Efficient bases of finite closure systems

K. Adaricheva

Nazarbayev University , Astana, and Yeshiva University, New York

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This presentation is based on:

(1) **"Ordered direct implicational basis of a finite closure system"**
joint work with J.B.Nation and R.Rand (ANR)
Discrete Applied Mathematics 161 (2013), pp. 707-723.

(2) **"On the implicational bases of closure systems with the unique criticals"**
joint work with J.B.Nation(AN)
Discrete Applied Mathematics
on-line September 23, 2013

(3) **"Optimum bases of convex geometries"** (A)
to appear; in arXiv

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Outline

Closure operators and lattices

Common knowledge about finite lattices and closure operators:

- Closed sets of any closure operator $\phi : 2^X \to 2^X$ on set X form a lattice.
- For any given (finite) lattice L there exist many pairs $\langle X, \phi \rangle$ for which L is the lattice of closed sets.
- The set X of smallest cardinality for L has $|Ji(L)|$ elements. ($Ji(L)$ is the set of *join-irreducible* elements of L)
- One can reduce any given closure space $\langle Y, \psi \rangle$ to $\langle X, \phi \rangle$, $X \subseteq Y$, without changing the lattice of closed sets L so that $|X| = |Ji(L)|$. Such space $\langle X, \phi \rangle$ is called *standard* for L.
- Space $\langle X, \phi \rangle$ is standard, when $\phi(\{x\})$, $x \in X$ are exactly join-irreducibles in lattice off closed sets.

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Common knowledge about closure operators and implications:

- Every set of implications $B \to A$, $A, B \in 2^X$, generates closure operator on X.
- Every closure operator ϕ on X could be defined by some set of implications on X.
- If $\langle X, \phi \rangle$ is standard for its lattice of closed sets L, then implications mean relation between join-irreducible elements: $B \to A$ means that $a \leq \bigvee B$, for every $a \in A$.
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OD graph

OD-graph of a finite lattice: J.B.Nation, *An approach to lattice varieties of finite height*, Alg. Univ. 1990

(I) partially ordered set of join irreducibles $\langle Ji(L), \leq \rangle$;

(II) *minimal join covers* $a \leq \bigvee B$, $a \in Ji(L)$, $B \subseteq Ji(L)$:

If $b \in B$, then $a \not\leq \bigvee \{b' \in Ji(L) : b' < b\} \vee \bigvee B \setminus b$.

Note for the future use: D-relation on $Ji(L)$ can be defined as aDb iff $b \in B$, for some minimal join cover $a \leq \bigvee B$.

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D-basis

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Canonical direct basis

K. Bertet, B. Monjardet, *The multiple facets of the canonical direct implicational basis*, Theor. Comp. Science, 2010:

- compare 5 implicational systems for general closure system introduced independently in the literature
- prove that they are the same, now called a *canonical direct basis* Σ_{CD}
- the main feature: $\phi(Y) = Y \cup \{a : (B \to a) \in \Sigma_{CD}$ and $B \subseteq Y\}$
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Canonical direct and the *D*-basis

Essentially, Σ_{CD} contains:

- implications $b \to a$, for join irreducibles $a \leq b$
- *non-redundant covers*: $B \to a$, where $a \leq \bigvee B$, but $a \not\leq \bigvee B \setminus b$.

The minimal covers in *OD*-graph are non-redundant. Hence:

$$\Sigma_D \subseteq \Sigma_{CD}.$$

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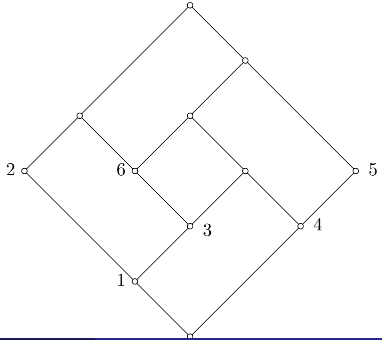
Ordered direct basis

The *D*-basis has a new feature: it is *ordered direct*. $\phi(Y)$ can be computed by applying implications in *particular order*, in a single iteration of the basis.

Example

Canonical direct basis Σ_{CD} for $\langle J(A_{12}), \phi \rangle$ has 13 implications.
$2 \to 1, 6 \to 1, 6 \to 3, 3 \to 1, 5 \to 4, 14 \to 3, 24 \to 3, 15 \to 3,$
$23 \to 6, 15 \to 6, 25 \to 6, 24 \to 5, 24 \to 6.$

D-basis has 9 implications.
$2 \to 1, 6 \to 3, 3 \to 1, 5 \to 4, 14 \to 3, 23 \to 6, 15 \to 6, 24 \to 5, 24 \to 6.$



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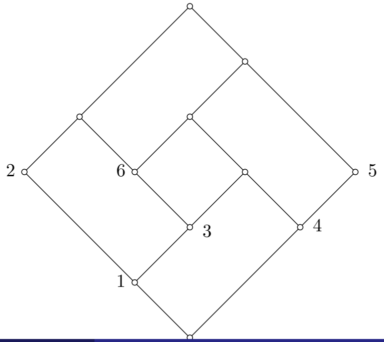

D-basis in representation of Galois lattice of a binary table

Algorithm for obtaining the D-basis of the concept lattice of a binary table:

K. Adaricheva, J.B. Nation, **Discover of the strong association rules in large binary table via hypergraph dualization**, submission to KDD-2014.

More questions

- What other types of "efficient" bases one can obtain for a closure system/finite lattice?
- How effectively this can be done? What are the complexity of the algorithms?

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“Efficient" bases

- A basis Σ' is *non-redundant*, if none of its implications can be removed to get another basis.
- A basis Σ' is *minimum*, if it has the minimal number of implications among all the set of implications for the same closure system.
- A basis $\Sigma' = \{X_i \to Y_i : i \leq n\}$ is called *optimum*, if number $s(\Sigma') = |X_1| + \cdots + |X_n| + |Y_1| + \cdots + |Y_n|$ is smallest among all sets of implications for the same closure system.
- A basis is called *right-side* *(left-side)* optimum basis, if the number $|Y_1| + \cdots + |Y_n|$ $(|X_1| + \cdots + |X_n|)$ is smallest among all sets of implications for the same closure system.
- The right-side optimum basis is connected to the problem of the shortest (i.e. with the minimal number of clauses) CNF-representation of a (definite) Horn function, also, minimal representations of the directed hypergraphs.

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Theorem ([D.Maier, 1983])

Optimum $\Longrightarrow$ minimum and left-side optimum $\Longrightarrow$ non-redundant.

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Theorem ([AN, 2013])

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Optimum and right-side optimum bases

Theorem ([D.Maier, 1983])

The problem of finding an optimum basis of a finite closure system is NP-complete.

Theorem ([G. Ausiello, A. D'Atri and D. Saccá, 1986])

The problem of finding a right-side optimum basis of a finite closure system is NP-complete.

Follow up on Cleaning N1

Corollary ([AN, 2013])

Theorem 1 follows from Theorem 2.

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Theorem 1 follows from Theorem 2.

Optimum and right-side optimum bases

Theorem ([D.Maier, 1983])

The problem of finding an optimum basis of a finite closure system is NP-complete.

Theorem ([G. Ausiello, A. D'Atri and D. Saccá, 1986])

The problem of finding a right-side optimum basis of a finite closure system is NP-complete.

Follow up on Cleaning N1

Corollary ([AN, 2013])

Theorem 1 follows from Theorem 2.

What can be done?

- Introduce new types of bases that are *near-optimum* but can be found quickly.
- Recognize *subclasses* of closure systems where the optimum basis can be found quickly.
- Combine both directions above.

Examples of the second direction:

- P. Hammer and A. Kogan, *Quasi-acyclic propositional Horn knowledge bases: optimal compression*, IEEE Transactions on knowledge and data engineering, 1995
- E. Boros, O. Čepek, A. Kogan and P. Kucěra, *A subclass of Horn CNFs optimally compressible in polynomial time*, Annals of Math and Artif. Intell., 2010
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Canonical basis

J.L. Guigues, V. Duquenne, *Familles minimales d'implications informatives résultant d'une tables de données binares*, Math. Sci. Hum. **95** (1986), 5–18.

- Defined *quasi-closed* and *critical* subsets of X for any given closure system $\langle X, \phi \rangle$.
- Canonical basis Σ_C is $\{A \to B : A \text{ is critical}, B = \phi(A) \setminus A\}$.
- Σ_C is a minimum basis among all the bases generating $\langle X, \phi \rangle$.
- Defined *saturation closure operator* σ associated with ϕ.
- Every other basis relates to Σ_C, via saturation operator σ.

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K-basis: approximation of optimum

- *K*-basis is inspired by minimal join representations of lattice elements.
- *K*-basis has the same number of implications as the canonical, i.e. it is a minimum basis.
- The size of *K*-basis is normally smaller than the size of the canonical.
- *K*-basis can be effectively obtained from the canonical.
- *K*-basis establishes the connection between the canonical basis and the *D*-relation on the set of join irreducibles of a lattice.

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K-basis

Essential idea: given $A \to B$ in Σ_C produce $A^* \to B^*$ in the K-basis, where $A^* \subseteq A$ gives a minimal join representation of element $x = \bigvee A$, and $B^* = max(B) \subseteq B$.

$x = \bigvee A^*$ is a minimal join representation of x, if for every $a \in A^*$, $x > \bigvee\{a' : a' < a\} \vee \bigvee A^* \setminus a$.

Comparison

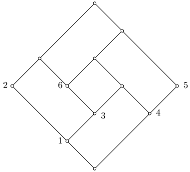

Figure : A_{12}

Canonical basis Σ_C:
$2 \to 1, 6 \to 13, 3 \to 1, 5 \to 4, 14 \to 3, 123 \to 6, 1345 \to 6, 12346 \to 5$
$s(\Sigma_C) = 27$
K-basis:
$2 \to 1, 6 \to 3, 3 \to 1, 5 \to 4, 14 \to 3, 23 \to 6, 15 \to 6, 24 \to 5$
$s(\Sigma_K) = 20$

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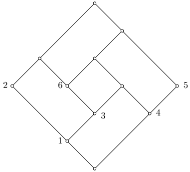

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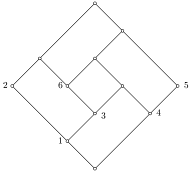

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Algorithmic aspect

Theorem ([A. Day, 1992])

Given any basis Σ' of a finite closure system, it requires time $O(s(\Sigma')^2)$ to obtain the canonical basis of Duquenne-Guigues.

Theorem ([AN, 2013])

A K-basis can be obtained from canonical basis Σ_C of Duquenne-Guigues in time $O(s(\Sigma_C)^2)$.

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K-basis

In general, the closure space may have more than one K-basis.

Definition

A closure system is called join semidistributive, if its closure lattice $Cl(X, \phi)$ satisfies the property:

$(SD_\vee)$ $$x \vee y = x \vee z \to x \vee y = x \vee (y \wedge z).$$

Theorem ([Jónsson and Kiefer, 1962])

Every element of a finite lattice has a unique minimal representation iff the lattice is join semidistributive.

Corollary

Every semidistributive closure system has a unique K-basis.

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Closure systems with the unique critical sets

Problem

Does there exist an effective algorithm to recognize that the closure systems is join semidistributive, given its canonical basis?

Some larger class of closure systems is easy to recognize from the canonical basis.

Definition

Closure system $\langle X, \phi \rangle$ has unique criticals, or it is UC-system, if $\phi(C_1) = \phi(C_2)$, for some critical sets C_1, C_2, implies $C_1 = C_2$.

$SD_\vee$ is UC

Proposition

Every join semidistributive closure system is a UC-system.

Proof.

Suppose there are two implications $C_1 \to B_1$ and $C_2 \to B_2$ in Σ_C with $\phi(C_1) = \phi(C_2)$. This means that in the closure lattice $x = \bigvee C_1 = \bigvee C_2$. One can find minimal representations $B_1 \subseteq C_1$ and $B_2 \subseteq C_2$ for x, i.e. $x = \bigvee B_1 = \bigvee B_2$. But $B_1 = B_2$, since x has a unique minimal representation. Hence, $\sigma(B_1) = C_1 = \sigma(B_2) = C_2$, which is needed. □

Lattice description of *UC*

There exists a *UC* closure system whose closure lattice is *not* join semidistributive.

Problem

Describe closure lattices of closure systems with the unique criticals.

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Describe closure lattices of closure systems with the unique criticals.

Join semidsitributive systems

Important subclasses of join semidsitributive closure systems:

- In lattice theory: lower bounded lattices (closure systems without *D*-cycles) (R.Freese, J.Jezek, J.B.Nation).
- In combinatorics: convex geometries and anti-matroids (P.Edelman and R. Jamison)
- In theory of Boolean functions: quasi-acyclic systems (P.Hammer and A.Kogan). This is a proper subclass of both: convex geometries and systems without *D*-cycles.

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Systems without *D*-cycles

Lower bounded lattices, or lattices without D-cycles: can be defined via D-relation on the set of join-irreducible elements (A.Day, 1979): aDb iff $a \leq \bigvee B$ is a minimal cover and $b \in B$.

Note that this corresponds to implication $B \rightarrow a$ in the D-basis.

Theorem (AN12)

Let D^* be a binary relation defined for any K-basis of the closure system:
aD^*b iff $B \rightarrow A$ is in the K-basis, $|B| > 1$, $a \in A$ and $b \in B$. Then

- $D^* \subseteq D$.
- $D \subseteq tr(D^*)$.

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Systems without *D*-cycles

Corollary

Given the canonical basis Σ_C of the closure system, there exists a polynomial time algorithm in $s(\Sigma_C)$ that recognizes whether the system is without D-cycles.

Bases in systems without D-cycles

This basis was introduced for the systems without D-cycles in: K.Adaricheva, J.B.Nation and R.Rand, *Ordered direct basis of a finite closure system*,

E-basis:

	Canonical basis	K – basis	E – basis
$\|A\| > 1$	$A \to B$	$A^* \to B^*$	$A^* \to B^{**}$
$\|A\| = 1$	$a \to B$	$a \to B^*$	$a \to B^*$

$$B^{**} \subseteq B^*$$

Proposition ([AN, 2013])

E-basis can be obtained from K-basis via polynomial time algorithm: if $b \in B_1^*, B_2^*$, for two implications $A_1^* \to B_1^*$, $A_2^* \to B_2^*$ in the K-basis, and $\phi(A_1^*) \subset \phi(A_2^*)$, then b can be removed from B_2^*.

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E-basis

Theorem ([AN, 2013])

The total right size $|B_1| + \cdots + |B_k|$ of all non-binary implications $A_i \to B_i$ in E-basis attains the minimum among all possible bases for the closure system.

Theorem ([ANR,2013])

The E-basis of a closure system without D-cycles is ordered direct.

4 parts of the optimum basis: systems without *D*-cycles

	Binary part	Non-binary part
the left side	$a \to B$ tractable	$A \to B$ NP
the right side	$a \to B$ NP	$A \to B$ tractable

Proposition ([AN, 2013])

Assume that the closure system is without D-cycles.
(1) Finding the optimum right-side in binary part of the basis is NP-complete.
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(1) Finding the optimum right-side in binary part of the basis is NP-complete.
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4 parts of the optimum basis: systems without D-cycles

	Binary part	Non-binary part
the left side	$a \to B$ tractable	$A \to B$ NP
the right side	$a \to B$ NP	$A \to B$ tractable

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Convex geometry

KA, *Optimum basis of a finite convex geometry*, arxiv (2012)
A closure system $\langle X, \phi \rangle$ is called a *convex geometry*, if $\phi(\emptyset) = \emptyset$, and *anti-exchange property* holds:

For every $A = \phi(A), x, y \notin A$, if $x \in \phi(A \cup y)$, then $y \notin \phi(A \cup x)$.

$x \in A$ is called *extreme point* of A, if $x \notin \phi(A \setminus x)$. $Ex(A)$ is a set of extreme points of A.

Theorem

[P. Edelman and R. Jamison, 1985] A closure system $\langle X, \phi \rangle$ is a convex geometry iff every closed set $A = \phi(Ex(A))$.

In particular, every convex geometry is join semidistributive.

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4 parts of the optimum basis: convex geometries

	Binary part	Non-binary part
the left side	$a \rightarrow B$ tractable	$A \rightarrow B$ tractable
the right side	$a \rightarrow B$ tractable	$A \rightarrow B$??

Proposition

Assume that the closure system is a convex geometry.
(1)[M.Wild, 1994] Finding the optimum left-side in non-binary part of the basis is tractable. $A = Ex(\phi(A))$.
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Optimum basis: convex geometries without D-cycles

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Corollary ([A,2013])

If a closure system is a convex geometry without D-cycles, then optimum basis can be obtained in polynomial time.

This class properly includes the quasi-acyclic closure systems defined in [P. Hammer and A. Kogan, 1995] , which are also G-geometries in [M.Wild, 1994].

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Optimum basis for convex geometries

Problem

Can the optimum basis be found effectively, for every convex geometry?

Conclusions

- *K*-basis might not be an optimum basis, but it is always the minimum basis whose size is smaller than or equal the size of the canonical basis.
- In semidistributive closure systems *K*-basis is unique and is a good approximation of optimum basis.
- If the closure system is without *D*-cycles, further refinement of the *K*-basis can be effectively obtained, giving right-side optimum in its non-binary part.
- If a closure system is a convex geometry, then many subclasses (without *D*-cycles, with n-Carusel rule etc) have tractable optimum bases.
- Thank you!

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