AMOUNT OF FAILURE OF UPPER-SEMICONTINUITY OF ENTROPY IN NONCOMPACT RANK ONE SITUATIONS, AND HAUSDORFF DIMENSION

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Abstract. Recently, Einsiedler and the authors provided a bound in terms of escape of mass for the amount by which upper-semicontinuity for metric entropy fails for diagonal flows on homogeneous spaces $\Gamma\backslash G$, where G is any connected semisimple Lie group of real rank 1 with finite center, and Γ is any nonuniform lattice in G. We show that this bound is sharp, and apply the methods used to establish bounds for the Hausdorff dimension of the set of points which diverge on average.

1. Introduction

Let G be a connected semisimple Lie group of $\mathbb{R}$-rank 1 with finite center and Γ a nonuniform lattice in G. Further let $a \in G \setminus \{1\}$ be chosen such that its adjoint action Ad_a on the Lie algebra $\mathfrak{g}$ of G is $\mathbb{R}$-diagonalizable. The element a acts on the homogeneous space $\mathcal{X} := \Gamma\backslash G$ by right multiplication, defining the (generator of the) discrete geodesic flow

$$T\colon \mathcal{X} \to \mathcal{X},\ x \mapsto xa. \tag{1}$$

The normalized Haar measure m on $\mathcal{X}$ uniquely realizes the maximal metric entropy $h_m(T)$ of T (see below for more details). The following relation between metric entropies of T and escape of mass along T-invariant probability measures on $\mathcal{X}$ has been proven in [EKP]. We note that the limit measure ν does not need to be a probability measure.

Theorem. *Let $(\mu_j)_{j\in\mathbb{N}}$ be a sequence of T-invariant probability measures on $\mathcal{X}$ which converges to the measure ν in the weak* topology. Then*

$$\nu(\mathcal{X})h_{\frac{\nu}{\nu(\mathcal{X})}}(T) + \frac{1}{2}h_m(T)\cdot(1-\nu(\mathcal{X})) \geq \limsup_{j\to\infty} h_{\mu_j}(T), \tag{2}$$

where it does not matter how we interpret $h_{\frac{\nu}{\nu(\mathcal{X})}}(T)$ if $\nu(\mathcal{X}) = 0$.

Since Γ is not cocompact, upper semi-continuity of metric entropy cannot be expected on $\mathcal{X}$. The theorem above shows that the amount by which it may fail is controlled by the escaping mass. In this formula, the factor $\frac{1}{2}$ is significant:

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it shows that the amount of failure is only half as bad as it could be *a priori* (which would be the factor 1).

The first aim of this article is to show that the factor $\frac{1}{2}$ is best possible. More precisely, we will establish the following theorem.

Theorem 1.1. *For any $c \in [\frac{1}{2}h_m(T), h_m(T)]$, there exists a convergent sequence of T-invariant probability measures $(\mu_j)_{j\in\mathbb{N}}$ on $\mathcal{X}$ with $\lim_{j\to\infty} h_{\mu_j}(T) = c$ such that its weak* limit ν satisfies*

$$\nu(\mathcal{X}) = \frac{2c}{h_m(T)} - 1.$$

For any such sequence (μ_j), equality holds in (2) as well as

$$h_{\frac{\nu}{\nu(\mathcal{X})}}(T) = h_m(T) \quad \text{for } \nu(\mathcal{X}) \neq 0$$

(and hence $\nu/\nu(\mathcal{X})$ is the normalized Haar measure on $\mathcal{X}$).

The second aim of this article is to relate the factor $\frac{1}{2}$ to the Hausdorff dimension of the set of points which diverge on average. We recall that a point $x \in \mathcal{X}$ is said to *diverge on average* (with respect to T) if for any compact subset $\mathcal{K}$ of $\mathcal{X}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{i \in \{0, 1, \ldots, n-1\} \mid T^i(x) \in \mathcal{K}\right\}\right| = 0.$$

It is said to be *divergent* (with respect to T) if its forward trajectory under T eventually leaves any compact subset. In other words, if for any compact subset $\mathcal{K}$ of $\mathcal{X}$ we find $N \in \mathbb{N}$ such that for $n > N$ we have $T^n x \notin \mathcal{K}$.

Obviously, each divergent point diverges on average. Let

$$U := \{u \in G \mid a^n u a^{-n} \to 1 \text{ as } n \to \infty\}$$

denote the unstable subgroup with respect to a. From [Dan85] and also from [EKP] it follows that the Hausdorff dimension of the set of divergent points is $\dim G - \dim U$. However, for the set of points diverging on average we prove that its Hausdorff dimension is strictly larger than $\dim G - \dim U$. Moreover, we also obtain an upper estimate showing that its dimension is strictly less than the full dimension. To state these results more precisely, let

$$\mathcal{D} := \{x \in \mathcal{X} \mid x \text{ diverges on average}\}.$$

The Lie group G has at most two positive roots, namely a short one, denoted α, and a long one 2α. Let

$$p_1 := \dim \mathfrak{g}_\alpha \quad \text{and} \quad p_2 := \dim \mathfrak{g}_{2\alpha}.$$

The group G has a single positive root if and only if it consists of isometries of a real hyperbolic space. In this case, we set $p_1 = 0$ or $p_2 = 0$ (both cases are possible and relevant, see Section 2).

Theorem 1.2. *For the Hausdorff dimension of $\mathcal{D}$ we have the estimates*

$$\dim G - \frac{1}{2}\dim U - \frac{p_2}{2} \leq \dim \mathcal{D} \leq \dim G - \frac{1}{2}\dim U + \frac{p_1}{4}.$$

The proof of Theorem 1.2 shows that the factor $\frac{1}{2}$ of $\dim U$ arises for the same reason as the factor $\frac{1}{2}$ in (2). If G consists of isometries of a real hyperbolic space, we obtain the following improvement. It is caused by the fact that in this case, the adjoint action of a has a single eigenvalue of modulus greater than 1.

Theorem 1.3. *Suppose that G consists of isometries of a real hyperbolic space. Then*

$$\dim \mathcal{D} = \dim G - \frac{1}{2} \dim U.$$

Therefore, it seems natural to expect the following precise value for the Hausdorff dimension of $\mathcal{D}$.

Conjecture 1.4. If G is any $\mathbb{R}$-rank 1 connected semisimple Lie group with finite center, then $\dim_H \mathcal{D} = \dim G - \frac{1}{2} \dim U$.

For the homogeneous spaces $\mathrm{SL}_{d+1}(\mathbb{Z}) \backslash \mathrm{SL}_{d+1}(\mathbb{R})$, $d \geq 1$, and the action of a certain singular diagonal element of $\mathrm{SL}_{d+1}(\mathbb{R})$, the analog of Theorem 1.1 has been proven in [Kad12]. For $d = 2$, the Hausdorff dimension of the set of points which diverge on average is shown in [EK12] to be $6 + 4/3$.

2. Preliminaries

The Lie algebra $\mathfrak{g}$ of the Lie group G is the direct sum of a simple Lie algebra of rank 1 and a compact one. The compact component does not have any influence on the dynamics considered here (cf. [EKP]). For this reason, we assume throughout that $\mathfrak{g}$ is a simple Lie algebra of rank 1 and, correspondingly, that G is a connected simple Lie group of $\mathbb{R}$-rank 1 with finite center. This allows us to work with a coordinate system for G which is adapted to the dynamics, and G can be realized as the isometry group of a Riemannian symmetric space of rank 1 and noncompact type. For more background information on this coordinate system we refer to [CDKR91, CDKR98].

Coordinate system. Let A be the maximal one-parameter subgroup of G of diagonalizable elements which contains a, the chosen generator for the discrete geodesic flow T. Then there exists a group homomorphism $\alpha\colon A \to (\mathbb{R}_{>0}, \cdot)$ such that $\alpha(a) > 1$ and $\mathfrak{g}$ decomposes into the direct sum

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{c} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \tag{3}$$

where

$$\mathfrak{g}_j := \left\{ X \in \mathfrak{g} \;\middle|\; \forall \widetilde{a} \in A\colon\ \mathrm{Ad}_{\widetilde{a}} X = \alpha(\widetilde{a})^{\frac{j}{2}} X \right\}, \quad j \in \{\pm 1, \pm 2\},$$

and $\mathfrak{c}$ is the Lie algebra of the centralizer $C = C_A(G)$ of A in G. The homomorphism α is the square root of the "group analog" of the root α in the Introduction. If $\mathfrak{g}$ is not isomorphic to $\mathfrak{so}(1, n)$, $n \in \mathbb{N}$, the decomposition (3) is the restricted root space decomposition of $\mathfrak{g}$. If $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(1, n)$ for some $n \in \mathbb{N}$ (which is equivalent to saying that G consists of isometries of a real hyperbolic space), either $\mathfrak{g}_1$ or $\mathfrak{g}_2$ is trivial. In this case, both

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{c} \oplus \mathfrak{g}_1 \quad \text{and} \quad \mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{c} \oplus \mathfrak{g}_2$$

are restricted root space decompositions of $\mathfrak{g}$. The first one corresponds to the Cayley-Klein models of real hyperbolic spaces, the second one to the Poincaré models (see [CDKR91, CDKR98]). In any case, let $\mathfrak{n} := \mathfrak{g}_2 \oplus \mathfrak{g}_1$ and let N be the connected, simply connected Lie subgroup of G with Lie algebra $\mathfrak{n}$. Further pick a maximal compact subgroup K of G such that

$$N \times A \times K \to G, \quad (n, \widetilde{a}, k) \mapsto n\widetilde{a}k \qquad \text{(Iwasawa decomposition)}$$

is a diffeomorphism, and let

$$M := K \cap C.$$

The semidirect product NA is parametrized by

$$\mathbb{R}_{>0} \times \mathfrak{g}_2 \times \mathfrak{g}_1 \to NA, \quad (s, Z, X) \mapsto \exp(Z + X) \cdot a_s$$

with $\alpha(a_s) = s$, $a_s \in A$. Let θ be a Cartan involution of $\mathfrak{g}$ such that the Lie algebra $\mathfrak{k}$ of K is its 1-eigenspace, and let B denote the Killing form. Further let

$$p_1 := \dim \mathfrak{g}_1 \quad \text{and} \quad p_2 := \dim \mathfrak{g}_2.$$

On $\mathfrak{n}$ we define an inner product via

$$\langle X, Y \rangle := -\frac{1}{p_1 + 4p_2} B(X, \theta Y) \qquad \text{for } X, Y \in \mathfrak{n}.$$

This specific normalization yields that the Lie algebra $[\cdot,\cdot]$ of $\mathfrak{g}$, even though it is indefinite, satisfies the Cauchy-Schwarz inequality

$$|[X, Y]| \le |X||Y|$$

for $X, Y \in \mathfrak{n}$ (see [Poh10]). We may identify $G/K \cong NA \cong \mathbb{R}_{>0} \times \mathfrak{g}_2 \times \mathfrak{g}_1$ with the space

$$D := \left\{ (t, Z, X) \in \mathbb{R} \times \mathfrak{g}_2 \times g_1 \;\middle|\; t > \frac{1}{4}|X|^2 \right\}$$

via

$$\mathbb{R}_{>0} \times \mathfrak{g}_2 \times \mathfrak{g}_1 \to D, \quad (t, Z, X) \mapsto (t + \tfrac{1}{4}|X|^2, Z, X).$$

With the linear map $J\colon \mathfrak{g}_2 \to \operatorname{End}(\mathfrak{g}_1), Z \mapsto J_Z$,

$$\langle J_Z X, Y \rangle := \langle Z, [X, Y] \rangle \quad \text{for all } X, Y \in \mathfrak{g}_1,$$

the geodesic inversion σ of D at the origin $(1, 0, 0)$ is given by (see [CDKR98])

$$\sigma(t, Z, X) = \frac{1}{t^2 + |Z|^2}\big(t, -Z, (-t + J_Z)X\big). \tag{4}$$

We shall identify σ with the element in K with acts as in (4). Then G has the Bruhat decomposition

$$G = NAM \cup NAM\sigma N. \tag{5}$$

To modify this Bruhat decomposition into one which is tailored to the dynamics on $\mathcal{X}$, we recall the following result on fundamental domains of Siegel type. For $s > 0$ let

$$A_s := \{a_t \in A \mid t > s\},$$

and for any compact subset η of N define the Siegel set

$$\Omega(s, \eta) := \eta A_s K.$$

Proposition 2.1 (Theorem 0.6 and 0.7 in [GR70])**.** *There exists $s_0 > 0$, a compact subset η_0 of N and a finite subset Ξ of G such that*

(i) $G = \Gamma \Xi \Omega(s_0, \eta_0)$,
(ii) *for all $\xi \in \Xi$, the group $\Gamma \cap \xi N \xi^{-1}$ is a cocompact lattice in $\xi N \xi^{-1}$,*

(iii) *for all compact subsets* η *of* N *the set*

$$\{\gamma \in \Gamma \mid \gamma\Xi\Omega(s_0,\eta) \cap \Omega(s_0,\eta) \neq \emptyset\}$$

is finite,

(iv) *for each compact subset* η *of* N *containing* η_0*, there exists* $s_1 > s_0$ *such that for all* $\xi_1, \xi_2 \in \Xi$ *and all* $\gamma \in \Gamma$ *with* $\gamma\xi_1\Omega(s_0,\eta) \cap \xi_2\Omega(s_1,\eta) \neq \emptyset$ *we have* $\xi_1 = \xi_2$ *and* $\gamma \in \xi_1 N M \xi_1^{-1}$.

Throughout we fix a choice for η_0, s_1 (with $\eta = \eta_0$) and Ξ. The elements of Ξ are representatives for the cusps of $\mathcal{X}$ (and will also be called cusps). Note that $\sigma N \sigma = U$ is the unstable subgroup with respect to a, and the conjugation of $\sigma(1, Z, X)\sigma \in U$ by a is given by

$$a^{-k}\sigma(1, Z, X)\sigma a^k = \sigma(1, \alpha(a)^{-k} Z, \alpha(a)^{-k/2} X)\sigma \qquad (k \in \mathbb{Z}).$$

Multiplying (5) with $\xi \in \Xi$ from the left and σ from the right yields

$$G = \xi NAM\sigma \cup \xi NAMU.$$

Maximal entropy. Let $\mathcal{M}_1(\mathcal{X})^T$ denote the set of T-invariant probability measures on $\mathcal{X}$. For each $\mu \in \mathcal{M}_1(\mathcal{X})^T$ let $h_\mu(T)$ denote the metric entropy of T with respect to μ. In [EL10, Section 7.8] it is shown that the maximal metric entropy

$$\max\{h_\mu(T) \mid \mu \in \mathcal{M}_1(\mathcal{X})^T\}$$

of T is uniquely realized by the normalized Haar measure m on $\mathcal{X}$, and it is given by

$$h_m(T) = \left(\frac{p_1}{2} + p_2\right) \log \alpha(a).$$

Normalization. If the element a in (1) changes (within A) then all metric entropies scale by the same factor. Thus, for proving Theorem 1.1-1.3 we may and will assume throughout that a is chosen such that

$$\alpha(a) = e, \qquad (e = \exp(1))$$

letting T result in the time-one geodesic flow.

The height function and an improved choice of s_1. In the following we recall the definition of the height function on $\mathcal{X}$ from [EKP] and its for us significant properties. For any $\xi \in \Xi$ consider the ξ-Iwasawa decomposition $G = \xi NAK$. For $g \in G$ let $s = s_\xi(g) > 0$ be given by $g = \xi N a_s K$. Then s is indeed well-defined. For $x \in \mathcal{X}$, its ξ-height is

$$\mathrm{ht}_\xi(x) = \sup\{s_\xi(g) \mid \Gamma g = x\}.$$

Its height is

$$\mathrm{ht}(x) = \max\{\mathrm{ht}_\xi(x) \mid \xi \in \Xi\}.$$

For $s > 0$ we set

$$\mathcal{X}_{<s} = \{x \in \mathcal{X} : \mathrm{ht}(x) < s\} \quad \text{and} \quad \mathcal{X}_{\geq s} = \{x \in \mathcal{X} : \mathrm{ht}(x) \geq s\}.$$

The constant s_1 in Proposition 2.1 can be chosen such that

(i) if for $x \in \mathcal{X}$ and $\xi \in \Xi$, we have $\mathrm{ht}_\xi(x) > s_1$, then $\mathrm{ht}(x) = \mathrm{ht}_\xi(x)$,

(ii) if for $x \in \mathcal{X}$, we have $\mathrm{ht}(x) > s_1$ and $\mathrm{ht}(x) > \mathrm{ht}(xa)$, then the T-orbit of x strictly descends below height s_1 before it can rise again. This means that there exists $n \in \mathbb{N}$ such that for $j = 0, \ldots, n-1$, we have $\mathrm{ht}(xa^j) > \mathrm{ht}(xa^{j+1})$ and $\mathrm{ht}(xa^n) \leq s_1$, and

(iii) if $x \in \mathcal{X}$ and $\mathrm{ht}_\xi(x) > s_1$ for some $\xi \in \Xi$, then there is (at least one) element $g = \xi n a_r m u \in \xi NAMU$ or $g = \xi n a_r m \sigma \in \xi NAM\sigma$ which realizes $\mathrm{ht}_\xi(x)$. That is, $x = \Gamma g$ and $\mathrm{ht}_\xi(x) = s_\xi(g)$. The components a_r and u do not depend on the choice of g.

We suppose from now on that s_1 satisfies these properties.

For any point $x \in \mathcal{X}$ which is high in some cusp, we have the following explicit formulas for the calculation of the height of the initial part of its orbit.

Proposition 2.2 ([EKP]). *Let $x \in \mathcal{X}$, $\xi \in \Xi$ and suppose that $\mathrm{ht}_\xi(xa^k) > s_1$ for all $k \in \{0, \ldots, n\}$.*

(i) *If $\mathrm{ht}_\xi(x)$ is realized by $g = \xi n a_r m \sigma \in \xi NAM\sigma$, then*

$$\mathrm{ht}_\xi(xa^k) = re^{-k}.$$

(ii) *If $\mathrm{ht}_\xi(x)$ is realized by $g = \xi n a_r m u \in \xi NAMU$ with $u = \sigma(1, Z, X)\sigma$, then*

$$\mathrm{ht}_\xi(xa^k) = r\frac{e^{-k}}{\left(e^{-k} + \frac{1}{4}|X|^2\right)^2 + |Z|^2}.$$

Riemannian metric on G and metric on $\mathcal{X}$. The isomorphism

$$\mathfrak{n} = \mathfrak{g}_2 \times \mathfrak{g}_1 \to N,\ (Z, X) \mapsto \exp(Z + X),$$

induces an inner product on N from the inner product on $\mathfrak{n}$. Then the isomorphism $N \to U$, $n \mapsto \sigma n \sigma$, induces an inner product on U, and using the inverse of the exponential map, also on $\overline{\mathfrak{n}} := \mathfrak{g}_{-2} \times \mathfrak{g}_{-1}$.

We pick a left G-invariant Riemannian metric on G, which on the tangent space $T_1G \cong \mathfrak{g}$ reproduces the inner products on $\mathfrak{n}$ and $\overline{\mathfrak{n}}$. Let d_G denote the induced left-G-invariant metric on G. For $r > 0$ let B_r^G, B_r^U, resp. B_r^{NAM} denote the r-balls in G, U, resp. NAM around $1 \in G$. We define

$$\lambda_0 := \min\{|\lambda| \mid \lambda \text{ is an eigenvalue of } \mathrm{Ad}_a \text{ with } |\lambda| > 1\}.$$

Thus,

$$\lambda_0 = \begin{cases} e & \text{if } \mathfrak{g}_1 = \{0\} \text{ (and hence } G/K \text{ is a real hyperbolic space)}, \\ e^{1/2} & \text{otherwise.} \end{cases}$$

Then for any $L \geq 0$ we have

$$a^L B_r^U a^{-L} \subseteq B^U_{\lambda_0^{-L} r}$$

or, in other words,

$$d_G(ua^{-L}, va^{-L}) \leq \lambda_0^{-L} d_G(u, v) \leq d_G(u, v)$$

for $u, v \in U$. Further

$$c \max\{|Z|, |X|\} \leq d_G(1, \sigma(1, Z, X)\sigma)$$

for some constant $c > 0$ and all $u = \sigma(1, Z, X)\sigma \in U$. In order to avoid carrying too many constants through the calculation, we may assume that $c = 1$. The induced metric $d_{\mathcal{X}}$ on $\mathcal{X}$ is given by

$$d_{\mathcal{X}}(x, y) := \min\{d_G(g, h) \mid x = \Gamma g,\ y = \Gamma h\}.$$

We usually omit the subscripts of d_G and $d_{\mathcal{X}}$.

Finally, to shorten notation, we use

$$[0, n] := \{0, \ldots, n\}$$

for $n \in \mathbb{N}$. The context will always clarify whether $[0, n]$ refers to this discrete interval or a standard interval in $\mathbb{R}$.

3. Upper bound on Hausdorff dimension

Recall that

$$\mathcal{D} = \{x \in \mathcal{X} \mid x \text{ diverges on average}\}.$$

Theorem 3.1. *The Hausdorff dimension of $\mathcal{D}$ is bounded from above by*

(i) $$\dim \mathcal{D} \le \dim G - \frac{1}{2}\dim U + \frac{p_1}{4}.$$

If $p_2 = 0$, then

(ii) $$\dim \mathcal{D} \le \dim G - \frac{1}{2}\dim U.$$

The proof of this theorem builds on Lemma 3.2 below, which easily follows from the contraction rate of the unstable direction under the action of a.

Lemma 3.2. *Let μ be a probability measure on $\mathcal{X}$ of dimension at most β. Then, for any $r > 0$, any $x \in \mathcal{X}$ and any $L \in \mathbb{N}$ we have*

$$\mu(xa^L B_r^U a^{-L} B_r^{NAM}) \le c r^\beta e^{\left(\dim NAM + \frac{p_1}{2} - \beta\right)L}.$$

If $p_2 = 0$, this bound can be improved to

$$\mu(xa^L B_r^U a^{-L} B_r^{NAM}) \le c r^\beta e^{(\dim NAM - \beta)\frac{L}{2}}.$$

Here, c is a constant only depending on μ.

Proof of Theorem 3.1. The claimed bound on the Hausdorff dimension of $\mathcal{D}$ follows using the method used to prove Theorem 1.4 and Corollary 1.5 in [EK12], using Lemmas 8.4 and 8.5 in [EKP] as well as Lemma 3.2. □

4. Lower bound on Hausdorff dimension

In this section we prove the following lower bound on Hausdorff dimension:

Theorem 4.1. *The Hausdorff dimension of the set of points in $\mathcal{X}$ which diverge on average is at least*

$$\dim G - \frac{1}{2}\dim U - \frac{p_2}{2}.$$

As a tool we use a lower estimate on the Hausdorff dimension of the limit set of strongly tree-like collections provided by [KM96, §4.1] (which goes back to [Fal86], [McM87], [Urb91], and [PW94]).

Let U_0 be a compact subset of U and let λ be the Lebesgue measure on U (using the identification $U \cong \mathbb{R}^{p_2} \times \mathbb{R}^{p_1}$). A countable collection $\mathcal{U}$ of compact subsets of U_0 (a subset of the power set of U_0) is said to be *strongly tree-like* if there exists a sequence $(\mathcal{U}_j)_{j\in\mathbb{N}_0}$ of finite nonempty collections on U_0 with $\mathcal{U}_0 = \{U_0\}$ such that

$$\mathcal{U} = \bigcup_{j\in\mathbb{N}_0} \mathcal{U}_j$$

and

$$\forall\, j \in \mathbb{N}_0\ \forall\, A, B \in \mathcal{U}_j \text{ either } A = B \text{ or } \lambda(A \cap B) = 0, \tag{6}$$

$$\forall\, j \in \mathbb{N}\ \forall\, B \in \mathcal{U}_j\ \exists\, A \in \mathcal{U}_{j-1} \text{ such that } B \subseteq A, \tag{7}$$

$$d_j(\mathcal{U}) := \sup_{A\in\mathcal{U}_j} \operatorname{diam}(A) \to 0 \text{ as } j \to \infty. \tag{8}$$

Note that (6) implies $\lambda(A) > 0$ for all $A \in \mathcal{U}$. For a strongly tree-like collection $\mathcal{U}$ with fixed sequence $(\mathcal{U}_j)_{j\in\mathbb{N}_0}$ we let

$$\mathbf{U_j} := \bigcup_{A\in\mathcal{U}_j} A \qquad \text{for any } j \in \mathbb{N}_0. \tag{9}$$

Clearly, $\mathbf{U}_j \subseteq \mathbf{U}_{j-1}$ for any $j \in \mathbb{N}$. Further we call the nonempty set

$$\mathbf{U}_\infty := \bigcap_{j\in\mathbb{N}_0} \mathbf{U}_j \tag{10}$$

the *limit set* of $\mathcal{U}$. For any subset B of U_0 and any $j \in \mathbb{N}$ we define the *j-th stage density* of B in $\mathcal{U}$ to be

$$\delta_j(B, \mathcal{U}) := \begin{cases} 0 & \text{if } \lambda(B) = 0 \\ \frac{\lambda(\mathbf{U}_j \cap B)}{\lambda(B)} & \text{if } \lambda(B) > 0. \end{cases}$$

Note that $\delta_j(B, \mathcal{U}) \le 1$. Finally, for any $j \in \mathbb{N}_0$ we define the *j-th stage density* of $\mathcal{U}$ to be

$$\Delta_j(\mathcal{U}) := \inf_{B\in\mathcal{U}_j} \delta_{j+1}(B, \mathcal{U}).$$

Lemma 4.2 ([KM96])**.** *For any strongly tree-like collection $\mathcal{U}$ of subsets of U_0 we have*

$$\dim_H(\mathbf{U}_\infty) \ge \dim U - \limsup_{j\to\infty} \frac{\sum_{i=0}^{j-1} \big|\log(\Delta_i(\mathcal{U}))\big|}{|\log(d_j(\mathcal{U}))|}.$$

4.1. **Construction of a strongly tree-like collection.** We construct a strongly tree-like collection such that its limit set consists only of points which diverge on average. This construction proceeds in several steps.

Proposition 4.3. *Let $s > 39s_1$ and $R \in \mathbb{N}$. Then there exists $x \in \mathcal{X}_{\le s}$ such that for any η in the interval $(0, \frac{1}{2})$ there exists a subset E of $\overline{B}^U_{\eta e^{-R/4}}$ with $S = \lfloor e^{R/2} \rfloor^{p_2} \lfloor e^{R/4} \rfloor^{p_1}$ elements such that*

(i) *for all $u \in E$, the points xu and $T^R(xu)$ are contained in $\mathcal{X}_{\le s}$,*

(ii) *for any two distinct elements* $u, v \in E$ *we have* $d\big(T^R(u), T^R(v)\big) \geq \eta$,

(iii) *for all* $u \in E$ *and all* $k \in [0,R]$ *we have* $T^k(xu) \in \mathcal{X}_{>s/39}$.

We may choose for x *any element* Γg *with*

$$g \in \{\xi n a_r m \sigma(1, Z_0, X_0)\sigma \mid n \in N, r \in I, m \in M\},$$

where $\xi \in \Xi$ *is any cusp,* I *is a specific interval in* $\mathbb{R}$ *of positive length and* $(1, Z_0, X_0)$ *is a specific point in* N*, both being specified in the proof. Thus, the dimension of the set of possible* x *is at least* $\dim(NAM)$.

Proof. Fix a cusp $\xi \in \Xi$ and pick an element $(Z_0, X_0) \in \mathfrak{g}_2 \times \mathfrak{g}_1$ with $|Z_0| = \frac{3}{2}e^{-R/2}$ and $|X_0| = \frac{3}{2}e^{-R/4}$. Define

$$g := \xi n a_r m \sigma(1, Z_0, X_0)\sigma \quad \text{and} \quad x := \Gamma g$$

with $n \in N$, $m \in M$. Set

$$B := \big\{(Z, X) \in \mathfrak{g}_2 \times \mathfrak{g}_1 \;\big|\; |Z| \leq \eta e^{-R/2},\; |X| \leq \eta e^{-R/4}\big\}.$$

In the following we will estimate the height of xa^k, $k \in [0,R]$, and deduce an allowed range for r such that x satisfies (iii) and (i) for all elements in $\sigma B \sigma$. Since the height does not depend on n and m, we omit these two elements. Let $(Z, X) \in B$. Recall that

$$g\sigma(1, Z, X)\sigma = \xi a_r \sigma(1, Z_0 + Z + \tfrac{1}{2}[X_0, X], X_0 + X)\sigma.$$

Then

$$e^{-R/4} < |X_0 + X| < 2e^{-R/4} \tag{11}$$

and, using $|[X_0, X]| \leq |X_0||X|$,

$$\frac{5}{8}e^{-R/2} < \left|Z_0 + Z + \frac{1}{2}[X_0, X]\right| < 3e^{-R/2}. \tag{12}$$

Let $k \in [0,R]$. Recall that

$$\operatorname{ht}_\xi\left(x\sigma(1, Z, X)\sigma a^k\right) = r \cdot \frac{e^{-k}}{\left(e^{-k} + \frac{1}{4}|X_0 + X|^2\right)^2 + \left|Z_0 + Z + \frac{1}{2}[X_0, X]\right|^2} \tag{13}$$

for sufficiently large r (calculated below). Using the upper bounds in (11) and (12) it follows that

$$\operatorname{ht}_\xi\left(x\sigma(1, Z, X)\sigma a^k\right) > \frac{r}{13}.$$

Hence, (iii) is satisfied for $r > \frac{s}{3}$ (note that then $\frac{r}{13} > \frac{s}{39} > s_1$). Moreover, for these r, [EKP, Proposition 5.5] shows that

$$\operatorname{ht}\left(x\sigma(1, Z, X)\sigma a^n\right) = \operatorname{ht}_\xi\left(x\sigma(1, Z, X)\sigma a^n\right).$$

Using the lower bounds in (11) and (12) we find

$$\operatorname{ht}(x\sigma(1, Z, X)\sigma a^k) \leq \frac{r}{e^{-k} + \frac{1}{2}e^{-R/2} + \frac{25}{64}e^{k-R}}.$$

For $r \leq \frac{25}{64}s$, this implies $\operatorname{ht}(x\sigma(1, Z, X)\sigma a^k) \leq s$ for $k \in \{0, R\}$ and hence (i).

To define the set E, we may pick pairwise disjoint elements

$$(Z_i, X_j) \in B, \quad i = 1, \ldots, \lfloor e^{R/2} \rfloor^{p_2},\; j = 1, \ldots, \lfloor e^{R/4} \rfloor^{p_1}$$

such that

$$|Z_k - Z_\ell| \geq \eta e^{-R}, \quad |X_k - X_\ell| \geq \eta e^{-R/2}$$

whenever $k \neq \ell$. Define

$$E := \left\{ \sigma(1, Z_i, X_j)\sigma \;\middle|\; i = 1, \dots \lfloor e^{R/2} \rfloor^{p_2},\ j = 1, \dots, \lfloor e^{R/4} \rfloor^{p_1} \right\}.$$

For any two distinct elements $\sigma(1, Z, X)\sigma, \sigma(1, Z', X')\sigma \in E$ we have

$$\begin{aligned} d(\sigma(1, Z, X)\sigma a^R, &\sigma(1, Z', X')\sigma a^R) \\ &\geq \max \left\{ \left| Z - Z' + \frac{1}{2}[X, X'] \right| e^R, |X - X'| e^{R/2} \right\} \end{aligned}$$

If $X \neq X'$, then

$$d(\sigma(1, Z, X)\sigma a^R, \sigma(1, Z', X')\sigma a^R) \geq |X - X'| e^{R/2} \geq \eta.$$

If $X = X'$, then

$$d(\sigma(1, Z, X)\sigma a^R, \sigma(1, Z', X')\sigma a^R) \geq |Z - Z'| e^R \geq \eta.$$

This completes the proof. □

To simplify notation we use the following convention: Given a sequence $(S_k)_{k \in \mathbb{N}}$ of positive natural numbers, for any $n \in \mathbb{N}$ we let

$$\mathcal{S}_n := \{(i_1, \dots, i_n) \mid 1 \leq i_j \leq S_j,\ j = 1, \dots, n\} = [1, S_1] \times \cdots \times [1, S_n]$$

be the set of n-multi-indices with entries $1, \dots, S_j$ in the j-th component. If $\mathbf{i} = (i_1, \dots, i_n) \in \mathcal{S}_n$ and $j \in [1, S_{n+1}]$, then we set

$$(\mathbf{i}, j) := (i_1, \dots, i_n, j) \quad \in \mathcal{S}_{n+1}.$$

Finally we let

$$\mathcal{S} := \bigcup_{n \in \mathbb{N}} \mathcal{S}_n.$$

We let

$$B_\varepsilon(\mathcal{K}) := \{x \in \mathcal{X} \mid d(\mathcal{K}, x) < \varepsilon\}$$

denote the ε-thickening of the set $\mathcal{K} \subseteq \mathcal{X}$.

Theorem 4.4. *Let $\mathcal{K}$ be a compact subset of $\mathcal{X}$. For any $k \in \mathbb{N}$ choose natural numbers $R_k, S_k \in \mathbb{N}$ such that there exist a subset $E^{(k)} \subseteq U$ of cardinality S_k and a point $x_k \in \mathcal{K}$ such that for any $u \in E^{(k)}$ we have*

$$x_k u, T^{R_k}(x_k u) \in \mathcal{K}. \tag{14}$$

Then for any $\mathbf{i} \in \mathcal{S}$ there exists $g_\mathbf{i} \in U$ such that, if we define

$$E'_n := \{g_\mathbf{i} \mid \mathbf{i} \in \mathcal{S}_n\} \quad \text{for } n \in \mathbb{N},$$

the following properties are satisfied:

(i) $E'_1 = E^{(1)}$,
(ii) *for any $m \in \mathbb{N}$ there exists an enumeration of $E^{(m)}$ by $[1, S_m]$, say*

$$E^{(m)} = \left\{ u_1^{(m)}, \dots, u_{S_m}^{(m)} \right\},$$

and for any $\eta > 0$ there exists $R' = R'(\eta, \mathcal{K}) \in \mathbb{N}$ (independent of the choice of the $g_{\mathbf{i}}$'s) such that with

$$F(k) := \sum_{i=1}^{k-1} R_i + (k-1)R', \quad k \in \mathbb{N},$$

we have

$$d\big(T^{F(n)+R_n} g_{\mathbf{i}}, T^{F(n)+R_n} g_{(\mathbf{i},j)}\big) < \eta \tag{15}$$

for any $n \in \mathbb{N}$, $\mathbf{i} \in \mathcal{S}_n$, and $j \in [1, S_{n+1}]$, and

$$T^{F(k)}(x_1 g_{\mathbf{i}}) \in x_k u_{i_k}^{(k)} B_{\eta/2}^{NAM} a^{R_k} B_{\eta/2}^{U} a^{-R_k} \tag{16}$$

for any $n \in \mathbb{N}$, any $\mathbf{i} = (i_1, \ldots, i_n) \in \mathcal{S}_n$ and any $k \in [1, n]$.

If, in addition, $\eta_0 > 0$ is an injectivity radius of $B_\varepsilon(\mathcal{K})$ for some (fixed) $\varepsilon > 0$, and

$$E^{(k)} \subseteq B_{\eta_0/4}^{U} \quad \text{for all } k \in \mathbb{N},$$

and

$$d\big(T^{R_k} u, T^{R_k} v\big) \geq \eta_0$$

for any distinct $u, v \in E^{(k)}$, any $k \in \mathbb{N}$, and in (ii) we have

$$\eta < \min\left\{\frac{\eta_0(\lambda_0 - 1)}{4\lambda_0}, \frac{\varepsilon}{2}\right\}$$

then

- (iii) *for any $n \in \mathbb{N}$, the set E'_n has the cardinality of $\mathcal{S}_n$, and*
- (iv) *for any $n \in \mathbb{N}$, any distinct $\mathbf{i}, \mathbf{j} \in \mathcal{S}_n$ we have*

$$\eta_0 > d(g_{\mathbf{i}}, g_{\mathbf{j}}) \quad \text{and} \quad d\big(T^{F(n)+R_n} g_{\mathbf{i}}, T^{F(n)+R_n} g_{\mathbf{j}}\big) > \frac{\eta_0}{2}.$$

The proof of Theorem 4.4 is based on Lemmas 4.5-4.7 below. Throughout these lemmas we let $\mathcal{K}$ be a fixed compact subset of $\mathcal{X}$.

Recall that the group $UNAM$ is a neighborhood of $1 \in G$. We fix $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}^G \subseteq UNAM$. The Shadowing Lemma 4.5 below uses the fact that the subgroups NAM and U intersect in the neutral element 1 only.

Lemma 4.5 (Shadowing Lemma)**.** *There exists $c > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$ and $x_-, x_+ \in \mathcal{X}$ with $d(x_-, x_+) < \varepsilon$ there exist $u^+ \in B_{c\varepsilon}^U$ and $u \in B_{c\varepsilon}^{NAM}$ such that*

$$x_- u^+ = x_+ u \tag{17}$$

Proof. There exists $g \in G$ with $d(g, 1) < \varepsilon$ such that $x_- g = x_+$. Write $g = u^+ u^{-1}$ with $u \in NAM$ and $u^+ \in U$. Then, $d(u^+, 1) < c\varepsilon$ and $d(u, 1) < c\varepsilon$ and $x_- u^+ = x_+ u$. Now continuity of the decomposition, continuous dependence of c on u^+ and u, and the bounded range for ε implies a uniform constant c. $\square$

The compactness of $\mathcal{K}$ and the topological mixing of T imply the following lemma.

Lemma 4.6. *For any $\eta > 0$ and any $\delta > 0$ there exists $R' = R'(\delta, \mathcal{K}, \eta) \in \mathbb{N}$ such that for any $z_-, z_+ \in B_\eta(\mathcal{K})$ and $\ell \geq R'$ there exists $z' \in \mathcal{X}$ such that $d(z', z_-) < \delta$ and $d(z_+, T^\ell(z')) < \delta$.*

The proof of the following lemma is a combination of Lemmas 4.5 and 4.6.

Lemma 4.7. *Let $\eta > 0$ and let z_- and z_+ be in $B_\eta(\mathcal{K})$. Let c be as in the Shadowing Lemma 4.5. For any $\delta > 0$ let $R' = R'(\delta, \mathcal{K}, \eta)$ be as in Lemma 4.6. Then there exist $u^+ \in B^U_{c(c+2)\delta}$ and $u \in B^{NAM}_{c(c+2)\delta}$ such that*

$$T^{R'}(z_- u^+) = z_+ u.$$

Proof. Throughout we will assume that $\delta < \frac{\varepsilon_1}{c+1}$ to be able to apply the Shadowing Lemma 4.5. If the statement is proven for these small δ, it holds *a fortiori* for larger δ. We first use Lemma 4.6 to obtain $z' \in \mathcal{X}$ such that

$$d(z', z_-) < \delta \quad \text{and} \quad d\big(z_+, T^{R'}(z')\big) < \delta. \tag{18}$$

Then we apply Lemma 4.5 with $x_- = z_-$, $x_+ = z'$ and $\varepsilon = \delta$ to obtain $u_1^+ \in B^U_{c\delta}$ and $u_1 \in B^{NAM}_{c\delta}$ such that

$$z_- u_1^+ = z' u_1. \tag{19}$$

The distance between $T^{R'}(z_- u_1^+)$ and z_+ is bounded as follows:

$$\begin{aligned} d\big(T^{R'}(z_- u_1^+), z_+\big) &= d\big(T^{R'}(z' u_1), z_+\big) \\ &\leq d\big(T^{R'}(z' u_1), T^{R'} z'\big) + d\big(T^{R'} z', z_+\big) \\ &< (c+1)\delta. \end{aligned}$$

We apply again Lemma 4.5, this time for $x_- = T^{R'}(z_- u_1^+)$, $x_+ = z_+$ and $\varepsilon = (c+1)\delta$ to obtain $u_2^+ \in B^U_{c(c+1)\delta}$ and $u \in B^{NAM}_{c(c+1)\delta}$ such that

$$T^{R'}(z_- u_1^+) u_2^+ = z_+ u.$$

Now $T^{R'}(z_- u_1^+) u_2^+ = T^{R'}\big(z_-(u_1^+ a^{R'} u_2^+ a^{-R'})\big)$. Setting $u^+ := u_1^+(a^{R'} u_2^+ a^{-R'})$ concludes the proof. $\square$

Proof of Theorem 4.4. We start by proving (i) and (ii). To that end let $\eta > 0$ be arbitrary and pick $c > 0$ as in the Shadowing Lemma 4.5. Set $D_\eta := B_\eta(\mathcal{K})$,

$$\delta := \frac{\eta}{2} \cdot \frac{\lambda_0 - 1}{c(c+2)\lambda_0}$$

and fix R' with the properties as in Lemma 4.6 applied for this δ. Instead of proving (16) we will prove the stronger statement

$$T^{F(k)}(x_1 g_{\mathbf{i}}) \in x_k u^{(k)}_{i_k} B^{NAM}_{c(c+2)\delta} a^{R_k} B^U_{r(n,k)} a^{-R_k} \tag{20}$$

for any $n \in \mathbb{N}$, any $\mathbf{i} = (i_1, \ldots, i_n) \in \mathcal{S}_n$ and any $k \in [1, n]$ where

$$r(n,k) := c(c+2)\delta \sum_{i=0}^{n-k-1} \lambda_0^{-i}$$

and $r(n,n) = 0$ by convention. Since $c(c+2)\delta < \eta/2$ and $r(n,k) < \eta/2$, this is indeed stronger than (16). For the proof of (20) we precede by induction on n. As a by-product, we will prove (i) and (15).

For $n=1$ and $j\in[1,S_1]$ we set $g_i = u_i^{(1)}$. Then (i) and (20) for $n=1$ are trivially satisfied. Suppose that for some $n\in\mathbb{N}$ we constructed the set E_n' fulfilling (20). We show how to construct E_{n+1}' from E_n' such that (20) is satisfied for $n+1$ and (15) for n.

Let $\mathbf{i}\in\mathcal{S}_n$ and $j\in[1,S_{n+1}]$. By the inductive hypothesis

$$T^{F(n)}(x_1 g_{\mathbf{i}}) \in x_n u_{i_n}^{(n)} B_{\frac{\eta}{2}}^{NAM} a^{R_n} B_{\frac{\eta}{2}}^{U} a^{-R_n}.$$

Thus,

$$T^{F(n)+R_n}(x_1 g_{\mathbf{i}}) \in T^{R_n}(x_n u_{i_n}^{(n)}) a^{-R_n} B_{\frac{\eta}{2}}^{NAM} a^{R_n} B_{\frac{\eta}{2}}^{U}.$$

From

$$a^{-R_n} B_{\frac{\eta}{2}}^{NAM} a^{R_n} B_{\frac{\eta}{2}}^{U} \subseteq B_{\eta}^{G}$$

and $T^{R_n}(x_n u_{i_n}^{(n)})\in\mathcal{K}$, it follows that $T^{F(n)+R_n}(x_1 g_{\mathbf{i}})\in D_\eta$. Further,

$$x_{n+1} u_j^{(n+1)} \in \mathcal{K} \subseteq D_\eta.$$

We apply Lemma 4.7 with

$$z_- := T^{F(n)+R_n}(x_1 g_{\mathbf{i}}) \quad \text{and} \quad z_+ := x_{n+1} u_j^{(n+1)}$$

to obtain $u_j^+ \in B_{c(c+2)\delta}^{U}$ and $u_j \in B_{c(c+2)\delta}^{NAM}$ satisfying

$$x_1 g_{\mathbf{i}} a^{F(n)+R_n} u_j^+ a^{R'} = T^{R'}(z_- u_j^+) = z_+ u_j = x_{n+1} u_j^{(n+1)} u_j. \tag{21}$$

We define

$$g_{(\mathbf{i},j)} := g_{\mathbf{i}} a^{F(n)+R_n} u_j^+ a^{-F(n)-R_n} \qquad \in U$$

and

$$E_{n+1}' := \{ g_{(\mathbf{i},j)} \mid \mathbf{i}\in\mathcal{S}_n,\ j\in[1,S_{n+1}]\}.$$

Clearly,

$$d\big(T^{F(n)+R_n}(g_{\mathbf{i}}), T^{F(n)+R_n}(g_{(\mathbf{i},j)})\big) = d(1,u_j^+) < \frac{\eta}{2},$$

which proves (15) for n.

We will now show (20) for $n+1$. Suppose first that $k=n+1$. From the definition of $F(n+1)$ and (21) it immediately follows that

$$T^{F(n+1)}(x_1 g_{(\mathbf{i},j)}) \in x_{n+1} u_j^{(n+1)} B_{c(c+2)\delta}^{NAM}.$$

Suppose now that $k\in[1,n]$. Then

$$\begin{aligned} T^{F(k)}(x_1 g_{(\mathbf{i},j)}) &= x_1 g_{\mathbf{i}} a^{F(n)+R_n} u_j^+ a^{F(k)-F(n)-R_n} \\ &= T^{F(k)}(x_1 g_{\mathbf{i}}) a^{-F(k)+F(n)+R_n} u_j^+ a^{F(k)-F(n)-R_n} \\ &\in T^{F(k)}(x_1 g_{\mathbf{i}}) a^{-F(k)+F(n)+R_n} B_{c(c+2)\delta}^{U} a^{F(k)-F(n)-R_n}. \end{aligned}$$

From the inductive hypothesis we have

$$T^{F(k)}(x_1 g_{\mathbf{i}}) \in x_k u_{i_k}^{(k)} B_{c(c+2)\delta}^{NAM} a^{R_k} B_{r(n,k)}^{U} a^{-R_k}.$$

Therefore

$$\begin{aligned} &T^{F(k)}(x_1 g_{(\mathbf{i},j)}) \\ &\quad\in x_k u_{i_k}^{(k)} B_{c(c+2)\delta}^{NAM} a^{R_k} B_{r(n,k)}^{U} a^{-F(k)-R_k+F(n)+R_n} B_{c(c+2)\delta}^{U} a^{F(k)-F(n)-R_n}. \end{aligned} \tag{22}$$

If $k = n$, then $r(n,k) = 0$. Hence (22) simplifies to

$$T^{F(n)}(x_1 g_{(\mathbf{i},j)}) \in x_n u_{i_n}^{(n)} B^{NAM}_{c(c+2)\delta} a^{R_n} B^{U}_{c(c+2)\delta} a^{-R_n}.$$

If $k \in [1, n-1]$, then

$$-F(k) - R_k + F(n) + R_n = \sum_{i=k+1}^{n} R_i + (n-k)R' =: p(k,n).$$

Hence

$$\begin{aligned} a^{-F(k)-R_k+F(n)+R_n} B^U_{c(c+2)\delta} a^{F(k)+R_k-F(n)-R_n} &\subseteq B^U_{c(c+2)\delta\lambda_0^{-p(k,n)}} \\ &\subseteq B^U_{c(c+2)\delta\lambda_0^{-(n-k)}}. \end{aligned}$$

With $r(n,k) + c(c+2)\delta\lambda_0^{-(n-k)} = r(n+1,k)$ it now follows that

$$T^{F(k)}(x_1 g_{(\mathbf{i},j)}) \in x_k u_{i_k}^{(k)} B^{NAM}_{c(c+2)\delta} a^{R_k} B^U_{r(n+1,k)} a^{-R_k}.$$

This completes the proof of (ii).

Since (iii) is an immediate consequence of (iv), it remains to prove the two statements in (iv). We start with the first one. Let $\mathbf{i} = (i_1, \ldots, i_n), \mathbf{j} = (j_1, \ldots, j_n) \in \mathcal{S}_n$. Then

$$d(g_{\mathbf{i}}, g_{\mathbf{j}}) \le d(g_{\mathbf{i}}, g_{i_1}) + d(g_{i_1}, g_{j_1}) + d(g_{j_1}, g_{\mathbf{j}}).$$

Since $g_{i_1}, g_{j_1} \in E^{(1)} \subseteq B^U_{\eta_0/4}$, we have $d(g_{i_1}, g_{j_1}) < \eta_0/2$. To bound the other two terms, let $k \in [1, S_{n+1}]$. Then by (15) we have

$$d\big(T^{F(n)+R_n} g_{\mathbf{i}}, T^{F(n)+R_n} g_{(\mathbf{i},k)}\big) < \eta.$$

Therefore,

$$d(g_{\mathbf{i}}, g_{(\mathbf{i},k)}) < \eta\lambda_0^{-F(n)-R_n}.$$

Applying this observation iteratively, we obtain

$$d(g_{i_1}, g_{\mathbf{i}}) < \eta \sum_{j=1}^{n-1} \lambda_0^{-F(j)-R_j} < \eta \cdot \frac{1}{\lambda_0 - 1} < \frac{\eta_0}{4}.$$

Thus,

$$d(g_{\mathbf{i}}, g_{\mathbf{j}}) < \eta_0$$

as claimed.

Finally, let $\mathbf{i}, \mathbf{j} \in \mathcal{S}_n$, $\mathbf{i} \ne \mathbf{j}$. It remains to show that

$$d(T^{F(n)+R_n} g_{\mathbf{i}}, T^{F(n)+R_n} g_{\mathbf{j}}) > \frac{\eta_0}{2}. \tag{23}$$

Suppose first that we find $k \in [1, n]$ such that

$$d(g_{\mathbf{i}} a^{F(k)}, g_{\mathbf{j}} a^{F(k)}) \ge \eta_0.$$

Since $F(k) - F(n) - R_n < 0$, the assumption

$$d(g_{\mathbf{i}} a^{F(n)+R_n}, g_{\mathbf{j}} a^{F(n)+R_n}) \le \frac{\eta_0}{2}$$

would result in

$$d(g_{\mathbf{i}} a^{F(k)}, g_{\mathbf{j}} a^{F(k)}) \le \frac{\eta_0}{2}.$$

Therefore, in this case, (23) is obviously satisfied.

To complete the proof pick $k \in [1, n]$ such that $i_k \neq j_k$ and suppose

$$d(g_{\mathbf{i}} a^{F(k)}, g_{\mathbf{j}} a^{F(k)}) < \eta_0.$$

Actually, we may suppose $\leq \eta_0/2$, but $< \eta_0$ turns out to be sufficient. By (16) we find $u_i^-, u_j^- \in B_{\eta/2}^{NAM}$ and $u_i^+, u_j^+ \in B_{\eta/2}^U$ such that

$$T^{F(k)}(x_1 g_{\mathbf{i}}) = x_k u_{i_k}^{(k)} u_i^- a^{R_k} u_i^+ a^{-R_k}$$

and

$$T^{F(k)}(x_1 g_{\mathbf{j}}) = x_k u_{j_k}^{(k)} u_j^- a^{R_k} u_j^+ a^{-R_k}.$$

Pick $h_0, h_k \in G$ such that $\Gamma h_0 = x_1$ and $x_k = x_1 h_k$. Further let $\gamma \in \Gamma$ be such that

$$\gamma h_0 g_{\mathbf{i}} a^{F(k)} = h_0 h_k u_{i_k}^{(k)} u_i^- a^{R_k} u_i^+ a^{-R_k}.$$

We will show that

$$\gamma h_0 g_{\mathbf{j}} a^{F(k)} = h_0 h_k u_{j_k}^{(k)} u_j^- a^{R_k} u_j^+ a^{-R_k} \tag{24}$$

(same γ!). To that end we note that

$$\begin{aligned} &d\big(h_0 h_k u_{i_k}^{(k)} u_i^- a^{R_k} u_i^+ a^{-R_k}, h_0 h_k u_{j_k}^{(k)} u_j^- a^{R_k} u_j^+ a^{-R_k}\big) \\ &\quad \leq d\big(u_{i_k}^{(k)} u_i^- a^{R_k} u_i^+ a^{-R_k}, u_{i_k}^{(k)}\big) + d\big(u_{i_k}^{(k)}, u_{j_k}^{(k)}\big) + d\big(u_{j_k}^{(k)}, u_{j_k}^{(k)} u_j^- a^{R_k} u_j^+ a^{-R_k}\big) \\ &\quad < \eta + \frac{\eta_0}{2} + \eta < \eta_0 \end{aligned}$$

and

$$d\big(\gamma h_0 g_{\mathbf{i}} a^{F(k)}, \gamma h_0 g_{\mathbf{j}} a^{F(k)}\big) < \eta_0.$$

Since η_0 is an injectivity radius of $\partial_{B_\varepsilon^G} \mathcal{K}$, equality (24) now follows. Finally,

$$\begin{aligned} &d\big(g_{\mathbf{i}} a^{F(n)+R_n}, g_{\mathbf{j}} a^{F(n)+R_n}\big) \\ &\quad \geq d\big(g_{\mathbf{i}} a^{F(k)+R_k}, g_{\mathbf{j}} a^{F(k)+R_k}\big) \\ &\quad = d\big(u_{i_k}^{(k)} u_i^- a^{R_k} u_i^+, u_{j_k}^{(k)} u_j^- a^{R_k} u_j^+\big) \\ &\quad \geq d\big(u_{i_k}^{(k)} a^{R_k}, u_{j_k}^{(k)} a^{R_k}\big) - d\big(u_{i_k}^{(k)} a^{R_k}, u_{i_k}^{(k)} u_i^- a^{R_k} u_i^+\big) \\ &\qquad\qquad - d\big(u_{j_k}^{(k)} a^{R_k}, u_{j_k}^{(k)} u_j^- a^{-R_k} u_j^+\big) \\ &\quad \geq \eta_0 - 2\eta > \frac{\eta_0}{2}. \end{aligned}$$

This completes the proof. □

Definition of the strongly tree-like collection. Fix $s_0 > 39 s_1$ and set $\mathcal{K} := \mathcal{X}_{\leq s_0}$. Further fix an injectivity radius η_0 of some neighborhood of $\mathcal{K}$ such that $\frac{1}{2} > \eta_0 > 0$ and choose

$$\eta < \frac{\eta_0(\lambda_0 - 1)}{4\lambda_0}$$

so small that we may apply Theorem 4.4. For $k \in \mathbb{N}$ we set $\widetilde{R}_k := k$ and

$$\widetilde{S}_k := \lfloor e^{k/2} \rfloor^{p_2} \cdot \lfloor e^{k/4} \rfloor^{p_1}.$$

For any $k \in \mathbb{N}$ we apply Proposition 4.3 with $\widetilde{R}_k$, $\widetilde{S}_k$, s_0 and η_0 to get a point $x_k \in \mathcal{K}$ and a subset $\widetilde{E}^{(k)} \subseteq \overline{B}_{\eta_0 e^{-k/4}}^U$ with the properties of this proposition.

For $k \geq k_0 := \lceil 4\log 4 \rceil$ we have $\widetilde{E}^{(k)} \subseteq B^U_{\eta_0/4}$. We set $E^{(k)} := \widetilde{E}^{(k+k_0-1)}$, $R_k := \widetilde{R}_{k+k_0-1}$, $S_k := \widetilde{S}_{k+k_0-1}$ for $k \in \mathbb{N}$ and apply Theorem 4.4 to these sequences to construct a sequence $(E'_n)_{n\in\mathbb{N}}$ of sets with the properties as in Theorem 4.4. For any $n \in \mathbb{N}$ we set

$$\mathcal{U}_n := \left\{ ua^{F(n)+R_n}\overline{B}^U_{\eta_0/4}a^{-F(n)-R_n} \;\middle|\; u \in E'_n \right\}.$$

Let

$$U_0 := \bigcup \mathcal{U}_1 = \bigcup_{u\in E'_1} ua^{k_0}\overline{B}^U_{\eta_0/4}a^{-k_0},$$

which is a compact non-null subset of U, and let $\mathcal{U}_0 := \{U_0\}$. We claim that

$$\mathcal{U} := \bigcup_{n\in\mathbb{N}_0} \mathcal{U}_n$$

is a strongly tree-like collection on U_0. To that end let $n \in \mathbb{N}$. Suppose that $g, h \in E'_n$, $g \neq h$. By Theorem 4.4 we have

$$d\big(ga^{F(n)+R_n}, ha^{F(n)+R_n}\big) > \frac{\eta_0}{2}.$$

Therefore

$$ga^{F(n)+R_n}\overline{B}^U_{\eta_0/4} \cap ha^{F(n)+R_n}\overline{B}^U_{\eta_0/4} = \emptyset,$$

and hence

$$ga^{F(n)+R_n}\overline{B}^U_{\eta_0/4}a^{-F(n)-R_n} \cap ha^{F(n)+R_n}\overline{B}^U_{\eta_0/4}a^{-F(n)-R_n} = \emptyset.$$

This shows (6) (and even a stronger disjointness). Now let $\mathbf{i} \in \mathcal{S}_n$ and $j \in [1, S_{n+1}]$. We claim that

$$g_{(\mathbf{i},j)}a^{F(n+1)+R_{n+1}}\overline{B}^U_{\eta_0/4}a^{-F(n+1)-R_{n+1}} \subseteq g_{\mathbf{i}}a^{F(n)+R_n}\overline{B}^U_{\eta_0/4}a^{-F(n)-R_n},$$

which is equivalent to

$$\begin{aligned} g_{(\mathbf{i},j)}a^{F(n)+R_n}a^{F(n+1)+R_{n+1}-F(n)-R_n}\overline{B}^U_{\eta_0/4}a^{-F(n+1)-R_{n+1}+F(n)+R_n} \\ \subseteq g_{\mathbf{i}}a^{F(n)+R_n}\overline{B}^U_{\eta_0/4}. \end{aligned} \tag{25}$$

Since

$$F(n+1) + R_{n+1} - F(n) - R_n = R_{n+1} + R' > 0,$$

we have

$$a^{F(n+1)+R_{n+1}-F(n)-R_n}\overline{B}^U_{\eta_0/4}a^{-F(n+1)-R_{n+1}+F(n)+R_n} \subseteq \overline{B}^U_{\lambda_0^{-1}\eta_0/4}.$$

Then (25) follows from

$$\lambda_0^{-1}\frac{\eta_0}{4} + d\big(g_{(\mathbf{i},j)}a^{F(n)+R_n}, g_{\mathbf{i}}a^{F(n)+R_n}\big) < \frac{\eta_0}{4}\cdot\frac{1}{\lambda_0} + \frac{\eta_0}{4}\cdot\frac{\lambda_0-1}{\lambda_0} = \frac{\eta_0}{4}.$$

Thus, the sets of the collection are nested in the required way. Finally,

$$ga^{F(n)+R_n}\overline{B}^U_{\eta_0/4}a^{-F(n)-R_n} \subseteq g\overline{B}^U_{\lambda_0^{-F(n)-R_n}\eta_0/4},$$

and hence

$$\operatorname{diam}\left(ga^{F(n)+R_n}\overline{B}^U_{\eta_0/4}a^{-F(n)-R_n}\right) \ll \lambda_0^{-F(n)-R_n}.$$

Therefore, the sequence of supremal diameters converges to 0 as $n \to \infty$. This completes the proof that $\mathcal{U} = \bigcup \mathcal{U}_n$ is a strongly tree-like collection.

Throughout we fix this choice of strongly tree-like collection. Moreover, we define the sets $\mathbf{U_n}$, $n \in \mathbb{N}_0$, and $\mathbf{U}_\infty$ as in (9) and (10).

Proposition 4.8. *Let $x_1 \in \mathcal{K} = \mathcal{X}_{\leq s_0}$ be as in Theorem 4.4. Then $x_1 g$ diverges on average for all $g \in \mathbf{U}_\infty$.*

Proof. The structure of the sets in $\mathcal{U}$ yields that $\mathbf{U}_\infty$ consists of the elements

$$g_\infty = \lim_{n\to\infty} g_{(i_1,\ldots,i_n)} = \bigcap_{n\in\mathbb{N}} g_{(i_1,\ldots,i_n)} a^{F(n)+R_n} \overline{B}^U_{\eta_0/4} a^{-F(n)-R_n},$$

where $(i_k)_{k\in\mathbb{N}}$ is any sequence such that $i_k \in [1, S_k]$ for $k \in \mathbb{N}$. Let $\mathcal{K}'$ be any compact subset of $\mathcal{X}$. Without loss of generality, we may assume that $\mathcal{K}' = \mathcal{X}_{\leq s}$ for some large s. In the following we will prove that the amount of time (discrete time steps) in $[0, F(n) + R_n]$ which is spend in $\mathcal{K}'$ by the points in

$$x_1 g_{(i_1,\ldots,i_n)} a^{F(n)+R_n} \overline{B}^U_{\eta_0/4} a^{-F(n)-R_n}$$

grows sublinearly as $n \to \infty$. This will then prove the proposition. To start we remark that for any given point in $x \in \mathcal{X}$, its T-orbit $(xa^k)_{k\in\mathbb{N}_0}$ stays in the strip $\mathcal{X}_{>s_1} \cap \mathcal{X}_{\leq s}$ for only a uniformly bounded number of consecutive steps (which is due to the space G/K being of rank one, see [EKP]). Let

$$\begin{aligned}\ell &:= \max\{k \in \mathbb{N} \mid \exists\, x \in \mathcal{X}_{\leq s_1} \colon Tx, \ldots, T^k x \in \mathcal{X}_{>s_1} \cap \mathcal{X}_{\leq s},\ T^{k+1}x \in \mathcal{X}_{>s}\}\\ &= \max\{k \in \mathbb{N} \mid \exists\, x \in \mathcal{X}_{>s} \colon Tx, \ldots, T^k x \in \mathcal{X}_{>s_1} \cap \mathcal{X}_{\leq s},\ T^{k+1}x \in \mathcal{X}_{\leq s_1}\}.\end{aligned}$$

By the choice of s_1, as soon as $\mathrm{ht}(xa^k) > \mathrm{ht}(xa^{k+1}) > s_1$, the orbit strictly descends until it is below height level s_1. Since $s_0/39 > s_1$, this means that as soon as the orbit stays above height s_1 for more than 2ℓ consecutive steps, say for m steps, it necessarily stays in $\mathcal{X}_{>s}$ for at least $m - 2\ell$ steps. To simplify the proof we may assume that s_0 is chosen such that

$$x\overline{B}^G_{\eta_0} \subseteq \mathcal{X}_{>s_1}$$

for all $x \in \mathcal{X}_{>s_0/39}$. We use the notation of the proof of Theorem 4.4. Let $n \in \mathbb{N}$ and $\mathbf{i} = (i_1, \ldots, i_n) \in \mathcal{S}_n$. We claim that

$$x_1 g_{\mathbf{i}} a^{F(n)+R_n} \overline{B}^U_{\eta_0/4} a^{-F(n)-R_n+k} \subseteq x_m u^{(m)}_{i_m} a^{k-F(m)} \overline{B}^G_{\eta_0} \tag{26}$$

for $k \in [F(m), F(m) + R_m]$ and $m = 1, \ldots, n$. For $n = 1$, this is clearly true. For $\mathbf{j} = (j_1, \ldots, j_{p+1}) \in \mathcal{S}_{p+1}$ for any $p \in \mathbb{N}$, the proof of Theorem 4.4 showed the identities

$$g_{\mathbf{j}} = g_{(j_1,\ldots,j_p)} a^{F(p)+R_p} u^+_{j_{p+1}} a^{-F(p)-R_p}$$

and

$$x_1 g_{\mathbf{j}} a^{F(p)+R_p} u^+_{j_{p+1}} a^{R'} = x_{p+1} u^{(p+1)}_{j_{p+1}} u_{j_{p+1}},$$

where $u^+_{j_{p+1}} \in B^U_{c(c+2)\delta}$ and $u_{j_{p+1}} \in B^{NAM}_{c(c+2)\delta}$. For $m = 1, \ldots, n-1$, these yield

$$\begin{aligned} x_1 g_{\mathbf{i}} &= x_1 g_{(i_1,\ldots,i_m)} \prod_{p=0}^{n-m-1} a^{F(m+p)+R_{m+p}} u^+_{i_{m+p+1}} a^{-F(m+p)-R_{m+p}}\\ &= x_{m+1} u^{(m+1)}_{i_{m+1}} a^{-F(m+1)} \prod_{p=1}^{n-m-1} a^{F(m+p)+R_{m+p}} u^+_{i_{m+p+1}} a^{-F(m+p)-R_{m+p}}. \end{aligned} \tag{27}$$

Therefore

$$
\begin{aligned}
(28)\qquad & x_1 g_{\mathbf{i}} a^{F(n)+R_n} \overline{B}^U_{\eta_0/4} a^{-F(n)-R_n+k} \\
&= \left(x_{m+1} u^{(m+1)}_{i_{m+1}} a^{k-F(m+1)} \right) \left(a^{F(m+1)-k} u_{i_{m+1}} a^{-F(m+1)+k} \right) \\
&\quad \times \prod_{p=1}^{n-m-1} \left(a^{F(m+p)+R_{m+p}-k} u^+_{i_{m+p+1}} a^{-F(m+p)-R-m+p+k} \right) \\
&\quad \times \left(a^{F(n)+R_n-k} \overline{B}^U_{\eta_0/4} a^{-F(n)-R_n+k} \right)
\end{aligned}
$$

for $m = 1, \dots, n-1$, and

$$
\begin{aligned}
(29)\qquad & x_1 g_{\mathbf{i}} a^{F(n)+R_n} \overline{B}^U_{\eta_0/4} a^{-F(n)-R_n+k} \\
&= x_1 g_{i_1} a^k \prod_{p=0}^{n-2} \left(a^{F(p+1)+R_{p+1}-k} u^+_{i_{p+2}} a^{-F(p+1)-R_{p+1}+k} \right) \\
&\quad \times \left(a^{F(n)+R_n-k} \overline{B}^U_{\eta_0/4} a^{-F(n)-R_n+k} \right).
\end{aligned}
$$

For $k \in [F(m+1), F(m+1)+R_{m+1}]$, we have

$$
\prod_{p=1}^{n-m-1} \left(a^{F(m+p)+R_{m+p}-k} u^+_{i_{m+p+1}} a^{-F(m+p)-R_{m+p}+k} \right) \in B^U_r
$$

with

$$
r = c(c+2)\delta \sum_{p=1}^{n-m-1} \lambda_0^{-(F(m+p)+R_{m+p}-k)} \le c(c+2)\delta \frac{1}{\lambda_0+1} \le \frac{\eta_0}{4},
$$

and

$$
a^{F(m+1)-k} u_{i_{m+1}} a^{-F(m+1)+k} \in B^{NAM}_{\eta_0/4}.
$$

Hence, (28) implies (26) for $2, \dots, n$. By the same argument, (29) implies (26) for 1 (note that $g_{i_1} = u^{(1)}_{i_1}$).

We consider (26) for $m \in \{1, \dots, n\}$ and $k \in [F(m), F(m)+R_m]$. Proposition 4.3 shows that $x_m u^{(m)}_{i_m} a^{k-F(m)} \in \mathcal{X}_{>\frac{s_0}{39}}$, and hence $x_m u^{(m)}_{i_m} a^{k-F(m)} \overline{B}^G_{\eta_0} \subseteq \mathcal{X}_{>s_1}$ for all $k \in [F(m), F(m)+R_m]$. As discussed above, this implies that for any point $y \in x_1 g_{\mathbf{i}} a^{F(n)+R_n} \overline{B}^U_{\eta_0/4} a^{-F(n)-R_n}$, its T-orbit $(ya^k)_{k\in\mathbb{N}_0}$ stays above height s for (at least) $k \in [F(m)+\ell, F(m)+R_m-\ell]$. Thus, in the time interval $[0, F(n)+R_n]$, this orbit stays above height s for at least $\sum_{j=1}^n R_j - 2n\ell$ steps. In turn, $(ya^k)_{k\in\mathbb{N}_0}$ visits $\mathcal{K}'$ for at most $(n-1)R' + 2n\ell$ values for k in $[0, F(n)+R_n]$. One easily sees that

$$
\lim_{n\to\infty} \frac{(n-1)R' + 2n\ell}{F(n)+R_n} = 0,
$$

which completes the proof. □

4.2. Hausdorff dimension.

Proposition 4.9. *We have*

$$
\dim_H \mathbf{U}_\infty \ge \frac{p_1}{2} = \frac{1}{2}\dim U - \frac{p_2}{2}.
$$

Proof. We apply Lemma 4.2. Let $k \in \mathbb{N}$ and $B \in \mathcal{U}_k$. Then

$$\delta_{k+1}(B,\mathcal{U}) = \frac{\lambda(\mathbf{U}_k \cap B)}{\lambda(B)} = \frac{S_{k+1} \cdot \lambda\big(a^{F(k+1)+R_{k+1}}\overline{B}^U_{\eta_0/4}a^{-F(k+1)-R_{k+1}}\big)}{\lambda\big(a^{F(k)+R_k}\overline{B}^U_{\eta_0/4}a^{-F(k)-R_k}\big)},$$

and hence

$$\Delta_k(\mathcal{U}) = \delta_{k+1}(B,\mathcal{U}).$$

For any $L \in \mathbb{N}$ we have

$$\lambda\big(a^L\overline{B}^U_{\eta_0/4}a^{-L}\big) = \left(\frac{\eta_0}{2}\right)^{p_1+p_2} e^{-L\left(p_2+\frac{p_1}{2}\right)} = \left(\frac{\eta_0}{2}\right)^{p_1+p_2} e^{-Lh_m(T)}.$$

Thus,

$$\Delta_k(\mathcal{U}) = S_{k+1}e^{-(R_{k+1}+R')h_m(T)}.$$

Note that $R_{k+1} = k + k_0$ and

$$e^{\frac{1}{2}R_{k+1}h_m(T)} \geq S_{k+1} = \left\lfloor e^{\frac{k+k_0}{2}}\right\rfloor^{p_2} \cdot \left\lfloor e^{\frac{k+k_0}{4}}\right\rfloor^{p_1} \geq e^{\frac{k}{2}h_m(T)}.$$

Then

$$1 \geq c_2e^{-\frac{k}{2}h_m(T)} \geq \Delta_k(\mathcal{U}) \geq c_1e^{-\frac{k}{2}h_m(T)}$$

for some constants c_1, c_2. It follows that

$$\sum_{k=1}^{n-1}\left|\log\left(\Delta_k(\mathcal{U})\right)\right| \asymp \frac{h_m(T)}{2}\sum_{k=1}^{n-1}k \asymp \frac{h_m(t)}{4}n^2.$$

Moreover

$$d_n(\mathcal{U}) \leq \frac{\eta_0}{2}e^{-\frac{1}{2}(F(n)+R_n)},$$

and hence

$$\left|\log\left(d_n(\mathcal{U})\right)\right| \geq c\frac{n^2}{4}$$

for some constant c and sufficiently large n. Then

$$\limsup_{n\to\infty}\frac{\sum_{k=1}^{n-1}\left|\log\left(\Delta_k(\mathcal{U})\right)\right|}{\left|\log\left(d_n(\mathcal{U})\right)\right|} \leq h_m(T).$$

Since $\dim U = p_1 + p_2$, this completes the proof. □

Proof of Theorem 4.1. The space of possible x in Proposition 4.3 (and hence of possible x_1 in Theorem 4.4 and Proposition 4.8) is at least of dimension $\dim(NAM)$. For the Hausdorff dimension of the set $\mathcal{D}$ of points in $\mathcal{X}$ which diverge on average this observation implies

$$\dim_H \mathcal{D} \geq \dim NAM + \dim \mathbf{U}_\infty.$$

Now using Proposition 4.9 completes the proof. □

5. Proof of Theorem 1.1

In [Kad12], the first named author proved the corresponding statement of Theorem 1.1 for $\mathrm{SL}_{d+1}(\mathbb{Z})\backslash \mathrm{SL}_{d+1}(\mathbb{R})$, $d \geq 1$, and the action of a certain diagonal element of $\mathrm{SL}_{d+1}(\mathbb{R})$ (which is singular in terms of symmetric spaces). For the proof he used the variational principle for entropy and established the existence of sufficiently large subsets of (n, ε)-separated points in $\mathrm{SL}_{d+1}(\mathbb{Z})\backslash \mathrm{SL}_{d+1}(\mathbb{R})$ whose trajectories are bounded but stay high up (near the bound) for a significant ratio of time (see [Kad12, Theorem 3.2]). These subsets are necessarily adapted to $\mathrm{SL}_{d+1}(\mathbb{Z})\backslash \mathrm{SL}_{d+1}(\mathbb{R})$. In Proposition 5.1 below we show the analogous statement for $\Gamma\backslash G$ and T being the time-one geodesic flow. After that, the proof of Theorem 1.1 is an adaption of [Kad12]. For the convenience of the reader, we provide some details.

Proposition 5.1. *Let $s > 39s_1$. Then there exists $R' \in \mathbb{N}$ such that for all $R \in \mathbb{N}$, $R > 4\log 4$, there is a subset $\widetilde{E}$ of $\mathcal{X}_{\leq s}$ such that the following properties are satisfied:*

(i) *There exists $s' > s$ such that*
$$T^\ell x \in \mathcal{X}_{\leq s'}$$
for all $x \in \widetilde{E}$ and all $\ell \in \mathbb{N}_0$.

(ii) *For any $m \in \mathbb{N}$ we find a subset $\widetilde{E}(m)$ of $\widetilde{E}$ such that*
   (1) *the cardinality of $\widetilde{E}(m)$ is S^m with $S = S(R) = \lfloor e^{\frac{R}{4}} \rfloor^{p_1} \cdot \lfloor e^{\frac{R}{2}} \rfloor^{p_2}$,*
   (2) *$\widetilde{E}(m)$ is $(mR + (m-1)R', \eta')$-separated for some $\eta' > 0$ not depending on m, and*
   (3) *for any $x \in \widetilde{E}(m)$ we have*
$$\left| \left\{ \ell \in [0, mR + (m-1)R' - 1] \;\middle|\; T^\ell x \in \mathcal{X}_{\geq \frac{s}{100}} \right\} \right| \geq mR.$$

To prove Proposition 5.1 we need the following lemma, which is similar to Lemma 5.2 in [Kad12]. We omit its proof. Let
$$\lambda_1 := \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } \mathrm{Ad}_a \text{ with } |\lambda| > 1\}.$$

Thus,
$$\lambda_1 = \begin{cases} e^{1/2} & \text{if } \mathfrak{g}_2 = \{0\} \text{ (and hence } G/K \text{ is real hyperbolic)}, \\ e & \text{otherwise.} \end{cases}$$

Lemma 5.2. *Let $s' > 0$ and pick an injectivity radius $\eta > 0$ of $\mathcal{X}_{\leq s'}$. Let $n \in \mathbb{N}$ and suppose that $g, h \in U$ and $x_0 \in \mathcal{X}$ are such that $T^\ell(x_0 g), T^\ell(x_0 h) \in \mathcal{X}_{\leq s'}$ for all $\ell \in [0, n]$. Further suppose that $d(g, h) = d(x_0 g, x_0 h)$ and that $d(T^n g, T^n h) > \frac{\eta}{\lambda_1}$. Then there exists $\ell \in [0, n]$ such that $d(T^\ell(x_0 g), T^\ell(x_0 h)) \geq \frac{\eta}{\lambda_1}$.*

Proof of Proposition 5.1. Let $\mathcal{K} := \mathcal{X}_{\leq s}$ and pick $\eta_0 \in (0, 1/2)$ such that it is an injectivity radius of $B_{\eta_0}(\mathcal{K})$. Apply Proposition 4.3 with η_0 and R to get a subset $E \subseteq B^U_{\eta_0/4}$ with
$$S = \lfloor e^{R/2} \rfloor^{p_2} \lfloor e^{R/4} \rfloor^{p_1}$$

elements and $x \in \mathcal{K}$ with properties as in that proposition. Let

$$0 < \eta < \frac{\eta_0(\lambda_0 - 1)}{4\lambda_0}$$

be small enough such that we may apply Theorem 4.4. In the following we will use the notation of Theorem 4.4. For $k \in \mathbb{N}$ define $R_k := R$, $S_k := S$, $E^{(k)} := E$ and $x_k := x$. Now Theorem 4.4 provides $R' = R'(\eta, \mathcal{K}) \in \mathbb{N}$ and a family of subsets

$$E'_n := \{g_{\mathbf{i}} \mid \mathbf{i} \in \mathcal{S}_n\}, \quad n \in \mathbb{N},$$

of U with the properties stated there. Let $\widetilde{\mathcal{S}} := [1, S]^{\mathbb{N}}$ and let

$$\mathbf{i}_\infty = (i_j)_{j \in \mathbb{N}} \in \widetilde{\mathcal{S}}.$$

As in the proof of Proposition 4.8, we see that $(g_{(i_1,\ldots,i_n)})_{n \in \mathbb{N}}$ is convergent. Let

$$g_{\mathbf{i}_\infty} := \lim_{n \to \infty} g_{(i_1,\ldots,i_n)}.$$

Define

$$\widetilde{E} := \left\{ x g_{\mathbf{i}_\infty} \,\middle|\, \mathbf{i}_\infty \in \widetilde{\mathcal{S}} \right\},$$

and

$$\widetilde{E}(m) := \left\{ x g_{\mathbf{i}_\infty} \,\middle|\, \mathbf{i}_\infty \in \widetilde{\mathcal{S}},\ i_j = 1 \text{ for } j > m \right\} \quad \text{for } m \in \mathbb{N}.$$

Since the maximal variation of height under one application of T is bounded, the sequence $(R_k)_k$ is constant (namely, R) and the starting points xu, $u \in E$, are contained in a compact set, we deduce from (27) in the proof of Proposition 4.8 (and a limit over n) that we find $s' > s$ such that the T-orbit of each element in $\widetilde{E}$ is contained in the compact set $\mathcal{X}_{\leq s'}$.

Let $n \in \mathbb{N}$, $\mathbf{i} \in \mathcal{S}_n$ and $m \in \{1, \ldots, n\}$. From (27) it follows that

$$x g_{\mathbf{i}} a^k \in x u_j a^{k - F(m)} \overline{B}^U_{\eta/2}$$

for some $j \in \{1, \ldots, S\}$ and all $k \in [F(m), F(m) + R]$. Since $x u_j a^{k-F(m)} \in \mathcal{X}_{\geq s/39}$, we have $x g_{\mathbf{i}} a^k \in \mathcal{X}_{\geq \frac{s}{39} - \frac{\eta}{2}}$. Note that η does not depend on n, m or $\mathbf{i}$. Thus, for any $x \in \widetilde{E}$ it follows that

$$\left| \left\{ \ell \in [0, mR + (m-1)R' - 1] \,\middle|\, T^\ell x \in \mathcal{X}_{\geq \frac{s}{39} + \frac{\eta}{2}} \right\} \right| \geq mR.$$

For η sufficiently small, this proves (ii3).

Obviously, the cardinality of $\widetilde{E}(m)$ is at most S^m. The equality follows from (ii2). For the proof of (ii2) we want to make use of Lemma 5.2. For $\mathbf{i}_\infty, \mathbf{j}_\infty \in \widetilde{\mathcal{S}}$, Theorem 4.4 yields $d(g_{\mathbf{i}_\infty}, g_{\mathbf{j}_\infty}) < \eta_0$. The proof of Proposition 4.8 shows

$$x g_{\mathbf{i}} \in x g_{i_1} B^U_{\eta_0/4}$$

for each $\mathbf{i} = (i_1, \ldots, i_n) \in \mathcal{S}_n$, $n \in \mathbb{N}$. It follows that $x g_{\mathbf{i}_\infty}, x g_{\mathbf{j}_\infty} \in B_{\eta_0}(\mathcal{K})$. Then η_0 being an injectivity radius of $B_{\eta_0}(\mathcal{K})$ yields

$$d(g_{\mathbf{i}_\infty}, \mathbf{j}_\infty) = d(x g_{\mathbf{i}_\infty}, x g_{\mathbf{j}_\infty}).$$

Now let $m \in \mathbb{N}$ and $\mathbf{i} = (i_1, \ldots, i_m), \mathbf{j} = (j_1, \ldots, j_m) \in \mathcal{S}_m$, $\mathbf{i} \neq \mathbf{j}$. We claim that

$$d\big(T^{F(m)+R} g_{(\mathbf{i},\mathbf{1})}, T^{F(m)+R} g_{(\mathbf{j},\mathbf{1})}\big) > \frac{\eta_0}{4},$$

where $(\mathbf{i},\mathbf{1})$ denotes the element in $\widetilde{\mathcal{S}}$ which extends $\mathbf{i}$ with 1's. We have

$$d\big(g_{\mathbf{i}}a^{F(m)+R}, g_{\mathbf{j}}a^{F(m)+R}\big) \leq d\big(g_{\mathbf{i}}a^{F(m)+R}, g_{(\mathbf{i},\mathbf{1})}a^{F(m)+R}\big) + d\big(g_{(\mathbf{i},\mathbf{1})}a^{F(m)+R}, g_{(\mathbf{j},\mathbf{1})}a^{F(m)+R}\big) + d\big(g_{(\mathbf{j},\mathbf{1})}a^{F(m)+R}, g_{\mathbf{j}}a^{F(m)+R}\big).$$

By Theorem 4.4(iv),

$$d\big(g_{\mathbf{i}}a^{F(m)+R}, g_{\mathbf{j}}a^{F(m)+R}\big) > \frac{\eta_0}{2}.$$

Let $\mathbf{1}_n := (1,\ldots,1) \in \mathcal{S}_n$. Then

$$d\big(g_{\mathbf{i}}a^{F(m)+R}, g_{(\mathbf{i},\mathbf{1})}a^{F(m)+R}\big) = \lim_{n\to\infty} d\big(g_{\mathbf{i}}a^{F(m)+R}, g_{(\mathbf{i},\mathbf{1}_n)}a^{F(m)+R}\big)$$

Since (see the proof of Proposition 4.8)

$$g_{(\mathbf{i},\mathbf{1}_n)} = g_{\mathbf{i}} \prod_{p=0}^{n-1} a^{F(m+p)+R} u^+_{i_{m+p+1}} a^{-F(m+p)-R}$$

we find

$$\begin{aligned} d\big(g_{\mathbf{i}}a^{F(m)+R}, g_{(\mathbf{i},\mathbf{1})}a^{F(m)+R}\big) &= \lim_{n\to\infty} d\Big(1, \prod_{p=0}^{n-1} a^{F(m+p)-F(m)} u^+_{i_{m+p+1}} a^{-F(m+p)+F(m)}\Big) \\ &= \lim_{n\to\infty} d\Big(1, \prod_{p=0}^{n-1} a^{p(R+R')} u^+_{i_{m+p+1}} a^{-p(R+R')}\Big) \\ &\leq c(c+2)\delta \sum_{p=0}^{\infty} \lambda_0^{-p(R+R')} \\ &< \frac{\eta_0}{8}\frac{(\lambda_0-1)^2}{\lambda_0^2}\frac{1}{1-\lambda_0^{-(R+R')}} < \frac{\eta_0}{8}. \end{aligned}$$

From this the claim follows. Pick now an injectivity radius η' of $\mathcal{X}_{\leq s'}$ such that $\eta_0/4 \geq \eta'$. Applying Lemma 5.2 with η' completes the proof. □

Lemma 5.3. *For any $\varepsilon > 0$ and any $s > s_1$ there exists a T-invariant probability measure μ on $\mathcal{X}$ such that*

$$h_\mu(T) > \frac{1}{2}h_m(T) - \varepsilon \quad \textit{and} \quad \mu(\mathcal{X}_{\geq s}) > 1-\varepsilon.$$

Proof. Throughout we use the notation of Proposition 5.1. We apply this proposition with $100s$ to obtain the constant $R' \in \mathbb{N}$. We pick $R \in \mathbb{N}$, $R > 4\log 4$, such that

$$\frac{R}{R+R'} > 1-\varepsilon \quad \text{and} \quad \frac{\log S(R)}{R+R'} > \frac{1}{2}h_m(T) - \varepsilon.$$

Note that this choice is possible since

$$\begin{aligned} S(R) &= \left\lfloor e^{\frac{R}{4}}\right\rfloor^{p_1} \cdot \left\lfloor e^{\frac{R}{2}}\right\rfloor^{p_2} > \left(e^{\frac{R}{4}}-1\right)^{p_1} \cdot \left(e^{\frac{R}{2}}-1\right)^{p_2} \\ &\asymp e^{R\left(\frac{p_1}{4}+\frac{p_2}{2}\right)} = e^{\frac{R}{2}h_m(T)} \quad \text{as } R\to\infty. \end{aligned}$$

Now we choose a subset $\widetilde{E}$ of $\mathcal{X}_{\leq 100s}$ and a family $(\widetilde{E}(m))_{m\in\mathbb{N}}$ of subsets of $\widetilde{E}$ with the properties as in Proposition 5.1. For $m \in \mathbb{N}$ let σ_m denote the uniform probability measure on $\widetilde{E}(m)$, that is,

$$\sigma_m := \frac{1}{S^m} \sum_{x\in\widetilde{E}(m)} \delta_x,$$

where δ_x denotes the Dirac measure with support $\{x\}$. Finite averaging of σ_m provides us with the probability measures

$$\mu_m := \frac{1}{mR+(m-1)R'} \sum_{i=0}^{mR+(m-1)R'-1} T^i_*\sigma_m$$

on $\mathcal{X}$ with support

$$\bigcup_{i=0}^{mR+(m-1)R'-1} T^i\widetilde{E}(m) \quad \subseteq \bigcup_{i\in\mathbb{N}_0} T^i\widetilde{E} =: \mathcal{E}.$$

By Proposition 5.1(i) we find $s' > 100s$ such that $\mathcal{E} \subseteq \mathcal{X}_{\leq s'}$. Let μ be any weak* limit of $(\mu_m)_{m\in\mathbb{N}}$. Then μ is T-invariant and, due to the compactness of $\mathcal{X}_{\leq s'}$, a probability measure. Note that

$$\mathcal{K} := \bigcap_{j\in\mathbb{N}_0} T^{-j}\mathcal{X}_{\leq s'}$$

is a compact subset of $\mathcal{X}$ on which T induces an action, and $\mathcal{E} \subseteq \mathcal{K}$. Thus, μ can be considered as a T-invariant probability measure on $\mathcal{K}$. Since each set $\widetilde{E}(m)$, $m \in \mathbb{N}$, is $(mR+(m-1)R', \eta')$-separated, respectively, the proof of the Variational Principle [Wal00, Theorem 8.6] shows

$$h_\mu(T) \geq \liminf_{m\to\infty} \frac{\log S^m}{mR+(m-1)R'} = \frac{\log S}{R+R'}.$$

By the choice of R, we have

$$h_\mu(T) > \frac{1}{2}h_m(T) - \varepsilon.$$

Moreover, Proposition 5.1(ii3) and the choice of R give

$$\mu_m(\mathcal{X}_{\geq s}) \geq \frac{mR}{mR+(m-1)R'} > \frac{R}{R+R'} > 1-\varepsilon.$$

Thus,

$$\mu(\mathcal{X}_{\geq s}) = \mu(\mathcal{K}\cap\mathcal{X}_{\geq s}) = \lim_{m\to\infty} \mu_m(\mathcal{K}\cap\mathcal{X}_{\geq s}) = \lim_{m\to\infty} \mu_m(\mathcal{X}_{\geq s}) > 1-\varepsilon.$$

This proves the lemma. □

For the proof of Theorem 1.1 we recall that m denotes the normalized Haar measure on $\mathcal{X}$.

Proof of Theorem 1.1. For sufficiently large $n \in \mathbb{N}$ we apply Lemma 5.3 with $\varepsilon = \frac{1}{n}$ and $s = n$ to obtain a T-invariant probability measure μ_n on $\mathcal{X}$ with $\mu_n(\mathcal{X}_{\geq n}) > 1 - \frac{1}{n}$ and

$$h_{\mu_n}(T) > \frac{1}{2}h_m(T) - \frac{1}{n}. \tag{30}$$

Then the weak* limit of the sequence $(\mu_n)_n$ is the zero measure. Now (30) and [EKP, Theorem 7.6] (the theorem presented in the Introduction) show

$$\lim_{n\to\infty} h_{\mu_n}(T) = \frac{1}{2}h_m(T).$$

Thus, Theorem 1.1 is proven for the case $c = \frac{1}{2}h_m(T)$. If c is any value in the interval $[\frac{1}{2}h_m(T), h_m(T)]$, then we consider the sequence $(\nu_n)_n$ of T-invariant probability measures on $\mathcal{X}$ given by the convex combination

$$\nu_n := \left(\frac{2c}{h_m(T)} - 1\right) m + \left(2 - \frac{2c}{h_m(T)}\right)\mu_n.$$

Its weak* limit ν satisfies

$$\nu = \lim_{n\to\infty}\nu_n = \left(\frac{2c}{h_m(T)} - 1\right) m,$$

hence

$$\nu(\mathcal{X}) = \frac{2c}{h_m(T)} - 1.$$

Moreover,

$$\begin{aligned}\lim_{n\to\infty} h_{\nu_n}(T) &= \left(\frac{2c}{h_m(T)} - 1\right) h_m(T) + \left(2 - \frac{2c}{h_m(T)}\right)\lim_{n\to\infty} h_{\mu_n}(T)\\ &= c.\end{aligned}$$

This finishes the proof. □

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