POSITIVE ENTROPY INVARIANT MEASURES ON THE SPACE OF LATTICES WITH ESCAPE OF MASS

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ABSTRACT. On the space of unimodular lattices, we construct a sequence of invariant probability measures under a singular diagonal element with high entropy and show that the limit measure is 0.

1. INTRODUCTION

Consider the homogeneous space $X_3 = \mathrm{SL}_3(\mathbb{Z})\backslash \mathrm{SL}_3(\mathbb{R})$ with the transformation T_3 acting as a right multiplication by $diag(e^{1/2}, e^{1/2}, e^{-1})$. In a joint work with M. Einsiedler in [2] we prove the following.

Theorem 1.1. *For any sequence of T_3-invariant probability measures μ_i on X_3 and $c \in [2, 3]$ with $h_{\mu_i}(\mathrm{T}_3) \geq c$ one has that any weak* limit μ of (μ_i) has $\mu(X_3) \geq c-2$.*

This shows that a lower bound on the entropy of a sequence of measures controls escape of mass in any weak* limit. We say that μ is a weak* limit of the sequence $(\mu_i)_{i\geq 1}$ if for some subsequence i_k and for all $f \in C_c(X)$ we have

$$\lim_{k\to\infty} \int_X f d\mu_{i_k} = \int_X f d\mu.$$

If $c < 2$ then the theorem does not tell us whether one should expect some positive mass left. In this paper we show that actually it is possible that if $c < 2$ then the limit measure could be zero, and also show this in higher dimension.

For $d \geq 1$ we let $G = \mathrm{SL}_{d+1}(\mathbb{R})$ and $\Gamma = \mathrm{SL}_{d+1}(\mathbb{Z})$. We consider the homogeneous space $X = \Gamma\backslash G$ and a transformation T defined by

$$\mathrm{T}(x) = xa$$

where $a = diag(e^{1/d}, e^{1/d}, ..., e^{1/d}, e^{-1}) \in G$.

Theorem 1.2. *There exists a sequence of T-invariant probability measures $(\mu_i)_{i\geq 1}$ on X whose entropies satisfy $\lim_{i\to\infty} h_{\mu_i}(\mathrm{T}) = d$ but the weak* limit μ is the zero measure.*

We note here that the maximum measure theoretic entropy, the entropy of T with respect to Haar measure on X, is $d + 1$. This follows for example from [3, Prop. 9.2 and 9.6]. An immediate consequence of Theorem 1.2 is the following corollary.

Corollary 1.3. *For any $c \in [0, 1]$ there exists a sequence of T-invariant probability measures $(\nu_i)_{i\geq 1}$ on X whose entropies satisfy $\lim_{\to\infty} h_{\mu_i}(\mathrm{T}) = d + c$ such that any weak* limit has mass c.*

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Theorem 1.1 and Corollary 1.3 suggest the following.

Conjecture 1.4. *Let* T *and* X *be as above with* $d \geq 3$ *and let* $c \in [d, d+1]$. *Then for* T*-invariant probability measures* μ_i *on* X *with* $h_{\mu_i}(\mathrm{T}) \geq c$ *one has that any weak* limit* μ *of* $(\mu_i)_{i\geq 1}$ *has* $\mu(X) \geq c - d$.

For more general conjecture of the similar spirit we refer to [1]. There, it is stated in terms of the Hausdorff dimension of the set of points that lie on divergent trajectories for the non-quasi-unipotent flow.

Let $M > 0$ be given. For a lattice $x \in X$, define the height $\mathrm{ht}(x)$ to be the inverse of the length of the shortest nonzero vector in x. Also, define the sets

$$X_{<M} := \{x \in X : \mathrm{ht}(x) < M\} \text{ and } X_{\geq M} := \{x \in X : \mathrm{ht}(x) \geq M\}.$$

We note that by Mahler's compactness criterion $X_{<M}$ is pre-compact. Theorem 1.2 follows from the following.

Theorem 1.5. *For any* $\epsilon > 0$ *and* $M \geq 1$ *there exists a* T*-invariant measure* μ *with* $h_\mu(\mathrm{T}) > d - \epsilon$ *such that* $\mu(X_{\geq M}) > 1 - \epsilon$.

We will construct infinitely many points in $X_{<M}$ whose forward trajectories mostly stay above height M. Taking union of the sets of forward trajectories of these points, we will construct a T-invariant set S_N with topological entropy greater than $d - \epsilon$ (cf. Theorem 3.2). To construct the T-invariant probability measures we want, we will make use of the Variational Principle. In the next section, we introduce preliminary definitions and deduce Theorem 1.2 and its corollary assuming Theorem 1.5. In § 3 we prove Theorem 1.5 assuming Theorem 3.2. In the last two sections we prove Theroem 3.2.

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2. Preliminaries

2.1. **Topological Entropy and Variational Principle.** In this section we will briefly introduce topological entropy and its relation to measure theoretic entropy which is called the Variational Principle. For details and proofs we refer to Chapter 7 and Chapter 8 of [5].

There are various definitions of topological entropy. Here, we will give the definition of topological entropy in terms of separated sets. Let (Y, d_0) be a compact metric space and let $\mathrm{T} : Y \to Y$ be a continuous map. Define a new metric d_n on Y by

$$d_n(x, y) := \max_{0 \leq i \leq n-1} d_0(\mathrm{T}^i(x), \mathrm{T}^i(y)).$$

For a given $\epsilon > 0$ and a natural number n, we say that the couple x, y is (n, ϵ)-*separated* if $d_n(x, y) \geq \epsilon$ and we say that the set E is (n, ϵ)-*separated* if any distinct $x, y \in E$ is (n, ϵ)-separated.

Now define $s_n(\epsilon, Y)$ to be the cardinality of the largest possible (n, ϵ)-separated set and let

$$s(\epsilon, Y) := \limsup_{n\to\infty} \frac{1}{n} \log s_n(\epsilon, Y).$$

Finally, we define the *topological entropy* of T with respect to Y by

$$h(\mathrm{T}) = \lim_{\epsilon \to 0} s(\epsilon, Y).$$

Here is the relation between the topological entropy and measure theoretic entropy:

Theorem 2.1 (Variational Principle)**.** *Topological entropy $h_{\mathrm{T}}(Y)$ of a T-invariant compact metric space Y is the supremum of measure theoretic entropies $h_\mu(Y)$ where supremum is taken over all T-invariant probability measures on the set Y.*

2.2. **Riemannian metric on X.** Let $G = \mathrm{SL}_{d+1}(\mathbb{R})$ and $\Gamma = \mathrm{SL}_{d+1}(\mathbb{Z})$. We fix a left-invariant Riemannian metric d_G on G and for any $x_1 = \Gamma g_1, x_2 = \Gamma g_2 \in X$ we define

$$d_X(x_1, x_2) := \inf_{\gamma\in\Gamma} d_G(g_1, \gamma g_2)$$

which gives a metric d_X on $X = \Gamma\backslash G$. For more information about the Riemannian metric, we refer [4, Chp. 2].

2.2.1. *Injectivity radius.* Let $B_r^H(x) := \{h \in H \mid d(h, x) < r\}$ where d is a metric defined in H and B_r^H is understood to be $B_r^H(1)$.

Lemma 2.2. *For any $x \in X$ there is an injectivity radius $r > 0$ such that the map $g \mapsto xg$ from B_r^G to $B_r^X(x)$ is an isometry.*

Note that since $X_{<M}$ is pre-compact we can choose $r > 0$ which is an injectivity radius for every point in $X_{<M}$. In this case, r is called *an injectivity radius of* $X_{<M}$.

2.3. **Relations between the metrics.** We endow $\mathbb{R}^d$, $\mathbb{R}^{d+1}$, and $\mathbb{R}^{(d+1)^2}$ with the maximum norm $\|\cdot\|$. Rescaling the Riemannian metric if necessary we will assume that there exists $\eta_0 \in (0, 1)$ and $c_0 > 1$ such that

$$d_G(1, g) < \|1 - g\| < c_0 d_G(1, g) \tag{2.1}$$

for any $g \in B_{\eta_0}^G$ where $G = \mathrm{SL}_{d+1}(\mathbb{R})$ as before.

2.4. **Some deductions.** Now we will deduce Corollary 1.3 from Theorem 1.2 and prove Theorem 1.2 assuming Theorem 1.5.

Proof of Corollary 1.3. Let $\{\mu_i\}$ be as in Theorem 1.2 and let λ be the Haar measure on X. We know that $h_\lambda(\mathrm{T}) = d + 1$ which is the maximum entropy. This follows for example from [3, Prop. 9.2 and 9.6]. Define $\nu_i = c\lambda + (1 - c)\mu_i$. Then we have $h_{\nu_i}(\mathrm{T}) = ch_\lambda(\mathrm{T}) + (1 - c)h_{\mu_i}(\mathrm{T})$ so that $\lim_{i\to\infty} h_{\nu_i}(\mathrm{T}) = d + c$. On the other hand, $\lim_{i\to\infty} \nu_i = c\lambda$. Hence, limiting measure has c mass left. □

Proof of Theorem 1.2. Now, let us assume Theorem 1.5. For any natural number i, we let μ_i to be the T-invariant measure with $h_{\mu_i} > d - \frac{1}{i}$ such that $\mu_i(X_{\geq i}) > 1 - \frac{1}{i}$ then any weak* limit has mass 0. □

3. The proof of Theorem 1.5

Before we start the construction, we would like to deduce Theorem 1.5 from Theorem 3.2 below.

Let $\delta > 0$ be an injectivity radius for $X_{<17M}$ with $\delta < \min\{\frac{1}{8M}, \eta_0\}$. Here is an easy lemma which will be used repeatedly in the last section.

Lemma 3.1. *There exists $N' > 0$ such that for any $x, y \in X_{<17M}$ there exists $z \in X_{<17M}$ such that $d(z, y) < \delta/(c_0^3 3^9)$ and $d(x, \mathrm{T}^{N'}(z)) < \delta/(c_0^3 3^9)$.*

Proof. Let λ be the Haar measure on X. Since $X_{<17M}$ is precompact we can cover it with open balls $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_k$ of diameter $\delta/(c_0^3 3^9)$. They have positive measure with respect to the Haar measure. Since T is mixing with respect to the Haar measure, for any $i, j \in \{1, 2, ..., k\}$ there exists $N_{ij} \geq 0$ with $\lambda(\mathrm{T}^{-l}(\mathcal{O}_j) \cap \mathcal{O}_i) > 0$ for any $l \geq N_{ij}$. Letting $N' = \max\{N_{ij} : i, j = 1, 2, ..., k\}$ we obtain the lemma. □

For a given $M \geq 1$ we fix N' as in Lemma 3.1.

Theorem 3.2. *Let $M \geq 1$ be given. For any large N let $K = \lfloor \frac{1}{13} e^{dN} \rfloor$. Then there exist a constant $M' > 1$ and a set S_N in $X_{<M}$ such that*

$$\mathrm{T}^l(x) \in X_{<M'} \text{ for all } x \in S_N \text{ and for all } l \geq 0.$$

Moreover, there exists a constant $s > 0$ such that for any $m \in \mathbb{N}$ there are subsets $S_N(m)$ of S_N with the following properties:

- **(i)** *cardinality of $S_N(m)$ is K^m*
- **(ii)** *$S_N(m)$ is $(mN + (m-1)N', s)$-separated and*
- **(iii)** *for any $x \in S_N(m)$ we have*

$$|\{l \in [0, mN + (m-1)N'] : \mathrm{T}^l(x) \in X_{\geq M/(c_0+1)}\}| \geq mN.$$

Now we deduce Theorem 1.5 from Theorem 3.2.

Proof of the Theorem 1.5. Let $\epsilon > 0$ be given and let N' be as in Lemma 3.1. Choose N large enough so that

$$\frac{1}{N+N'} \log\lfloor \frac{1}{13} e^{dN} \rfloor > d - \epsilon \text{ and } \frac{N'}{N+N'} < \epsilon$$

and let S_N be the set as in Thereom 3.2.

To obtain a T-invariant probability measure with high entropy we would like to make use of Variational Principle 2.1. For this, we need a compact T-invarinat subspace of X. We define

$$Y_{\leq M'} = \{x \in X_{\leq M'} \mid \mathrm{T}^l(x) \in X_{\leq M'}, \text{ for } l \geq 0\}.$$

Clearly, we obtain a T-invariant compact subspace containing $\mathrm{T}^l(S_N)$ for all $l \geq 0$.

We have $h_{\mathrm{T}}(Y_{\leq M'}) > d - \epsilon$ since $Y_{\leq M'}$ contains the sets $S_N(m)$ which are $(mN+(m-1)N', s)$-separated by Theorem 3.2. Now, from Variational Principle 2.1 we know that there is a T-invariant measure μ on $Y_{\leq M'}$, hence on X, with $h_\mu(\mathrm{T}) > d - \epsilon$. In order to obtain the theorem, we want to have $\mu(X_{\geq M/(c_0+1)}) > 1 - \epsilon$, but we do not get this from Variational Principle itself. Thus, we need to look into the proof of Variational Principle and see how the measures are constructed.

Let $S_N(m)$ be the subset of $Y_{\leq M'}$ as in Theorem 3.2. We have that $S_N(m)$ is $(mN+(m-1)N', s)$-separated and has cardinality K^m where $K = \lfloor \frac{1}{13} e^{dN} \rfloor$. Define a probability measure

$$\sigma_m = \frac{1}{K^m} \sum_{x \in S_N(m)} \delta_x \text{ where } \delta_x(A) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}.$$

Now, let a probability measure μ_m be defined by

$$\mu_m = \frac{1}{mN + (m-1)N'} \sum_{i=0}^{mN+(m-1)N'-1} \sigma_m \circ \mathrm{T}^{-i}$$

where $\sigma_m \circ \mathrm{T}^{-i}(A) = \sigma_m(\mathrm{T}^{-i}(A))$ for any measurable set A. We know that $\mathcal{M}(Y_{\leq M'})$, the space of Borel probability measures, is compact in the weak* topology [5, Theorem 6.5]. We obtained a set of measures $\mu_m \in \mathcal{M}(Y_{\leq M'})$. If necessary going into subsequence, we have that $\{\mu_m\}$ converges to some probability measure μ in $\mathcal{M}(Y_{\leq M'})$. The measure μ we obtained is T-invariant [5, Theorem 6.9]. From the proof of Variational Principle [5, Theorem 8.6], we know that μ has

$$\begin{aligned} h_\mu(\mathrm{T}_{|Y_{\leq M'}}) &\geq \lim_{m\to\infty} \frac{1}{mN+(m-1)N'} \log s_m(\epsilon, Y_{\leq M'}) \\ &\geq \lim_{m\to\infty} \frac{1}{mN+(m-1)N'} \log K^m \\ &= \frac{1}{N+N'} \log K. \end{aligned}$$

On the other hand, by assumption we have $\frac{1}{N+N'} \log K > d-\epsilon$ and hence we obtain

$$h_\mu(\mathrm{T}) \geq h_\mu(\mathrm{T}_{|Y_{\leq M'}}) > d-\epsilon.$$

We have $\mu_m(X_{<M/(c_0+1)}) = \frac{1}{mN+(m-1)N'} \sum_{i=0}^{mN+(m-1)N'-1} \sigma_m \circ \mathrm{T}^{-i}(X_{<M/(c_0+1)})$. Hence, from part (iii) of Theorem 3.2

$$\mu_m(X_{<M/(c_0+1)}) \leq \frac{(m-1)N'}{mN+(m-1)N'} < \frac{N'}{N+N'} < \epsilon.$$

It is easy to see, approximating $X_{<M/(c_0+1)}$ by continuous functions with compact support, that

$$\mu(X_{\geq M/(c_0+1)}) > 1-\epsilon.$$

So, we obtain the theorem if we apply Theorem 3.2 for $(c_0+1)M$ instead of M. □

4. Initial setup and shadowing lemma

In this section we will construct about e^{dN} lattices whose forward trajectories stay above height M in the time interval $[1, N]$ for some large number N. Later we prove the shadowing lemma 4.3, which will be used in the proof of Theorem 3.2 in the next section.

Fix a height $M > 0$. Let $N \in \mathbb{N}$ be given. For $t = (t_1, t_2, ..., t_d) \in [0, e^{-N/d}]^d$ consider the lattice $x_t = \Gamma g_t$ where

$$g_t = \begin{pmatrix} M^{1/d} & 0 & \dots & 0 & 0 \\ 0 & M^{1/d} & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & M^{1/d} & 0 \\ \frac{t_1}{M} & \frac{t_2}{M} & \dots & \frac{t_d}{M} & \frac{1}{M} \end{pmatrix}. \tag{4.1}$$

We would like to consider those lattices that stay above height M in $[1, N]$ and are in $X_{<16M}$ at time N. We start with first considering the set

$$A_N := \{t \in [0, e^{-N/d}]^d : \mathrm{T}^N(x_t) \in X_{<16M}\}.$$

We claim that A_N is significant in size.

Lemma 4.1. *For $d \geq 2$ let $m_{\mathbb{R}^d}$ be the Lebesgue measure on $\mathbb{R}^d$. Then*

$$m_{\mathbb{R}^d}(A_N) \geq (\frac{15^d}{16^d} - \frac{1}{4^d})e^{-N}.$$

The explicit constant $(\frac{15^d}{16^d}-\frac{1}{4^d})$ has no importance to us. All we need is that $m_{\mathbb{R}^d}(A_N) \gg e^{-N}$. However, the explicit constant simplifies the later work. We can think of A_N as a subset of the unstable subgroup U^+ of G with respect to a. Although A_N has small volume in $\mathbb{R}^d$, it gets expanded by T^N to a set of volume $\gg e^{dN}$ which will give us an (N,s)-separated set of cardinality $\gg e^{dN}$.

Proof. We will prove that $m_{\mathbb{R}^d}(A'_N) \geq (\frac{15^d}{16^d}-\frac{1}{4^d})e^{-N}$ where

$$A'_N = A_N \cap [\frac{1}{16}e^{-N/d}, e^{-N/d}]^d. \tag{4.2}$$

Assume that $\mathrm{ht}(\mathrm{T}^N(x_t)) > 16M$. So, for some nonzero $(p_1,p_2,...,p_d,q)\in\mathbb{Z}^{d+1}$ with $\gcd(p_1,p_2,...,p_d,q)=1$ and $q>0$ we must have

$$\begin{aligned}&\|(p_1,p_2,...,p_d,q)g_t a^N\|\\ &= \|(p_1M^{1/d}+q\frac{t_1}{M})e^{N/d},(p_2M^{1/d}+q\frac{t_2}{M})e^{N/d},...,(p_dM^{1/d}+q\frac{t_d}{M})e^{N/d},q\frac{1}{M}e^{-N})\|\\ &< \frac{1}{16M}.\end{aligned}$$

So, letting $\epsilon = \frac{e^{-N/d}}{16M^{(d+1)/d}}$ we have

$$|p_i + q\frac{t_i}{M^{(d+1)/d}}| < \epsilon \text{ for all } i=1,2,...,d \text{ and } q < \frac{e^N}{16}. \tag{4.3}$$

We have $t_i \in [\frac{1}{16}e^{-N/d}, e^{-N/d}]$. For a fixed q, we will calculate the Lebesgue measure of $(t_1,t_2,...,t_d)\in[\frac{1}{16}e^{-N/d},e^{-N/d}]^d$ for which (4.3) hold for some p_i's.

We have

$$q\frac{t_i}{M^{(d+1)/d}} \in [q\epsilon, 16q\epsilon].$$

If $16q\epsilon \leq \frac{1}{2}$ then $(p_1,p_2,...,p_d)=0$ and since we only need to consider the primitive vectors in x_t we have $q=1$. In this case, $q\frac{t_i}{M^{(d+1)/d}}\in[\epsilon,16\epsilon]$ and hence (4.3) does not hold. So, we can assume that

$$16q\epsilon > \frac{1}{2}.$$

We note that $q\frac{t_i}{M^{(d+1)/d}}$ must be in the ϵ-neighborhood of an integer point. If $16q\epsilon\in(1/2,1)$ then $[q\epsilon,16q\epsilon]$ does not contain any integers and only possible way for (4.3) to hold is when $q\frac{t_i}{M^{(d+1)/d}}$ is in $(1-\epsilon,1+\epsilon)$ so that t_i must be in

$$(\frac{(1-\epsilon)M^{(d+1)/d}}{q}, \frac{(1+\epsilon)M^{(d+1)/d}}{q}).$$

Thus, for a fixed $q\in(\frac{1}{32\epsilon},\frac{1}{16\epsilon})$ we have that the Lebesgue measure of points that satisfy (4.3) is

$$\leq \left(\frac{2\epsilon M^{(d+1)/d}}{q}\right)^d = \frac{2^d\epsilon^d M^{d+1}}{q^d}.$$

Now, for $16q\epsilon\geq 1$ we have that $[q\epsilon,16q\epsilon]$ has at most $\leq 15q\epsilon+1$ integer points. Thus, there could be $\leq 15q\epsilon+2$ integers for which $q\frac{t_i}{M^{(d+1)/d}}$ can be ϵ-close for some t_i. Since $16q\epsilon\geq 1$ we have $15q\epsilon+2\leq 48q\epsilon$. Hence, arguing as in the previous case, for a fixed $q\geq\frac{1}{16\epsilon}$ we have that the Lebesgue measure of points satisfying (4.3) is

$$\leq \left((48q\epsilon)(2\epsilon)(\frac{M^{(d+1)/d}}{q})\right)^d = 96^d\epsilon^{2d}M^{d+1}.$$

Thus, we obtain that the Lebesgue measure of points for which (4.3) hold is

$$\leq \sum_{q=\lceil \frac{1}{32\epsilon}\rceil}^{\lfloor \frac{1}{16\epsilon}\rfloor} \frac{2^d\epsilon^d M^{d+1}}{q^d} + \sum_{q=\lceil \frac{1}{16\epsilon}\rceil}^{\lfloor \frac{e^N}{16}\rfloor} 96^d\epsilon^{2d}M^{d+1}.$$

Since $\epsilon^d = \frac{e^{-N}}{16^d M^{d+1}}$, the above inequality simplifies to

$$\leq e^{-N}\left(\sum_{q=\lceil \frac{1}{32\epsilon}\rceil}^{\lfloor \frac{1}{16\epsilon}\rfloor} \frac{2^d}{16^d q^d} + \sum_{q=\lceil \frac{1}{16\epsilon}\rceil}^{\lfloor \frac{e^N}{16}\rfloor} \frac{96^d e^{-N}}{16^{2d}M^{d+1}}\right). \tag{4.4}$$

We want to show that, independent of N, the term inside the parenthesis is strictly less than 1.

$$\sum_{q=\lceil \frac{1}{32\epsilon}\rceil}^{\lfloor \frac{1}{16\epsilon}\rfloor} \frac{2^d}{16^d q^d} \leq \sum_{q=\lceil \frac{1}{32\epsilon}\rceil}^{\lfloor \frac{1}{16\epsilon}\rfloor} \frac{2^d}{16^d q} \leq \frac{1}{8^d \frac{1}{32\epsilon}}(\lfloor \frac{1}{16\epsilon}\rfloor - \lceil \frac{1}{32\epsilon}\rceil) \leq \frac{1}{8^d}.$$

On the other hand,

$$\sum_{q=\lceil \frac{1}{16\epsilon}\rceil}^{\lfloor \frac{e^N}{16}\rfloor} \frac{96^d e^{-N}}{16^{2d}M^{d+1}} \leq \frac{96^d e^{-N}}{16^{2d}M^{d+1}}\frac{e^N}{16} < \frac{1}{2^{d+4}M^{d+1}}.$$

Together, we see that the inequality (4.4) is

$$< (\frac{1}{8^d} + \frac{1}{2^{d+4}M^{d+1}})e^{-N} \leq \frac{e^{-N}}{4^d}.$$

Thus, we conclude that $m_{\mathbb{R}^d}(A_N) \geq m_{\mathbb{R}^d}(A'_N) > (\frac{15^d}{16^d} - \frac{1}{4})e^{-N}$. □

From the set A_N, in fact from A'_N as in (4.2), we want to pick about e^{dN} elements which are not too close to each other so that within N iterations under T they get apart from each other. For this purpose, let us partition $[\frac{1}{16}e^{-N/d}, e^{-N/d}]^d$ into $\lfloor e^N \rfloor^d$ small d-cubes of side length $\frac{15}{16}e^{-N(d+1)/d}$.

Now, consider even smaller d-cubes of side length $\frac{13}{16}e^{-N(d+1)/d}$ each lying at the center of one of the small d-cubes. We need to find a lower bound for the number of these smaller d-cubes that intersect with the set A'_N. Each of these d-cubes has volume equal to $(\frac{13}{16})^d e^{-N(d+1)}$. Thus, there could be at most

$$\left\lceil \frac{(\frac{1}{4^d})e^{-N}}{(\frac{13}{16})^d e^{-N(d+1)}} \right\rceil = \left\lceil \frac{4^d}{13^d}e^{dN} \right\rceil$$

many that do not intersect with A'_N. Therefore, for N large, at least

$$\lfloor e^N \rfloor^d - \left\lceil \frac{4^d}{13^d}e^{dN} \right\rceil \geq \frac{1}{13}e^{dN}$$

of these smaller d-cubes do intersect with A'_N.

Let us pick one element t from each of these smaller d-cubes that is also contained in A'_N and consider the set $S'_N(1)$ of these lattices $x_t = \Gamma g_t$ where g_t is as in (4.1). To simplify notation we let

$$S'_N(1) = \{x_1, x_2, ..., x_K\} = \{\Gamma g_1, \Gamma g_2, ..., \Gamma g_K\} \tag{4.5}$$

where

$$K = \lfloor \frac{1}{13} e^{dN} \rfloor.$$

We note that for elements t, t' that are picked from different d-cubes one has

$$\frac{1}{4} e^{-N(d+1)/d} \le \|t - t'\| < \frac{15}{16} e^{-N/d}. \tag{4.6}$$

Proposition 4.2. *For a given large N the set $S'_N(1) = \{x_1, x_2, ..., x_K\}$ has the following properties:*

(i) $\mathrm{ht}(\mathrm{T}^l(x_i)) \ge M$ *for* $l \in [1, N]$ *and* $i \in [1, K]$,

(ii) $\mathrm{ht}(x_i) < M$ *and* $\mathrm{ht}(\mathrm{T}^N(x_i)) < 16M$ *for any* $i \in [1, K]$,

(iii) *for* $i \neq j$ *we have* $d(g_i, g_j) < \frac{30}{16} e^{-N/d}$ *and* $d(\mathrm{T}^N(g_i), \mathrm{T}^N(g_j)) \ge \frac{1}{8M}$.

Proof. Let $x_i = x_t = \Gamma g_t$ for some $t = (t_1, t_2, ..., t_d) \in [\frac{1}{16} e^{-N/d}, e^{-N/d}]^d$ (cf. (4.1)). It is easy to see that $x_t \in X_{<M}$. On the other hand, by construction $t \in A_N$ so that $\mathrm{T}^N(x_t) \in X_{<16M}$.

Now, consider the vector $v = (\frac{t_1}{M}, \frac{t_2}{M}, ..., \frac{t_d}{M}, \frac{1}{M}) \in x_t$. We have

$$\mathrm{T}(v) = (\frac{t_1 e^{1/d}}{M}, \frac{t_2 e^{1/d}}{M}, ..., \frac{t_d e^{1/d}}{M}, \frac{e^{-1}}{M})$$

so that

$$\| \mathrm{T}(v) \| \le \max\{\frac{e^{-(N-1)/d}}{M}, \frac{e^{-1}}{M}\} < \frac{1}{M}.$$

Also,

$$\mathrm{T}^N(v) = (\frac{t_1 e^{N/d}}{M}, \frac{t_2 e^{N/d}}{M}, ..., \frac{t_d e^{N/d}}{M}, \frac{e^{-N}}{M})$$

which implies

$$\| \mathrm{T}^N(v) \| \le \max\{\frac{1}{M}, \frac{e^{-N}}{M}\} \le \frac{1}{M}.$$

Since the function $\| \mathrm{T}^l(v) \|$ in l has only one critical point we conclude that for $l = 1, 2, ..., N$

$$\mathrm{ht}(\mathrm{T}^l(x_t)) \ge M.$$

Let x_j be another element and let $t' \in [\frac{1}{16} e^{-N/d}, e^{-N/d}]^d$ be such that $x_j = x_{t'} = \Gamma g_{t'}$. From (4.6) together with left invariance of the metric we have

$$d(\mathrm{T}^N(g_t), \mathrm{T}^N(g_{t'})) = d(a^N a^{-N} g_t a^N, a^N a^{-N} g'_t a^N) \ge \frac{\|t - t'\|}{2M} e^{N(d+1)/d} \ge \frac{1}{8M}.$$

The fact that $d(g_i, g_j) < \frac{30}{16} e^{-N/d}$ follows from (4.6) also. □

Our main tool for the construction of lattices is the shadowing lemma:

Lemma 4.3 (Shadowing lemma)**.** *Let $\epsilon \in (0, \eta_0/(3c_0))$ be given. If $d(x_-, x_+) < \epsilon$ for some $x_-, x_+ \in X$ then there exists $y \in X$ such that*

(i) $d(\mathrm{T}^l(y), \mathrm{T}^l(x_-)) < 2c_0 \epsilon e^{l(d+1)/d}$ *for all* $l \le 0$ *and*

(ii) $d(\mathrm{T}^l(y), \mathrm{T}^l(x_+)) < 3c_0 \epsilon$ *for all* $l \ge 0$.

Moreover, there exists c in the centralizer C of a with $d(c, 1) < 3c_0 \epsilon$ such that $d(\mathrm{T}^l(y), \mathrm{T}^l(x_+ c)) < 6c_0^2 \epsilon e^{-l(d+1)/d}$ for all $l \ge 0$.

Proof. We have $x_- = x_+ g$ for some $g = (g_{ij}) \in \mathrm{SL}(d+1, \mathbb{R})$ with $d(g, 1) < \epsilon$. Consider

$$u^+ := \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ u_1 & u_2 & \dots & u_d & 1 \end{pmatrix}$$

and let $y = x_- u^+$. For $\|(u_1, u_2, \dots, u_d)\| < 2c_0\epsilon$ we have

$$\begin{aligned} d(\mathrm{T}^l(y), \mathrm{T}^l(x_-)) &= d(x_- u^+ a^l, x_- a^l) \\ &= d(x_- a^l a^{-l} u^+ a^l, x_- a^l) \\ &\le d\left(\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ u_1 e^{l(d+1)/d} & u_2 e^{l(d+1)/d} & \dots & u_d e^{l(d+1)/d} & 1 \end{pmatrix}, 1 \right) \\ &< \|(u_1, u_2, \dots, u_d)\| e^{l(d+1)/d} < 2c_0 \epsilon e^{l(d+1)/d}. \end{aligned}$$

This establishes part (i). Now, we let

$$\begin{aligned} g' &:= g u^+ \\ &= \begin{pmatrix} g_{11} + g_{1(d+1)} u_1 & \dots & g_{1d} + g_{1(d+1)} u_d & g_{1(d+1)} \\ g_{21} + g_{2(d+1)} u_1 & \dots & g_{2d} + g_{2(d+1)} u_d & g_{2(d+1)} \\ \vdots & & \vdots & \vdots \\ . & \dots & . & . \\ g_{(d+1)1} + g_{(d+1)(d+1)} u_1 & \dots & g_{(d+1)d} + g_{(d+1)(d+1)} u_d & g_{(d+1)(d+1)} \end{pmatrix}. \end{aligned}$$

Since $d(g, 1) < \epsilon$, from (2.1) we have that

$$|g_{(d+1)(d+1)} - 1| \le \|g - 1\| < c_0 d(g, 1) < 1/2.$$

In particular, $g_{(d+1)(d+1)} \neq 0$. Letting $u_i = -\frac{g_{(d+1)i}}{g_{(d+1)(d+1)}}$ for $i = 1, 2, ..., d$ we can make sure that the unstable part with respect to a is 0. For any $i \in [1, d]$ we have $|g_{(d+1)i}| \le \|g - 1\| < c_0\epsilon$. Hence, we have

$$||(u_1, u_2, \dots, u_d)|| = \frac{1}{|g_{(d+1)(d+1)}|} \max_i \{|g_{(d+1)i}|\} < \frac{c_0 \epsilon}{1/2} = 2c_0\epsilon.$$

Now,

$$d(\mathrm{T}^l(y), \mathrm{T}^l(x_+)) = d(\mathrm{T}^l(x_+ g u^+), \mathrm{T}^l(x_+)) = d(x_+ a^l a^{-l} g' a^l, x_+ a^l) \le d(a^{-l} g' a^l, 1).$$

Since unstable part of g' is 0, for $l \ge 0$ we obtain

$$d(\mathrm{T}^l(y), \mathrm{T}^l(x_+)) \le d(g', 1) = d(g u^+, 1) \le d(u^+, 1) + d(1, g) < \|u^+\| + \epsilon < 3c_0\epsilon.$$

For the last part, let

$$c := \begin{pmatrix} g_{11} + g_{1(d+1)} u_1 & \dots & g_{1d} + g_{1(d+1)} u_d & 0 \\ g_{21} + g_{2(d+1)} u_1 & \dots & g_{2d} + g_{2(d+1)} u_d & 0 \\ \vdots & & \vdots & \vdots \\ . & \dots & . & 0 \\ 0 & \dots & 0 & g_{(d+1)(d+1)} \end{pmatrix},$$

then we have that $c \in C$ with $d(c,1) \le d(g',1) < 3c_0\epsilon$, and hence $d(c^{-1},1) < 3c_0\epsilon$. On the other hand, if we let $u^- = c^{-1}g'$ then, $u^- \in U^-$ and

$$\|u^- - 1\| < c_0 d(u^-,1) \le c_0 d(g',1) + c_0 d(1,c) < 6c_0^2\epsilon.$$

Thus, $d(\mathrm{T}^l(y), \mathrm{T}^l(x_+c)) = d(x_+gu^+a^l, x_+ca^l) = d(x_+g'a^l, x_+ca^l) \le d(g'a^l, ca^l) = d(a^{-l}c^{-1}g'a^l, 1) = d(a^{-l}u^-a^l, 1) < \|u^- - 1\|e^{-l(d+1)/d} < 6c_0^2\epsilon e^{-l(d+1)/d}$. □

5. Construction

In this section we construct the set S_N mentioned in the introduction with the properties as in Theorem 3.2. Repeatedly using both the shadowing lemma and K lattices constructed in the previous section we obtain more and more lattices that in the limit gives the set S_N.

Recall the set $S'_N(1)$ constructed in § 4 (see (4.5)). Let $M' > 0$ be a height that depends on N such that for any $x_i \in S'_N(1)$ and for any $l = 0, 1, ..., N$ we have $\mathrm{T}^l(x_i) \in X_{<M'}$. Recall that $\delta > 0$ is an injectivity radius for $X_{<17M}$ with $\delta < \min\{\frac{1}{8M}, \eta_0\}$. Now, let $\eta \in (0, \delta)$ be such that 2η is an injectivity radius of $X_{<M'}$. Recall that $K = \lfloor \frac{1}{13} e^{dN} \rfloor$. We will prove Theorem 3.2 with choice of $s = \eta/e^2$ and with the choice of M' as defined above.

Theorem 3.2 follows from the following proposition.

Proposition 5.1. *As before, let N be sufficiently large. For any positive integer m, there is a subset*

$$S'_N(m) = \{x_{i_1 i_2 \ldots i_m} : i_1, i_2, ..., i_m \in \{1, 2, ..., K\}\}$$

of $X_{<M}$ with the following properties:

(i) *for any $x \in S'_N(m)$ we have*

$$|\{l \in [0, mN + (m-1)N'] : \mathrm{T}^l(x) \in X_{\ge M/(c_0+1)}\}| \ge mN,$$

(ii) *for any $x \in S'_N(m)$ we have $\mathrm{T}^{mN+(m-1)N'}(x) \in X_{<17M}$,*

(iii) *for any distinct $x_{i_1 i_2 \ldots i_m}, x_{j_1 j_2 \ldots j_m} \in S'_N(m)$, say $i_n \ne j_n$, there exist $g, h \in G$ such that*

$$\mathrm{T}^{(n-1)(N+N')}(x_{i_1 i_2 \ldots i_m}) = \Gamma g \text{ and } \mathrm{T}^{(n-1)(N+N')}(x_{j_1 j_2 \ldots j_m}) = \Gamma h$$

with $d(\Gamma g, \Gamma h) = d(g,h)$ and that

$$d(\mathrm{T}^N(g), \mathrm{T}^N(h)) > \delta - \frac{\delta}{3^4} \text{ if } n = m \text{ and}$$

$$d(\mathrm{T}^N(g), \mathrm{T}^N(h)) > \delta - \delta \sum_{l=3}^{m-n+2} 3^{-l} \text{ if } n \in [1, m).$$

Moreover, we can make sure that for $x_{i_1 i_2 \ldots i_m} \in S'_N(m)$ and for $x_{i_1 i_2 \ldots i_{m+1}} \in S'_N(m+1)$ we have $d(x_{i_1 i_2 \ldots i_m}, x_{i_1 i_2 \ldots i_{m+1}}) < \delta e^{-m}$.

To derive Theorem 3.2 from Proposition 5.1 we need the lemma below which helps us to determine when two lattices get separated.

Lemma 5.2. *For $\Gamma g, \Gamma h \in X$ with $\mathrm{T}^l(\Gamma g), \mathrm{T}^l(\Gamma h) \in X_{<M'}$ in $[0, N]$ assume that $d(g,h) < \frac{\eta}{e^2}$ and $d(\mathrm{T}^N(g), \mathrm{T}^N(h)) \ge \frac{\eta}{e^2}$. Then $\Gamma g, \Gamma h$ is $(N, \frac{\eta}{e^2})$-separated, that is, there exists $l \in [1, N]$ with $d(\mathrm{T}^l(\Gamma g), \mathrm{T}^l(\Gamma h)) \ge \frac{\eta}{e^2}$.*

Proof. Since we have $d(g,h) < \frac{\eta}{e^2}$ and that $d(\mathrm{T}^N(g), \mathrm{T}^N(h)) > \frac{\eta}{e^2}$, there exists $l \in [1, N]$ such that

$$d(\mathrm{T}^{l-1}(g), \mathrm{T}^{l-1}(h)) < \frac{\eta}{e^2} \le d(T^l(g), T^l(h)).$$

We have $d(\mathrm{T}(g), \mathrm{T}(h)) = d(a^{-1}h^{-1}ga, 1) = d(a^{-1}u^+aa^{-1}u^-ca, 1)$. On the other hand, we note that any two elements of the unstable subgroup with respect to a gets expanded at most by the factor of $e^{(d+1)/d}$ under the action of T. Together with triangle inequality we have

$$\begin{aligned} d(a^{-1}u^+aa^{-1}u^-ca, 1) &\le d(a^{-1}u^+aa^{-1}u^-ca, a^{-1}u^+a) + d(a^{-1}u^+a, 1) \\ &= d(a^{-1}u^-ca, 1) + d(a^{-1}u^+a, 1) \\ &\le d(u^-c, 1) + e^{(d+1)/d}d(u^+, 1) \\ &\le e^2(d(u^-c, 1) + d(u^+, 1)) \\ &\le 2e^2 d(u^+u^-c, 1). \end{aligned}$$

Thus, $d(\mathrm{T}^l(g), \mathrm{T}^l(h)) \le 2e^2 d(\mathrm{T}^{l-1}(g), \mathrm{T}^{l-1}(h)) < 2\eta$. On the other hand, $\mathrm{T}^l(\Gamma g)$, $\mathrm{T}^l(\Gamma h)$ are in $X_{<M'}$ and 2η is an injectivity radius of $X_{<M'}$. Hence,

$$d(\mathrm{T}^l(\Gamma g), \mathrm{T}^l(\Gamma h)) = d(\mathrm{T}^l(g), T^l(h)) \ge \frac{\eta}{e^2}.$$

□

Proof of Theorem 3.2. For any m let us pick a set

$$S'_N(m) = \{x_{i_1 i_2 \dots i_m} : i_1, i_2, \dots, i_m \in \{1, 2, \dots, K\}\}$$

as in Proposition 5.1. Also, assume for $x_{i_1 i_2 \dots i_m} \in S'_N(m)$ and for $x_{i_1 i_2 \dots i_{m+1}} \in S'_N(m+1)$ we have $d(x_{i_1 i_2 \dots i_m}, x_{i_1 i_2 \dots i_{m+1}}) < \delta e^{-m}$. If we fix a sequence $\{i_l\} \subset \{1, 2, \dots, K\}^{\mathbb{N}}$, then the sequence $\{x_{i_1}, x_{i_1 i_2}, x_{i_1 i_2 i_3}, \dots\}$ becomes a Cauchy sequence and hence converges. So, we let $x_{\{i_l\}} = \lim_{n \to \infty} x_{i_1 i_2 \dots i_m}$. Varying the sequence $\{i_l\}$ we define the set

$$S_N = \left\{x_{\{i_l\}} : \{i_l\} \subset \{1, 2, \dots, K\}^{\mathbb{N}}\right\}.$$

Also, define subsets $S_N(m)$'s of S_N

$$S_N(m) = \left\{x_{\{i_l\}} : \{i_l\} \subset \{1, 2, \dots, K\}^{\mathbb{N}} \text{ with } i_l = 1 \text{ for all } l > m\right\}.$$

By definition of $S_N(m)$ and by (i) of Proposition 5.1, for any $x_{\{i_l\}} \in S_N(m)$ we have

$$|\{l \in [0, mN + (m-1)N'] : \mathrm{T}^l(\{x_I\}) \in X_{\ge M/(c_0+1)}\}| \ge mN.$$

As for part (ii), again from the construction of the set $S_N(m)$ and from (iii) of Proposition 5.1 we conclude that for any distinct $x_{\{i_l\}}, x_{\{j_l\}} \in S_N(m)$, say $i_n \neq j_n$, there exist $g, h \in G$ with $\mathrm{T}^{(n-1)(N+N')}(x_{\{i_l\}}) = \Gamma g$, $\mathrm{T}^{(n-1)(N+N')}(x_{\{j_l\}}) = \Gamma h$ and $d(\Gamma g, \Gamma h) = d(g, h)$ such that

$$d(\mathrm{T}^N(g), \mathrm{T}^N(h)) > \delta - \delta \sum_{l=3}^{\infty} 3^{-l} = \frac{17}{18}\delta.$$

If $d(\Gamma g, \Gamma h) \ge \frac{\eta}{e^2}$ then there is nothing to show, if not then from Lemma 5.2 for some $s \in [1, N]$ we conclude that $d(\mathrm{T}^s(\Gamma g), \mathrm{T}^s(\Gamma h)) \ge \frac{\eta}{e^2}$ since $\frac{\eta}{e^2} < \frac{17}{18}\delta$. Thus, for some $s \in [1, N]$ we have

$$d\left(\mathrm{T}^{(n-1)(N+N')+s}(x_{\{i_l\}}), \mathrm{T}^{(n-1)(N+N')+s}(x_{\{j_l\}})\right) \geq \frac{\eta}{e^2}$$

and hence the set $S_N(m)$ is $(mN + (m-1)N', \eta/e^2)$-separated since $n \leq m$. This concludes the proof. □

Now, we will make use of what we obtained in the previous section to prove Proposition 5.1.

Proof of Proposition 5.1. We inductively prove (ii) and (iii) and briefly discuss how these arguments imply (i). Let us fix some large N.

For $m = 1$ let $S'_N(1) = \{x_1, x_2, ..., x_K\}$ be the set as in Proposition 4.2. It is clear that (i) and (ii) are satisfied. Let $x_i = \Gamma g_i, x_j = \Gamma g_j$ be distinct elements (cf. (4.5)). Then letting $g = g_i$ and $h = g_j$ we obtain (iii) since the part (iii) of Proposition 4.2 gives

$$d(\mathrm{T}^N(g_i), \mathrm{T}^N(g_j)) \geq \frac{1}{8M} > \delta.$$

Now, assume that the proposition holds for $m = k \geq 1$, we have the set $S'_N(k) = \{x_{i_1 i_2 \ldots i_k} : i_1, i_2, ..., i_k = 1, ..., K\}$. Let us construct the set $S'_N(k+1)$.

For any $x_{i_1 i_2 \ldots i_k} \in S'_N(k)$, we have $\mathrm{T}^{kN+(k-1)N'}(x_{i_1 i_2 \ldots i_k}) \in X_{<17M}$. Hence, applying Lemma 3.1 we have that for x_j there exists z with

$$d(\mathrm{T}^{kN+(k-1)N'}(x_{i_1 i_2 \ldots i_k}), z) < \delta/(c_0^3 3^9) \text{ and } d(x_j, \mathrm{T}^{N'}(z)) < \delta/(c_0^3 3^9).$$

Now, we apply shadowing lemma with $x_- = \mathrm{T}^{kN+(k-1)N'}(x_{i_1 i_2 \ldots i_k})$ and $x_+ = z$ and $\epsilon = \delta/(c_0^3 3^9)$. There exists y such that

$$d(\mathrm{T}^l(y), \mathrm{T}^l(T^{kN+(k-1)N'}(x_{i_1 i_2 \ldots i_k}))) < \frac{\delta}{c_0^2 3^8} e^{l(d+1)/d} \text{ for } l \leq 0 \text{ and} \tag{5.1}$$

$$d(\mathrm{T}^l(y), \mathrm{T}^l(z)) < \frac{\delta}{c_0^2 3^8} \text{ for } l \geq 0. \tag{5.2}$$

We have $d(x_j, \mathrm{T}^{N'}(y)) < d(x_j, \mathrm{T}^{N'}(z)) + d(\mathrm{T}^{N'}(z), \mathrm{T}^{N'}(y)) < \delta/(c_0^4 3^9) + \delta/(c_0^2 3^8) < \delta/(c_0^2 3^7)$. We apply shadowing lemma once more with $x_- = \mathrm{T}^{N'}(y)$ and $x_+ = x_j$ and $\epsilon = \delta/(c_0^2 3^7)$. There exists y' such that

$$d(\mathrm{T}^l(y'), \mathrm{T}^l(\mathrm{T}^{N'}(y))) < \frac{\delta}{c_0 3^6} e^{l(d+1)/d} \text{ for } l \leq 0 \text{ and} \tag{5.3}$$

$$d(\mathrm{T}^l(y'), \mathrm{T}^l(x_j)) < \frac{\delta}{c_0 3^6} \text{ for } l \geq 0 \tag{5.4}$$

Also, there exists $c_j \in C$ with $d(c_j, 1) < \frac{\delta}{c_0 3^6}$ such that

$$d(\mathrm{T}^l(y'), \mathrm{T}^l(x_j c_j)) < \frac{\delta}{3^5} e^{-l(d+1)/d} \text{ for } l \geq 0 \tag{5.5}$$

Now we let $x_{i_1 i_2 \ldots i_k j} = \mathrm{T}^{-k(N+N')}(y')$ and varying j we obtain the set

$$S'_N(k+1) = \{x_{i_1 i_2 \ldots i_k j} : j \in \{1, 2, ..., K\}\}.$$

Let us justify part (ii) first. Let us fix some $j = 1, 2, ..., K$. Recalling that $x_{i_1 i_2 \ldots i_k j} = \mathrm{T}^{-k(N+N')}(y')$ we obtain from (5.4) with $l = N$ that

$$d(\mathrm{T}^{(k+1)N+kN'}(x_{i_1 i_2 \ldots i_k j}), \mathrm{T}^N(x_j)) < \frac{\delta}{c_0 3^6}.$$

Moreover, from Proposition 4.2 we have $\mathrm{T}^N(x_j) \in X_{<16M}$ so that

$$\mathrm{ht}(\mathrm{T}^{(k+1)N+kN'}(x_{i_1 i_2 \dots i_k j})) \le \frac{\mathrm{ht}(\mathrm{T}^N(x_j))}{1-\frac{\delta}{3^6}} < 17M.$$

To prove (iii) let us consider any distinct pairs $x_{i_1 i_2 \dots i_k i_{k+1}}$ and $x_{j_1 j_2 \dots j_k j_{k+1}}$ in $S'_N(k+1)$. First, assume that $i_{k+1} \neq j_{k+1}$ and let $g, h \in G$ be such that

$$\mathrm{T}^{k(N+N')}(x_{i_1 i_2 \dots i_k i_{k+1}}) = \Gamma g, \ \ \mathrm{T}^{k(N+N')}(x_{j_1 j_2 \dots j_k j_{k+1}}) = \Gamma h$$

with

$$\begin{aligned}(5.6)\quad & d(\mathrm{T}^{k(N+N')+N}(x_{i_1 i_2 \dots i_k i_{k+1}} c_{i_{k+1}}), \mathrm{T}^N(x_{i_{k+1}})) \\ &\qquad = d(\mathrm{T}^N(g c_{i_{k+1}}), \mathrm{T}^N(g_{i_{k+1}})) < \frac{\delta}{3^5} e^{-N(d+1)/d} \text{ and}\end{aligned}$$

$$\begin{aligned}(5.7)\quad & d(\mathrm{T}^{k(N+N')+N}(x_{j_1 j_2 \dots j_k j_{k+1}} c_{j_{k+1}}), \mathrm{T}^N(x_{j_{k+1}})) \\ &\qquad = d(\mathrm{T}^N(h c_{j_{k+1}}), \mathrm{T}^N(g_{j_{k+1}})) < \frac{\delta}{3^5} e^{-N(d+1)/d}\end{aligned}$$

for some $c_{i_{k+1}}, c_{j_{k+1}} \in C$ with $d(c_{i_{k+1}}, 1) < \frac{\delta}{c_0 3^6}$ and $d(c_{j_{k+1}}, 1) < \frac{\delta}{c_0 3^6}$ as in (5.5). Thus, we have

$$d(g_{i_{k+1}}, g c_{i_{k+1}}) < \frac{\delta}{3^5} \text{ and } d(g_{j_{k+1}}, h c_{j_{k+1}}) < \frac{\delta}{3^5}.$$

We also note from Proposition 4.2 that $d(g_{i_{k+1}}, g_{j_{k+1}}) < \frac{30}{16} e^{-N/d}$. Thus, for N large enough we get

$$\begin{aligned} & d(g,h) \\ & < d(g, g c_{i_{k+1}}) + d(g c_{i_{k+1}}, g_{i_{k+1}}) + d(g_{i_{k+1}}, g_{j_{k+1}}) + d(g_{j_{k+1}}, h c_{j_{k+1}}) + d(h c_{j_{k+1}}, h) \\ & < \frac{\delta}{3^6} + \frac{\delta}{3^5} + \frac{30}{16} e^{-N/d} + \frac{\delta}{3^5} + \frac{\delta}{3^6} \\ & < \frac{\delta}{3^4}. \end{aligned}$$

In particular, $d(\Gamma g, \Gamma h) = d(g,h)$ since δ is an injectivity radius for $X_{<17M}$. On the other hand, from Proposition 4.2 we know that

$$d(\mathrm{T}^N(g_{i_{k+1}}), \mathrm{T}^N(g_{j_{k+1}})) > \frac{1}{8M} > \delta.$$

So, together with (5.6) and (5.7) we conclude that

$$\begin{aligned} & d(\mathrm{T}^N(g), \mathrm{T}^N(h)) \\ & > d(\mathrm{T}^N(g_{i_{k+1}}), \mathrm{T}^N(g_{j_{k+1}})) - d(\mathrm{T}^N(g_{i_{k+1}}), \mathrm{T}^N(g)) - d(\mathrm{T}^N(g_{j_{k+1}}), \mathrm{T}^N(h)) \\ & > \delta - \frac{\delta}{3^5} e^{-N(d+1)/d} - \frac{\delta}{c_0 3^6} - \frac{\delta}{3^5} e^{-N(d+1)/d} - \frac{\delta}{c_0 3^6} \\ & > \delta - \frac{\delta}{3^4}. \end{aligned}$$

Now, assume that $i_n \neq j_n$ for some $n \leq k$. By replacing l in (5.1) by $l-(k-n)(N+N')$ we obtain

$$(5.8) \quad d(\mathrm{T}^{l-(k-n)(N+N')}(y), \mathrm{T}^{l+n(N+N')-N'}(x_{i_1 i_2 \ldots i_k})) < \frac{\delta}{c_0^2 3^8} e^{(l-(k-n)(N+N'))(d+1)/d} \text{ for } l \leq 0.$$

On the other hand, if we replace l in (5.3) by $l-(k-n)(N+N')-N'$ we get

$$(5.9) \quad d(\mathrm{T}^{l-(k-n)(N+N')-N'}(y'), \mathrm{T}^{l-(k-n)(N+N')}(y)) < \frac{\delta}{c_0 3^6} e^{(l-(k-n)(N+N')-N')(d+1)/d} \text{ for } l \leq 0.$$

Thus, (5.8) and (5.9) together with the triangular inequality give

$$d(\mathrm{T}^{l-(k-n)(N+N')-N'}(y'), \mathrm{T}^{l+n(N+N')-N'}(x_{i_1 i_2 \ldots i_k})) < \frac{\delta}{c_0 3^5} e^{(l-(k-n)(N+N')-N')(d+1)/d}$$

for $l \leq 0$ where $y' = \mathrm{T}^{-k(N+N')}(x_{i_1 i_2 \ldots i_k j})$ for $j = 1, 2, \ldots, K$. Thus, we have

$$(5.10) \quad d(\mathrm{T}^{n(N+N')-N'+l}(x_{i_1 i_2 \ldots i_k}), \mathrm{T}^{n(N+N')-N'+l}(x_{i_1 i_2 \ldots i_{k+1}})) < \frac{\delta}{c_0 3^5} e^{(l-(k-n)(N+N'))(d+1)/d}$$

and

$$(5.11) \quad d(\mathrm{T}^{n(N+N')-N'+l}(x_{j_1 j_2 \ldots j_k}), \mathrm{T}^{n(N+N')-N'+l}(x_{j_1 i_2 \ldots j_{k+1}})) < \frac{\delta}{c_0 3^5} e^{(l-(k-n)(N+N'))(d+1)/d}.$$

Now, from the induction hypothesis we have that there are g', h' with

$$\mathrm{T}^{n(N+N')}(x_{i_1 i_2 \ldots i_k}) = \Gamma g', \quad \mathrm{T}^{n(N+N')}(x_{j_1 j_2 \ldots j_k}) = \Gamma h'$$

such that $d(\Gamma g', \Gamma h') = d(g', h')$ and that

$$d(\mathrm{T}^N(g'), \mathrm{T}^N(h')) > \delta - \frac{\delta}{3^4} \text{ if } n = k \text{ and}$$

$$d(\mathrm{T}^N(g'), \mathrm{T}^N(h')) > \delta - \delta \sum_{l=3}^{k-n+2} 3^{-l} \text{ if } n \in [1, k).$$

Let $g, h \in G$ be such that

$$\mathrm{T}^{(n-1)(N+N')}(x_{i_1 i_2 \ldots i_{k+1}}) = \Gamma g \text{ and } \mathrm{T}^{(n-1)(N+N')}(x_{j_1 j_2 \ldots j_{k+1}}) = \Gamma h$$

with

$$d(g, g') < \frac{\delta}{c_0 3^5} e^{[-(k-n)(N+N')-N](d+1)/d},$$

$$d(h, h') < \frac{\delta}{c_0 3^5} e^{[-(k-n)(N+N')-N](d+1)/d}.$$

This can be done using (5.10) and (5.11) with $l = -N$. In particular,

$$d(\mathrm{T}^N(g), \mathrm{T}^N(g')) < \frac{\delta}{c_0 3^5} e^{-(k-n)(N+N')(d+1)/d},$$
$$d(\mathrm{T}^N(h), \mathrm{T}^N(h')) < \frac{\delta}{c_0 3^5} e^{-(k-n)(N+N')(d+1)/d}.$$

Also, since by construction

$$\mathrm{T}^{(n-1)(N+N')}(x_{i_1 i_2 \dots i_{k+1}}), \mathrm{T}^{(n-1)(N+N')}(x_{j_1 j_2 \dots j_{k+1}}) \in X_{<17M}$$

and since $\frac{\delta}{3^5} e^{[-(k-n)(N+N')-N](d+1)/d}$ is less than the injectivitiy radius δ for $X_{<17M}$ we have

$$d\left(\mathrm{T}^{(n-1)(N+N')}(x_{i_1 i_2 \dots i_{k+1}}), \mathrm{T}^{(n-1)(N+N')}(x_{i_1 i_2 \dots i_k})\right) = d(g, g') \text{ and}$$
$$d\left(\mathrm{T}^{(n-1)(N+N')}(x_{j_1 j_2 \dots j_{k+1}}), \mathrm{T}^{(n-1)(N+N')}(x_{j_1 j_2 \dots j_k})\right) = d(h, h').$$

Now, if $n = k$ then

$$\begin{aligned} d(\mathrm{T}^N(g), \mathrm{T}^N(h)) &\geq d(\mathrm{T}^N(g'), \mathrm{T}^N(h')) - d(\mathrm{T}^N(g'), \mathrm{T}^N(g)) - d(\mathrm{T}^N(h'), \mathrm{T}^N(h)) \\ &> \delta - \frac{\delta}{3^4} - \frac{\delta}{c_0 3^5} - \frac{\delta}{c_0 3^5} \\ &> \delta - \frac{\delta}{3^3} \\ &= \delta - \delta \sum_{l=3}^{k+1-n+2} 3^{-l}. \end{aligned}$$

Otherwise, if $n < k$ then

$$\begin{aligned} d(\mathrm{T}^N(g), \mathrm{T}^N(h)) &\geq d(\mathrm{T}^N(g'), \mathrm{T}^N(h')) - d(\mathrm{T}^N(g'), \mathrm{T}^N(g)) - d(\mathrm{T}^N(h'), \mathrm{T}^N(h)) \\ &> \delta - \delta \sum_{l=3}^{k-n+2} 3^{-l} - 2\frac{\delta}{c_0 3^5} e^{-(k-n)(N+N')(d+1)/d} \\ &> \delta - \delta \sum_{l=3}^{k-n+2} 3^{-l} - \delta \cdot 3^{-(k-n+3)} \\ &= \delta - \delta \sum_{l=3}^{k+1-n+2} 3^{-l}. \end{aligned}$$

This concludes the proof of (iii) for $n = k + 1$ and the inductive argument.

Now, we will briefly point out why (i) holds. Clearly it is true for the elements of $S'_N(1)$ as suggested by Proposition 4.2. In the inductive step, to estimate the distance between the elements of $S'_N(m)$ and $S'_N(m+1)$ under the action of T we made use of (5.6), (5.7) , (5.10), and (5.11) and obtained part (iii). Arguing in the same way, we can inductively prove for any $m \geq 1$ and for any $x \in S'_N(m)$ that

$$d(\mathrm{T}^{l+n(N+N')}(x), \mathrm{T}^l(x_j)) < \delta \sum_{k=3}^{m-n+3} 3^{-k}$$

for some $x_j \in S'_N(1)$ and for $l \in [n(N+N'), (n+1)N + nN')]$ with $n \le m$. In particular,

$$d(\mathrm{T}^{l+n(N+N')}(x), \mathrm{T}^l(x_j)) < \delta \sum_{k=3}^{\infty} 3^{-k} = \frac{\delta}{18}$$

for some $x_j \in S'_N(1)$ and for $l \in [n(N+N'), (n+1)N + nN')]$. Together with (i) of Proposition 4.2 we obtain

$$\mathrm{ht}(\mathrm{T}^{l+n(N+N')}(x)) \ge \frac{\mathrm{ht}(\mathrm{T}^l(x_j))}{\frac{c_0\delta}{18}+1} > \frac{M}{c_0+1}$$

for $l \in [n(N+N'), (n+1)N + nN')]$. This justifies (i).

Finally, from (5.10) with $n = 1$ and $l = -N$ we have

$$d(x_{i_1 i_2 \dots i_k}, x_{i_1 i_2 \dots i_{k+1}}) < \frac{\delta}{c_0 3^5} e^{(-N-(k-1)(N+N'))(d+1)/d} < \delta e^{-k}$$

which concludes the proof. □

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