i$ we have that:

$$\frac{\partial P}{\partial z_i}(\delta_1,\delta_2) = 2a_i\frac{i-\delta_i}{i+\delta_i}r_i + b\frac{i-\delta_j}{i+\delta_j}r_j + c =$$

$$= 2a_i\frac{i-\delta_i}{i+\delta_i}r_i + b\frac{i-\delta_j}{i+\delta_j}\sqrt{1-r_i} + c_i =$$

$$= \frac{a(1,1)\delta_i\delta_j + ia(1,0)\delta_i + ia(0,1)\delta_j + a(0,0)}{(i+\delta_i)(i+\delta_j)} = \frac{Q(\delta_1,\delta_2)}{(i+\delta_i)(i+\delta_j)}$$

The coefficients of polynomial $Q(\delta_1\delta_2)$ are:
$a(1,1) = -2a_ir_i - b\sqrt{1-r_i^2} + c_i,$
$a(1,0) = -2a_ir_i + b\sqrt{1-r_i^2} + c_i,$
$a(0,1) = 2a_ir_i - b\sqrt{1-r_i^2} + c_i,$
$a(0,0) = -2a_ir_i - b\sqrt{1-r_i^2} - c_i,$
From the above, $Q(\delta_1,\delta_2) = (a(1,1)\delta_i\delta_j + a(0,0)) + i(a(1,0)\delta_i + a(0,1)\delta_j)$.
From Lemma 14, it is known that $z_i = \frac{i-\delta_i}{i+\delta_i}r_i$ transforms Ω-stability of $\frac{\partial P}{\partial z}$ to H_U^2-stability of $Q(\delta,\tau)$ (dividing by $\frac{i+\delta}{i+\tau}$ does not affect stability).
So, $Q(\delta_1,\delta_2) = (a(1,1)\delta_i\delta_j + a(0,0)) + i(a(1,0)\delta_i + a(0,1)\delta_j)$ is H_U^2-stable.
Let $Q(\delta_1,\delta_2) = h+ig$, where $h = a(1,1)\delta_i\delta_j+a(0,0)$ and $g = a(1,0)\delta_i+a(0,1)\delta_j$.
Obviously, h and g are polynomials with real coefficients.

From Theorem 5, we have that Q is H_U^2-stable if only if $\gamma h + \beta g$ is real stable on H_U^2 for all $\gamma, \beta \in \mathbb{R}$.

$f(\delta_1, \delta_2) = \gamma h + \beta g = \gamma a(1,1)\delta_i\delta_j + \beta i a(1,0)\delta_i + \beta i a(0,1)\delta_j + \gamma a(0,0)$ is real stable on H_U^2 because all coefficients are real.
From Theorem 6 and Corollary 7, $f(\delta_1, \delta_2)$ is real stable if only if:

$$\beta a(1,0) \cdot \beta a(0,1) - \gamma a(1,1) \cdot \gamma a(0,0) \geq 0 \iff \beta^2 a(1,0)a(0,1) \geq \gamma^2 a(1,1)a(0,0)$$

If both $a(1,0)a(0,1)$ and $a(1,1)a(0,0)$ are of one sign, then it is possible to find such real γ and β that the inequality will fail. So, they have different signs, and $a(1,0)a(0,1) \geq 0 \geq a(1,1)a(0,0)$.
Also, from Theorem 5, we have that Q is H_U^2-stable if only if

$$\frac{\partial h}{\partial \delta_i}(x) \cdot g(x) - h(x) \cdot \frac{\partial g}{\partial \delta_i}(x) \geq 0 \quad \forall x \in \mathbb{R}.$$

Applying this we have:

$$\frac{\partial h}{\partial \delta_i} \cdot g - h \cdot \frac{\partial g}{\partial \delta_i} = a(1,1)\delta_j(\delta_i a(1,0) + \delta_j a(0,1)) - (a(1,1)\delta_i\delta_j + a(0,0))a(1,0) =$$

$$= a(1,1)a(0,1)\delta_j^2 - a(0,0)a(1,0),$$

so:

$$\frac{\partial h}{\partial \delta_i}(x) \cdot g(x) - h(x) \cdot \frac{\partial g}{\partial \delta_i}(x) = a(1,1)a(0,1)x^2 - a(0,0)a(1,0) \geq 0 \iff$$

$$\iff a(1,1)a(0,1)x^2 \geq a(0,0)a(1,0).$$

Before we obtained, that $a(1,0)a(0,1) \geq 0 \geq a(1,1)a(0,0)$, which means that there is odd number of negatives among these four coefficients, so one of the $a(1,1)a(0,1)$ and $a(0,0)a(1,0)$ must be negative. By the positivity of square of any real number,

$$a(1,1)a(0,1) \geq 0 \geq a(0,0)a(1,0).$$

Similarly, as in the above,

$$\frac{\partial h}{\partial \delta_j}(x) \cdot g(x) - h(x) \cdot \frac{\partial g}{\partial \delta_j}(x) = a(1,1)a(1,0)x^2 - a(0,0)a(0,1) \geq 0 \iff$$

$$\iff a(1,1)a(1,0)x^2 \geq a(0,0)a(0,1) \Rightarrow a(1,1)a(1,0) \geq 0 \geq a(0,0)a(0,1).$$

Finally, we have three inequalities,

$$\begin{cases} a(1,0)a(0,1) \geq 0 \geq a(1,1)a(0,0) \\ a(1,1)a(0,1) \geq 0 \geq a(0,0)a(1,0) \\ a(1,1)a(1,0) \geq 0 \geq a(0,0)a(0,1) \end{cases}$$

By simple calculations, this system leads to one of the following:

(i) $a(1,1), a(1,0), a(0,1) \geq 0$, and $a(0,0) \leq 0$;

(ii) $a(1,1), a(1,0), a(0,1) \leq 0$, and $a(0,0) \geq 0$.

Case (i):

$a(1,1) = -2a_ir_i - b\sqrt{1-r_i^2} + c_i \geq 0,$
$a(1,0) = -2a_ir_i + b\sqrt{1-r_i^2} + c_i \geq 0,$
$a(0,1) = 2a_ir_i - b\sqrt{1-r_i^2} + c_i \geq 0,$
$a(0,0) = -2a_ir_i - b\sqrt{1-r_i^2} - c_i \leq 0 \iff 2a_ir_i + b\sqrt{1-r_i^2} + c_i \geq 0.$

These four inequalities above are equivalent to $c_i \pm 2a_ir_i \pm b\sqrt{1-r_i^2} \geq 0$ which is always true if

$$c_i - |2a_ir_i| - |b\sqrt{1-r_i^2}| \geq 0 \Rightarrow c_i \geq 2r_i|a_i| + |b|\sqrt{1-r_i^2}.$$

Obviously, there $c_i \geq 0$.

Case (ii): This case, similarly as Case (i), leads to the inequality $c_i \pm 2a_ir_i \pm b\sqrt{1-r_i^2} \leq 0$ which is always true if

$$c_i + |2a_ir_i| + |b\sqrt{1-r_i^2}| \leq 0 \Rightarrow -c_i \geq 2r_i|a_i| + |b|\sqrt{1-r_i^2}.$$

Obviously, there $c_i \leq 0$.

The combination of the two cases gives that $\frac{\partial P}{\partial z_i}$ is real stable on Ω_r if and only if

$$|c_i| \geq 2r_i|a_i| + |b|\sqrt{1-r_i^2}.$$

The converse can be proved by going up, where all statement are related to each other in "if and only if" manner. ■

Remark: Theorem 15 shows that derivative operator does not preserve stability of polynomials. The only condition for stability of the derivative of a polynomial is condition (2) in Theorem 15. As an example, we can consider

$P(z_1, z_2) = z_1^2 + z_1 z_2 + z_2^2 + 1000 \in \Omega(z_1, z_2)$ which is clearly stable on Ω. However, we can check that the derivative of $P(z_1, z_2)$ is not real stable on Ω because the condition (2) of Theorem 15 does not hold.

Corollary 16. Let $P(z_1, z_2) = a_1 z_1^2 + b z_1 z_2 + a_2 z_2^2 + c_1 z_1 + c_2 z_2 + d$ be polynomial with real coefficients. Then $\frac{\partial P}{\partial z_i}$ is real stable on Ω if and only if $|c_i| \geq \sqrt{4a_i^2 + b^2}$.

The statement of the Corollary 16 follows from Theorem 15 by looking for the maximum value of $f(r_i) = 2r_i|a_i| + |b|\sqrt{1 - r_i^2}$ for $r_i \in [0, 1]$ that can be found by simple calculus, searching critical points of the function and checking behavior of each interval. If c_i from the Theorem 15 is not less than max of $f(r_i)$ then $\frac{\partial P}{\partial z_i}$ is stable on any Ω_r, i.e. stable on the whole unit ball.

6. Conclusion and further research

In conclusion, the paper presented results of investigation of polynomials stable on the unit ball. Some statements were obtained and the theorem related to partial derivatives was proved. In addition to this result, one might look for other differential operators which do preserve stability on the unit ball. As a further possible research objects, we advise to consider the ideas given in the Section 4, through transformation of the variables. Moreover, other possible research ideas may include following topics which we did not attempt yet:

(i) Ω-stability of quadratic polynomials of higher dimensions,

(ii) Ω-stability of polynomials of higher degrees,

(iii) Ω-stability of homogeneous polynomials.

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