

ALGEBRAIC CONVEX GEOMETRIES REVISITED

K. ADARICHEVA

ABSTRACT. Representation of convex geometry as an appropriate join of compatible orderings of the base set can be achieved, when closure operator of convex geometry is algebraic, or finitary. This bears to the finite case proved by P. Edelman and R. Jamison to the greater extent than was thought earlier.

1. INTRODUCTION

This paper is stimulated by the work of the author on the chapters for the new edition of G. Grätzer's book [13], whose first volume appeared recently. Observations about extension of result of P. Edelman and R. Jamison were written in notes for a number of years, until the publication of N. Wahl [15] came to our attention. While the current paper revisits the main topic of [15], it contains mostly new results, which also go beyond just representation, and establish important properties of convex geometries in algebraic case. The only borrowed result is Lemma 22, which is proved in [15, Theorem 2]. In particular, our Lemma 18 and Theorem 24 seem to navigate proper sail from the finite case into realm of algebraic.

2. PRELIMINARIES

Let X be a non-empty set, and $\text{Pow } X$ be the set of all subsets of X . We know that with respect to the order relation \subseteq , $\text{Pow } X$ has a structure of a complete boolean lattice. In this paper, we will be interested in considering algebraic closure operators, or some of their generalizations.

Definition 1. A mapping $\phi : \text{Pow } X \rightarrow \text{Pow } X$ is called an *algebraic (or finitary) closure operator* on set X , if for all $A, B \subseteq X$

- (1) $A \subseteq \phi(A)$;
- (2) if $A \subseteq B$, then $\phi(A) \subseteq \phi(B)$;
- (3) $\phi(\phi(A)) = \phi(A)$;
- (4) $\phi(A) = \bigcup \{ \phi(B) : B \subseteq A, |B| < \omega \}$.

While the first three properties say that ϕ is a closure operator on set X , the last property indicates that closures of arbitrary sets are fully defined by closures of their finite subsets.

We know that every closure operator ϕ on a set is uniquely associated with the family $\text{Cld}(X, \phi) \subseteq \text{Pow } X$ of its closed subsets: $Y \in \text{Cld}(X, \phi)$ iff $\phi(Y) = Y$. We

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would like to identify the families $\mathcal{C} \subseteq \text{Pow } X$ that are represented as $\text{Cld}(X, \phi)$, for some algebraic closure operator ϕ on X .

A subset $\mathcal{F} \subseteq \text{Pow } X$ is called *algebraic*, if

- (i) $\bigcap X_i \in \mathcal{F}$, for any $X_i \in \mathcal{F}$, $i \in I$;
- (ii) $\bigcup X_i \in \mathcal{F}$, for any non-empty up-directed family $X_i \in \mathcal{F}$, $i \in I$.

We recall that family X_i , $i \in I$, of elements $\text{Pow } X$ is called *up-directed*, if for any X_i, X_j there exists another member of the family X_k such that $X_i \cup X_j \subseteq X_k$. Item (i) allows empty family, for which $\bigcap \emptyset = X$, thus, $X \in \mathcal{F}$, for every algebraic \mathcal{F} . The following statement represents the common knowledge, the proof may be checked, for example, in [6].

Lemma 2. *Family $\mathcal{F} \subseteq \text{Pow } X$ is represented as $\text{Cld}(X, \phi)$, for some algebraic closure operator ϕ iff \mathcal{F} is an algebraic subset of boolean lattice $\text{Pow } X$.*

As in G. Grätzer [13, Lemma 28], one would observe that Lemma 2 actually establishes Galois correspondence between algebraic closure operators on set X and algebraic subsets of $\text{Pow } X$. Firstly, two mappings defined in Lemma 2 are the inverses of each other, and secondly, each of them reverses a natural order, defined below, on the set of all closure operators and on the set of all algebraic subsets.

Definition 3. Given two algebraic closure operators Δ and ϕ on set X , we set $\Delta \leq \phi$ iff $\Delta(Y) \subseteq \phi(Y)$, for every $Y \subseteq X$. The partially ordered set of all algebraic closure operators is denoted $(\text{AClo } X, \leq)$.

Given two algebraic subsets \mathcal{G} and \mathcal{F} , we define $\mathcal{G} \leq \mathcal{F}$ iff $\mathcal{G} \subseteq \mathcal{F}$. The partially ordered set of all algebraic subsets of $\text{Pow } X$ is denoted $(\text{S}_p(\text{Pow } X), \leq)$.

We note that $\Delta \leq \phi$, for closure operators Δ, ϕ implies that every ϕ -closed set is Δ -closed, and that the lattice of closed sets of Δ will include the lattice of closed sets of ϕ as a lower subsemilattice.

Theorem 4. *Both $(\text{AClo } X, \leq)$ and $(\text{S}_p(\text{Pow } X), \leq)$ are complete lattices, moreover, $(\text{AClo } X, \leq) \cong^\delta (\text{S}_p(\text{Pow } X), \leq)$ as complete lattices.*

We also observe the relation between $\text{S}_p(\text{Pow } X)$ and $\text{Sg}_\wedge(\text{Pow } X)$. The latter notation is for the lattice of complete meet-subsemilattices with 1 of $\text{Pow } X$. Each element $\mathcal{F} \in \text{Sg}_\wedge(\text{Pow } X)$ represents the family of closed sets of some closure operator on X . The following result is proved in V. Gorbunov [12, Theorem 6.9], see also a slightly stronger version in K. Adaricheva [1, Theorem 3.2] that avoids direct reference to quasi-varieties. For any family $\mathcal{F} \subseteq \text{Pow } X$, we denote by $\text{Sg}_\wedge(\mathcal{F})$ the family generated by \mathcal{F} and closed under arbitrary intersections, while $\widetilde{\mathcal{F}}$ is the family closed under the unions of up-directed subfamilies in \mathcal{F} .

Theorem 5. *Let X be an arbitrary set.*

- (1) $\text{S}_p(\text{Pow } X)$ is a complete \wedge -subsemilattice and \vee -subsemilattice of $\text{Sg}_\wedge(\text{Pow } X)$.
- (2) For any $\mathcal{F} \subseteq \text{Pow } X$, the minimal element $\mathcal{F}^* \in \text{S}_p(\text{Pow } X)$, containing \mathcal{F} , can be obtained as $\mathcal{F}^* = \widetilde{\text{Sg}_\wedge(\mathcal{F})}$.

The implication from this Theorem is that the join of arbitrary collection of algebraic subsets is generally larger than closing it by arbitrary intersections: it requires to add the unions of up-directed subfamilies. Therefore, $\text{S}_p(\text{Pow } X)$ does not form a complete \vee -subsemilattice in $\text{Sg}_\wedge(\text{Pow } X)$.

Now we introduce the convex geometries. A pair (X, ϕ) will be called a *closure space*, if ϕ is a closure operator on X ; also, ϕ is *zero-closure operator*, if $\phi(\emptyset) = \emptyset$.

Definition 6. [2] A zero-closure space (X, ϕ) satisfies the *anti-exchange property* if the following statement holds,

$$(AEP) \quad \begin{array}{l} x \in \phi(A \cup \{y\}) \text{ and } x \notin A \text{ imply that } y \notin \phi(A \cup \{x\}) \\ \text{for all } x \neq y \text{ in } X \text{ and all closed } A \subseteq X. \end{array}$$

We then say that (X, ϕ) is a *convex geometry*.

When X is finite, this definition corresponds to condition (1) of Theorem 2.1 in P. Edelman and R. Jamison [9] given below. The equivalence of (3) and (4) can be achieved through the statements proved in R.P. Dilworth [7] and S.P. Avann [5]. The reference to other descriptions can be found in B. Monjardet [14].

A decomposition $y = \bigvee \{y_i : i \leq n\}$ of an element $y \in L$ into a join of join irreducible elements $y_i \in L$ is called *irredundant* if $y > \bigvee \{y_i : i \leq n, i \neq j\}$, for all $j \leq n$.

A finite lattice is said to be *locally distributive* if for any $x \in L$ the interval $[x', x]$ with $x' = \bigwedge \{y : y \prec x\}$ is a distributive lattice, where $y \prec x$ means that x covers y .

Finally, for any $A \subseteq X$, $x \in A$ is called an *extreme points* of A , if $x \notin \phi(A \setminus x)$. The set of extreme points of A is denoted $Ex(A)$.

Theorem 7. *Let L be a finite lattice. Then the following are equivalent:*

- (1) L is the closure lattice $\text{Cld}(X, \Phi)$ of a closure space (X, Φ) with the anti-exchange property;
- (2) L is the closure lattice of a closure space (X, Φ) with the property that for any closed subset $A \neq X$ of X there exists $x \in X \setminus A$ such that $A \cup \{x\}$ is closed.
- (3) L is a locally distributive lattice;
- (4) every element of L has a unique irredundant decomposition;
- (5) every element $A \in L$ is a closure of $Ex(A)$.

It remains to be seen whether any of these properties may remain equivalent in case of algebraic closure operators. We propose in the next section a new equivalent property for convex geometry that can be generalized to algebraic case.

We finish this section by the important statement that finite convex geometries, considered from the point of view of their closure lattices, are always join-semidistributive. Recall that a lattice is called *lower semi-modular*, if $a \prec b$ and $c \leq b$ imply $a \wedge c \prec c$ or $a \wedge c = c$. In explicit form, the theorem first appeared in V. Duquenne [8].

Theorem 8. *A finite lattice L is isomorphic to the closure lattice of some finite convex geometry iff L is join-semidistributive and lower semimodular.*

It was shown in K. Adaricheva, V.A. Gorbunov, V.I. Tumanov [2] that in infinite convex geometries the join-semidistributive law does not necessarily hold. But we will see in the next section that semimodularity remains to be the property of all convex geometries.

3. ALGEBRAIC CONVEX GEOMETRIES

It is well-known that a convex geometry on a finite set X is always a *standard* closure system, which means that $\phi(\{x\}) \setminus \{x\}$ is closed, for every $x \in X$. This observation can be generalized to algebraic convex geometries.

Proposition 9. *Let (X, ϕ) be an algebraic convex geometry. Then $\phi(\{x\}) \setminus \{x\}$ is closed for every $x \in X$.*

Proof. Suppose $\phi(\{x\}) \setminus \{x\} = P \neq \phi(P)$, for some $x \in X$. Since $\phi(P) = \phi(\{x\})$ and ϕ is an algebraic operator, there exists a finite subset $P' \subseteq P$ such that $\phi(P') = \phi(\{x\})$. We may assume that P' is minimal with this property. Note that $P' \neq \emptyset$ due to definition of convex geometry. Then for every $p \in P'$ we have $\phi(P' \setminus \{p\}) \subset \phi(\{x\})$ and $p \notin \phi(P' \setminus \{p\})$. Denoting $A = \phi(P' \setminus \{p\})$, we have $x, p \notin A$ and $x \in \phi(A \cup \{p\})$, $p \in \phi(A \cup \{x\})$, which contradict (AEP). \square

The next statement immediately follows from properties of standard closure operator.

Corollary 10. *In every algebraic convex geometry, for every $x \in X$, $\phi(\{x\})$ is a completely join irreducible element of $\text{Cld}(X, \phi)$.*

The statement of Proposition 9 is no longer true in non-algebraic convex geometries.

Example 11.

Let $X = N \cup \{x\}$, for some countable set N . Then define closure operator ϕ as follows:

$$\phi(Y) = \begin{cases} X & \text{if } Y \text{ is co-finite or contains } x; \\ Y & \text{otherwise.} \end{cases}$$

It is easy to verify that ϕ satisfies (AEP), thus, it is a convex geometry. We also observe that $\phi(\{x\}) = X$, and $N = X \setminus \{x\}$ is not closed.

We note that every standard closure system is *reduced* and *zero-closure*. The first property means that the closures of different singletons must be different, and the second that the closure of empty set is empty; check K. Adaricheva, J.B. Nation and R. Rand [3, Section 2]. Zero-closure property is adopted in the definition of arbitrary convex geometry, but the closure operator of convex geometry might not be reduced in non-algebraic case.

In K. Adaricheva and M. Pouzet [4], there were some further observations on even wider class of convex geometries, those with *weakly atomic* lattice of closed sets. The latter property means that every interval $[a, b]$ has a pair of elements c, d that form a cover: $c \prec d$. We call such elements a *covering pair*. It is well-known that every algebraic lattice is weakly atomic, see, for example, [10]. Thus, the following statement is a form of generalization from the algebraic to weakly atomic case. We recall that a (complete) lattice is called *spatial*, if every element is an (infinite) join of completely join irreducible elements.

Lemma 12. *Suppose convex geometry $C = (X, \phi)$ satisfies the property that every interval $[A, B] \subseteq L = \text{Cld}(X, \phi)$ of closed sets has a covering pair: $A \subseteq A' \prec B' \subseteq B$. Then L is spatial.*

The part of the proof of Lemma 12 was to show that if two closed sets of any convex geometry form a covering pair $X_1 \prec X_2$, then $|X_2 \setminus X_1| = 1$. The next statement shows such a property of closed sets in algebraic closure systems holds only in convex geometries. While an observation about the covering pair in finite convex geometry was done in the proof of Theorem 5.2 in S.P. Avann [5], the property was never listed among equivalent for convex geometry.

Proposition 13. *For algebraic closure system $S = (X, \phi)$, the following are equivalent:*

- (1) S is a convex geometry;
- (2) If $X_1 \prec X_2$ in $\text{Cld}(X, \phi)$, then $|X_2 \setminus X_1| = 1$.

Proof. That (1) implies (2) follows from the proof of Lemma 12, and we include it here for completeness. Note that assumption about algebraicity of the operator is not needed here.

Indeed, let $c = X_1 = \phi(X_1) \prec d = X_2 = \phi(X_2)$ be a covering pair in $\text{Cld}(X, \phi)$. Pick any $x \in X_2 \setminus X_1$. Then $X_2 = \phi(X_1 \cup \{x\})$. If there is another $y \in X_2 \setminus X_1$, $y \neq x$, then $y \in \phi(X_1 \cup x)$ implies $x \notin \phi(X_1 \cup y)$. Hence $X_1 < \phi(X_1 \cup \{y\}) < \phi(X_1 \cup \{x\}) = X_2$, a contradiction to $X_1 \prec X_2$.

Now assume (2) and consider $x \neq y \in X \setminus A$, for some $A = \phi(A)$, such that $x \in \phi(A \cup \{y\}) = A_y$. Take a maximal chain in $\mathcal{C} \subseteq [A, A_y] \setminus \{A_y\}$, and consider $\bigvee \mathcal{C} = C_0 \in \text{Cld}(X, \phi)$. By the algebraicity of the operator, we get $C_0 = \bigvee \mathcal{C} = \bigcup \mathcal{C}$. If $C_0 = A_y$, then we get a contradiction between $\phi(\{y\}) \leq \bigcup \mathcal{C}$ and $y \notin \mathcal{C}$, for every $C \in \mathcal{C}$. Hence, $C_0 \prec A_y$, and, by assumption, $|A_y \setminus C_0| = 1$. This implies $A_y = C_0 \cup \{y\}$. Moreover, every maximal chain in $[A, A_y]$ contains C_0 , therefore, C_0 is a unique lower cover of A_y in interval $[A, A_y]$.

If we assume that $y \in \phi(A \cup \{x\}) = A_x$, then $A_y = A_x$, so the same argument as above leads to $A_y = C_0 \cup \{x\}$, a contradiction. Hence, $A_x \subseteq C_0$ and $y \notin A_x$. This implies (AEP). \square

Corollary 14. *If $L = \text{Cld}(X, \phi)$ for convex geometry (X, ϕ) , then L is lower semimodular and locally distributive.*

4. GENERALIZATION OF EDELMAN-JAMISON THEOREM FOR CONVEX GEOMETRIES ON A FIXED SET

In this section we want to consider all possible *algebraic* convex geometries defined on a given set X . The focus of this section is the generalization of representation of convex geometry via compatible orders, given in finite case in P. Edelman and R. Jamison [9]. There were further efforts, for the case of algebraic convex geometries, see N. Wahl [15]. We will fine-tune latter results and provide some new observations.

If \mathcal{G} is the family of closed sets of an algebraic convex geometry, then \mathcal{G} is an element of $\text{S}_p(\text{Pow } X)$, as shown in Lemma 2. We recall that $\text{S}_p(\text{Pow } X)$ itself is contained in $\text{Sg}_\wedge(\text{Pow } X)$, the latter representing all complete meet subsemilattices in $\text{Pow } X$, or the families of closed sets of closure operators on X .

We will denote $\text{ACG } X \subseteq \text{S}_p(\text{Pow } X)$ the collection of all algebraic convex geometries defined on X , ordered by the containment order on their families of closed sets, see Definition 3.

Our first effort is to show that $\text{ACG } X$ is a complete \bigvee -subsemilattice in $\text{S}_p(\text{Pow } X)$. For this, we will use Theorem 5 that tells that obtaining the smallest family

$\mathcal{F}^* \in \text{S}_p(\text{Pow } X)$, containing any given family $\mathcal{F} \subseteq \text{Pow } X$ can be done in two steps:

- build family $\text{Sg}_\wedge(\mathcal{F})$ adding arbitrary intersections of subfamilies in \mathcal{F} ;
- build family $\widetilde{\text{Sg}_\wedge(\mathcal{F})}$ adding the unions of non-empty up-directed subfamilies in $\text{Sg}_\wedge(\mathcal{F})$.

The following result shows that if we start from families \mathcal{G}_i , $i \in I$, of closed sets of some convex geometries on X , and $\mathcal{F} = \bigcup_{i \in I} \mathcal{G}_i$, then both $\text{Sg}_\wedge(\mathcal{F})$ and $\widetilde{\text{Sg}_\wedge(\mathcal{F})}$ will represent families of closed sets of another convex geometry on X .

Theorem 15. *Let X be an arbitrary set. If $\mathcal{G}_i = \text{Cld}(X, \phi_i)$, $i \in I$, are families of closed sets of some algebraic convex geometries on X , then the smallest element $\mathcal{F}^* \in \text{S}_p(\text{Pow } X)$, containing $\mathcal{F} = \bigcup_{i \in I} \mathcal{G}_i$ is also a convex geometry.*

Remark 16. Using notation \bigvee_{S_p} for the join operator in $\text{S}_p(\text{Pow } X)$, we could write more compactly that $\mathcal{F}^* = \bigvee_{\text{S}_p} \{\mathcal{G}_i : i \in I\}$. Analogously, \bigvee_{S_g} will stand for the join operator in $\text{Sg}_\wedge(\text{Pow } X)$.

Proof. We split the argument into two parts.

First, we show that if \mathcal{G}_i , $i \in I$, are convex geometries on X , then $\text{Sg}_\wedge(\bigcup_{i \in I} \mathcal{G}_i) = \bigvee_{\text{S}_g} \{\mathcal{G}_i : i \in I\}$ is also a convex geometry on X . Apparently, the latter family comprises the closed sets of some closure operator ψ , and, for every $A \subseteq X$, we have $\psi(A) = \bigcap_{i \in I} \phi_i(A)$. Moreover, $\psi(\emptyset) = \emptyset$. Thus, we only need to show that ψ satisfies (AEP).

Take any $A = \psi(A)$, $x, y \notin A$ and $x \in \psi(A \cup \{y\})$. We claim that there exists $i \in I$ such that $x, y \notin \phi_i(A)$. Indeed, suppose not, and $I = I_1 \cup I_2$, where $x \in \phi_i(A)$, for $i \in I_1$, and $y \in \phi_j(A)$, for $j \in I_2$. Define closure operators τ and ϕ on X as follows: $\tau(A) = \bigcap_{i \in I_1} \phi_i(A)$ and $\phi(A) = \bigcap_{j \in I_2} \phi_j(A)$, $A \subseteq X$. Apparently, $\psi(A) = \tau(A) \cap \phi(A)$ and $x \in \tau(A)$, $y \in \phi(A)$. Then $x \in \psi(A \cup \{y\}) \subseteq \phi(\phi(A) \cup \{y\}) \subseteq \phi(\phi(A)) = \phi(A)$, which implies $x \in \phi(A) \cap \tau(A) = \psi(A)$, a contradiction.

Thus, we may assume that $x, y \notin \phi_i(A)$, for some $i \in I$. Since $x \in \phi_i(\phi_i(A) \cup \{y\})$, we apply (AEP) that holds for ϕ_i to conclude $y \notin \phi_i(\phi_i(A) \cup \{x\})$. This implies $y \notin \psi(A \cup \{x\})$, which is needed.

Secondly, consider $\mathcal{F}^* = \widetilde{\text{Sg}_\wedge(\mathcal{F})}$, assuming that $\text{Sg}_\wedge(\mathcal{F})$ represents the family of closed sets of convex geometry (X, ψ) . According to Theorem 5, \mathcal{F}^* represents the family of closed sets of some algebraic closure operator ρ on X . Apparently, $\rho(A) \subseteq \psi(A)$, for every $A \subseteq X$. We need to show that ρ satisfies (AEP).

So take some $A = \rho(A)$, $x, y \notin A$ and $x \in \rho(A \cup \{y\})$. If $A = \psi(A)$, then we use (AEP) for ψ to conclude that $y \notin \psi(A \cup \{x\})$. This implies $y \notin \rho(A \cup \{x\})$.

Otherwise, $A = \bigcup_{i \in I} A_i$, for some up-directed family of ψ -closed sets A_i , $i \in I$. Since $x, y \notin A$, we have $x, y \notin A_i$, for every $i \in I$.

The sub-family $\psi(A_i \cup \{y\})$, $i \in I$, is up-directed in $\text{Sg}_\wedge \mathcal{F}$, hence, $\bigcup_{i \in I} \psi(A_i \cup \{y\}) \in \mathcal{F}^*$. Moreover, $A \cup \{y\} \subseteq \bigcup_{i \in I} \psi(A_i \cup \{y\})$. Therefore, $\rho(A \cup \{y\}) \subseteq \bigcup_{i \in I} \psi(A_i \cup \{y\})$. This implies $x \in \psi(A_i \cup \{y\})$, for some $i \in I$. Pick any $j \in I$. Since the family $(A_i, i \in I)$ is up-directed, we can find another $k \in I$ such that $A_j \subseteq A_k$ and $x \in \psi(A_k \cup \{y\})$. Applying (AEP) that holds for ψ , we obtain $y \notin \psi(A_k \cup \{x\})$, thus, also $y \notin \psi(A_j \cup \{x\})$. We conclude that $y \notin \bigcup_{j \in I} \psi(A_j \cup \{x\})$, hence, also $y \notin \rho(A \cup \{x\})$, which is due to $\rho(A \cup \{x\}) \subseteq \bigcup_{i \in I} \psi(A_i \cup \{x\})$. \square

Corollary 17. *ACG X is a complete \vee -subsemilattice in $S_p(\text{Pow } X)$.*

We note that while $\mathcal{G} \cap \mathcal{F}$ is a family of some algebraic closure operator, if \mathcal{G} and \mathcal{F} are such, it does not necessarily gives the family of closed sets of a convex geometry, when both \mathcal{G}, \mathcal{F} are convex geometries, i.e., ACG X does not form a meet subsemilattice in $S_p(\text{Pow } X)$. Indeed, if $X = \{1, 2\}$, and $\mathcal{G} = \{\emptyset, \{1\}, X\}$, $\mathcal{F} = \{\emptyset, \{2\}, X\}$, then both \mathcal{G}, \mathcal{F} are convex geometries, while $\mathcal{G} \cap \mathcal{F} = \{\emptyset, X\}$ is not.

It was proved in [9, Theorem 2.2] that every maximal chain of a finite convex geometry on set X has the length equal to $|X|$. Equivalently, each maximal chain has $|X|$ covering pairs. Similar result holds in case of algebraic convex geometries.

Lemma 18. *Let $G = (X, \phi)$ be an algebraic convex geometry. For a maximal chain \mathcal{C} in $L = \text{Cld}(X, \phi)$, let $\mathcal{C}^* = \{D \in \mathcal{C} : D_* \prec D \text{ for some } D_* \in \mathcal{C}\}$. Define a mapping $h_{\mathcal{C}} : X \rightarrow \mathcal{C}$ as*

$$h_{\mathcal{C}}(x) = \bigcap \{C' \in \mathcal{C} : x \in C'\}, x \in X.$$

Then $h_{\mathcal{C}}$ is one-to-one and onto mapping from X to \mathcal{C}^ .*

Proof. First, we observe that $h_{\mathcal{C}}$ is well-defined, due to maximality of chain \mathcal{C} . Secondly, if $C_0 = h_{\mathcal{C}}(x)$ does not have a lower cover in \mathcal{C} , then $C_0 = \bigvee_L \{C'' \in \mathcal{C} : x \notin C''\}$. This contradicts to the fact that $\phi(x) \subseteq C_0$ is a compact element of L . Therefore, C_0 has a lower cover $C_* \prec C_0$ in \mathcal{C} , and $C_0 = C_* \cup \{x\}$, by Proposition 13. This implies that $h_{\mathcal{C}}(x) \neq h_{\mathcal{C}}(y)$, for $x \neq y$.

Finally, if $D_* \prec D$ is any covering pair from \mathcal{C} , then $D = D_* \cup \{t\}$, for some $t \in X$. Hence, $h_{\mathcal{C}}(t) = D$, and $h_{\mathcal{C}}$ is onto. \square

Our next goal will be to establish stronger connection between maximal chains of (algebraic) convex geometry and *compatible ordering* of the base set of the geometry.

There is natural way to define algebraic convex geometry for any partially ordered (in particular, total ordered) set (X, \leq) that is generalization of *downset alignment* (correspondingly, *monotone alignment*) of [9].

Definition 19. Given partially ordered set (P, \leq) , a pair (P, ϕ) , where $\phi(Q) = \downarrow Q = \{p \in P : p \leq q \text{ for some } q \in Q\}$, is called an *ideal closure system* and its lattice closed sets is denoted $\text{Id}(P, \leq)$.

Evidently, the lattice of closed sets of $\text{Id}(P, \leq)$ is algebraic. It is also straightforward to check that operator ϕ in Definition 19 satisfies (AEP), i.e., (P, ϕ) is a convex geometry. Moreover, if (P, \leq) is a chain, then $\text{Id}(P, \leq)$ coincides with the lattice of ideals of this chain, where the latter is treated as a lattice.

Definition 20. Given closure system (X, ϕ) , the total ordering \leq of the base set X is called *compatible* with the system, if $\text{Id}(X, \leq) \subseteq \text{Cld}(X, \phi)$.

The following statement is a part of [15, Theorem 1], but without any assumptions on the closure system.

Lemma 21. *If (X, \leq) is a total ordering compatible with closure system (X, ϕ) , then $\text{Id}(X, \leq)$ is a maximal chain in $\text{Cld}(X, \phi)$.*

Proof. Suppose the compatible ordering gives the chain $\text{Id}(X, \leq)$ in $\text{Cld}(X, \phi)$, which is not maximal, i.e. there exists closed set T such that $\{T\} \cup \text{Id}(X, \leq)$ is a chain as well. Since $\text{Id}(X, \leq)$ is stable under arbitrary joins and meets, there

exists $C_1, C_2 \in \text{Id}(X, \leq)$ such that they form a covering pair in $\text{Id}(X, \leq)$, and $C_1 \subset T \subset C_2$. If $C_2 \setminus C_1$ has two different elements x_1, x_2 , then, assuming that $x_1 \leq x_2$ in (X, \leq) , we obtain $C_1 \subset \downarrow x_1 \subset C_2$, so that $C_1 \subset C_2$ cannot be a covering pair in $\text{Id}(X, \leq)$. Hence, $C_2 = C_1 \cup \{x\}$, for some $x \in X$, and $T = C_1$ or $T = C_2$. \square

Inverse statement is also true, under additional assumption on closure system.

Lemma 22. [15] *If (X, ϕ) is an algebraic convex geometry, then, for every maximal chain $\mathcal{C} \subseteq \text{Cld}(X, \phi)$ there exists the total ordering $\leq_{\mathcal{C}}$ on X such that $\text{Id}(X, \phi) = \mathcal{C}$.*

Indeed, the total ordering $\leq_{\mathcal{C}}$ can be defined using mapping $h_{\mathcal{C}}$ from Lemma 18: $x \leq_{\mathcal{C}} y$ iff $h_{\mathcal{C}}(x) \subseteq h_{\mathcal{C}}(y)$.

Corollary 23. *The minimal elements of ACG X are $\text{Id}(X, \leq)$, where \leq ranges over all possible total orders on X .*

The following result extends P. Edelman and R. Jamison [9, Theorem 5.2] to the case of algebraic convex geometries. We note that join operator in the statement below differs from one defined in [15, Theorem 3]. More precisely, we replace operator \bigvee_{Sg} in the latter publication by \bigvee_{Sp} . Apparently, operator \bigvee_{Sg} cannot be used in a property sufficient for *algebraic* closure system, since it only produces a minimal closure system from the given joinands. The algebraicity may not be achieved as it was manifested in the example given in [15].

Theorem 24. *Let $G = (X, \phi)$ be a closure system. The following are equivalent:*

- (1) *G is an algebraic convex geometry;*
- (2) *$G = \bigvee_{Sp} \{\text{Id}(X, \leq_i) : i \in I\}$, where $\{\leq_i : i \in I\}$ is some set of total orderings on set X .*

Proof. (2) implies (1) due to Theorem 15. In other direction, one can take as set $\{\leq_i : i \in I\}$ all compatible orderings of given convex geometry. According to Lemma 22, all such orderings correspond to maximal chains in $\text{Cld}(X, \phi)$. Since every $Y \in \text{Cld}(X, \phi)$ belongs to some maximal chain \mathcal{C} , Y will be in $\text{Id}(X, \leq_{\mathcal{C}})$ for the corresponding ordering $\leq_{\mathcal{C}}$. With this choice of set of total orderings, we obtain $G = \bigcup \{\text{Id}(X, \leq_i) : i \in I\} = \bigvee_{Sp} \{\text{Id}(X, \leq_i) : i \in I\}$. \square

It will be an interesting direction of future studies to explore the possibility to represent algebraic convex geometry by the means of the minimal number of total orderings on its base set. We consider a few examples below.

Example 25.

Consider the convex geometry $G = (\mathbb{R}, \phi)$ of convex sets of the chain of real numbers (\mathbb{R}, \leq) . Natural ordering of real numbers \leq is compatible with G via maximal chain $C_{\leq} = \{(-\infty, r) : r \in \mathbb{R}\}$. Reversed ordering \leq^r ($t_1 \leq^r t_2$ iff $t_2 \leq t_1$) is also compatible via maximal chain $C_{\leq^r} = \{(r, \infty) : r \in \mathbb{R}\}$. Apparently, $G = C_1 \vee C_2 = \text{Id}(\mathbb{R}, \leq) \vee_{Sp} \text{Id}(\mathbb{R}, \leq)$. Chains C_1, C_2 contain all completely meet irreducible elements of $\text{Cld}(\mathbb{R}, \phi)$, and operator \bigvee_{Sp} in this case is reduced to taking *finite intersections* of elements in $C_1 \cup C_2$.

Example 26.

Consider convex geometry $G = \text{Pow } \mathbb{N}$, i.e. the closure system on the set of natural numbers \mathbb{N} , with identical closure operator. In order to represent this geometry by means of maximal chains, one can take one maximal chain C_n per each meet irreducible element $k_n = \mathbb{N} \setminus \{n\}$, so that $k_n \in C_n$. In this case $G = \bigvee_{Sp} \{C_n : n \geq 1\}$, and operator \bigvee_{Sp} is reduced to taking arbitrary meets of elements from $\bigcup \{C_n : n \in \mathbb{N}\}$, i.e., it acts equivalently to \bigvee_{Sg} .

However, the number of chains in this representation may be reduced. For example, one can choose C_2, C_3, \dots in such a way that $\mathbb{N} \setminus \{1, 2\}$ is in C_2 , $\mathbb{N} \setminus \{1, 3\}$ is in C_3 etc. Then $k_1 = \mathbb{N} \setminus \{1\}$ can be represented as the union of sets, each of which is intersection of sets from $\bigcup \{C_n : n \geq 2\}$. In other words, $G = \bigvee_{Sp} \{C_n : n \geq 2\}$.

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SCHOOL OF SCIENCE AND TECHNOLOGY, NAZARBAYEV UNIVERSITY, ASTANA, REPUBLIC OF KAZAKHSTAN, 010000

E-mail address: kira.adaricheva@nu.edu.kz

DEPARTMENT OF MATHEMATICAL SCIENCES, YESHIVA UNIVERSITY, NEW YORK, NY 10016, USA

E-mail address: adariche@yu.edu