Scalar field collapse and cosmic censorship

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We analyze here the final fate of complete gravitational collapse of a massless scalar field within general relativity. A class of dynamical solutions with initial data close to the Friedmann-Lemaître-Robertson-Walker (FLRW) collapse model is explicitly given and the Einstein equations are integrated numerically in a neighborhood of the center. We show that the initial data space is evenly divided between the dynamical evolutions that terminate in a black hole final state and those that produce a locally naked singularity. We comment on the genericity aspects of the collapse end-states and the connection to cosmic censorship conjecture is pointed out.

PACS numbers: 04.20.Dw,04.20.Jb,04.70 Bw
Keywords: Gravitational collapse, black holes, naked singularities, scalar fields

The Cosmic Censorship Conjecture (CCC) [1] has been one of the most prominent open problems of modern general relativity. The original idea of Penrose was that the evolution of a generic and physically realistic matter cloud, subject to Einstein equations, would never lead to spacetime singularities that are visible to faraway observers. Over the last few decades much work has been done on CCC, with multiple formulations being proposed towards a possible proof, and on what is meant by genericity of a certain collapse outcome, a concept for which there is no precise definition in gravitation theory today.

While a proof of the CCC has not been found despite many efforts, and its very formulation has been questioned, many models of physically realistic gravitational collapse leading to the formation of naked singularities have been obtained and studied (see for example [2] and references therein). These models mainly deal with fluids such as dust, perfect fluids, and other more general matter fields, which have been used widely in astrophysical considerations. These are typically described by averaged properties where the microscopic structure of the fluid itself may have been neglected in the very final stages of collapse (see, however, [3]). In any case, it is not clear whether the fluid approximation will continue to hold till the ultra-high densities that characterize the singularity are reached, or what is the final form of matter and equation of state valid at such ultra-high densities.

On the other hand, much work has also been done on scalar field collapse ([4]-[9]). Scalar fields are fundamental fields satisfying the Klein-Gordon equation and therefore, despite the fact that they have never been observed in nature, constitute a perfect candidate for a fundamental field whose validity will hold at any scale. Also, as shown by Choptuik [6], the black hole threshold in the space of initial data for classes of massless scalar fields show universality and power-law scaling of the black hole mass, which correspond to critical phenomena. These are explained by the existence of exact solutions which are attractors within the black hole threshold and are typically self-similar. The critical phenomena evolve a smooth initial data to arbitrarily large curvatures visible from infinity, via the formation of infinite time naked singularities that form at the boundary between the continual collapse to a black hole and the region where collapse reverses, turning to a dispersal (see [9] for details). The genericity and stability aspects of naked singularities arising in certain classes of collapsing massless scalar fields were investigated by Christodoulou [5], showing these to be non-generic within the context of a certain parent space.

We investigate here complete collapse of a class of spherically symmetric massless scalar fields within general relativity. We show explicitly that there exist sets of non-zero measure of initial data, which are arbitrarily close to the FLRW homogeneous collapse scenario when the collapse begins, and which terminate in a naked singularity. We note that the naked singularities considered here are of a different nature compared to the infinite time singularity in the Choptuik like analysis in the fact that in our models the spacetime is no longer assumed to be self-similar, and these are much similar to the central singularities of fluid models that may be locally visible.

The energy-momentum tensor for a massless scalar field $\phi(r,t)$ is given by,

$$T_{ab} = \phi_{;a}\phi_{;b} - \frac{1}{2}g_{ab}(\phi_{;c}\phi_{;d}g^{cd}) \ . \tag{1}$$

We consider here the class of scalar fields which admit one timelike and three spacelike eigenvectors (see e.g. [10]), and consider the continual gravitational collapse of such a field. The FLRW scalar field collapse is an example of such a scenario. Then T_{ab} can be written in comoving coordinates and becomes diagonal. The spherically symmetric line element then depends on three functions of t and r (which are the comoving time and radius respectively), which are $g_{00} = -e^{2\nu(r,t)}$, $g_{11} = e^{2\psi(r,t)}$ and $g_{22} = g_{33}\sin^2\theta = R(r,t)^2$. As we can always choose here a frame along the eigenvectors of the energy-momentum tensor of the scalar field, we have $T^1_0 = 0$. We then get $\dot{\phi}\phi' = 0$. We shall consider here the class of scalar fields for which $\phi(t) \neq 0$ [4]. Then the components of the energy momentum tensor become $T^a_b = \frac{1}{2}e^{-2\nu}\dot{\phi}^2[-1,1,1,1]$, from which we see that the matter behaves like a stiff perfect fluid with an equation of state

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$p = \rho$, with ρ given by the equation above.

The system of Einstein equations in units where $8\pi G = c = 1$, can be written as

$$F' = \rho R^2 R' \ , \tag{2}$$
$$\dot{F} = -p R^2 \dot{R} \ , \tag{3}$$
$$0 = \partial_t (R^2 e^{\psi-\nu} \dot{\phi}) \ , \tag{4}$$
$$\dot{G} R' = 2\nu' \dot{R} G \ . \tag{5}$$

where primed and dotted quantities denote partial derivatives with respect to r and t respectively, and we have defined $G = e^{-2\psi} R'^2$. Here $F(r,t)$ is the Misner-Sharp mass of the system describing the amount of matter enclosed within the shell labeled by r at the time t.

$$F = R(1 - G + e^{-2\nu} \dot{R}^2) \ . \tag{6}$$

Note that equation (4) together with the definition of the scalar field implies that ϕ must obey the Klein-Gordon equation. The system of Einstein equations together with the equation of state and the Misner-Sharp mass is a set of six equations in the six unknowns ρ, p, ψ, ν, R and F, and is thus a fully closed system for this case.

There is a scaling degree of freedom that we can use to set $R(r, t_i) = r$. We then write $R(r,t) = r v(r,t)$ and rewrite all the equations in terms of the variable v. Thus $v(r,t)$ is such that $v = 1$ corresponds to the initial time and $v = 0$ to the time of occurrence of the singularity, that is, the shell labeled by r becomes singular at the time $t_s(r)$ given by $v(r, t_s(r)) = 0$ ([11], [12]). The condition for continual collapse is then given by $\dot{v} < 0$, while the situation where $\dot{v}$ changes sign at a finite time corresponds to a halting and dispersal of the scalar field. We can rewrite all the relevant functions in terms of the new coordinates $\{r, v\}$, therefore for any function $X(r,t)$ we have $X' = X_{,r} + X_{,v} v'$ where $v'(r,t)$ itself is treated as a function of r and v.

Avoidance of shell-crossing singularities, which are generally believed to be a breakdown of the coordinate system rather than true singularities, is obtained if we impose $R' > 0$, which corresponds to $v + r v' > 0$. Since v is always positive, there is always a small neighborhood of the center $r = 0$ for which no shell-crossing occurs, and our analysis holds good for such a collapsing cloud.

In order to describe physically valid scenarios the energy conditions must be fulfilled throughout collapse. In this case these are always satisfied once ρ is taken to be positive, which, provided $R' > 0$, corresponds to $F' > 0$. Furthermore regularity must be imposed on the initial data in order for ρ and the metric functions to be well behaved. We shall require that the Misner-Sharp mass behaves like $F = r^3 M(r,v)$ with $M'(0,v) = 0$ in order for the matter density to be regular and without cusps at the center at all epochs of collapse earlier than the singularity.

An important point here is, to solve the system of Einstein equations we write all equations in terms of the mass function $M(r,v)$ and its derivatives. From equation (4) we get $R^2 e^{\psi-\nu}\dot{\phi} = f(r)$ where $f(r)$ is an arbitrary function related to the kinetic energy of the collapsing cloud. From the definition of ρ given by the energy-momentum tensor we can write $e^{2\nu} = -v^2 \dot{\phi}^2/(2M_{,v})$, while using the equation of state and the above definition for f allows us to write G as $G(r,v) = -v^2(vM_{,v} - 3M - rM_{,r})^2/2f^2 M_{,v}$. Now using equation (2) and (3) we can write the differential equation for v' as

$$v' = W(r,v) = -\frac{vM_{,v} + 3M + rM_{,r}}{2M_{,v}} \ , \tag{7}$$

while from the Misner equation (6) we obtain

$$\dot{v} = V(r,v) = -e^{\nu} \sqrt{\frac{M}{v} + \frac{G-1}{r^2}} \ , \tag{8}$$

with negative sign taken in order to describe collapse. From the above we see that for collapse to occur we must require a 'reality condition', namely $\frac{M}{v} + \frac{G-1}{r^2} > 0$. If this condition is not satisfied throughout the evolution then the system will reach $\dot{v} = 0$ in a finite time and collapse will then reverse to dispersal. Finally the whole system reduces to a second order partial differential equation in M and its derivatives, which can be written as

$$V_{,v} W - V W_{,v} = V_{,r} \ . \tag{9}$$

This equation gives the integrability condition of (7) and (8) that must be satisfied by the mass function $M(r,v)$ in order for it to be a solution to the collapsing system. Once a function $M(r,v)$ solving equation (9) is found the system of Einstein equations is fully solved. It is clear that the above equation has a non-empty class of solutions to which the self-similar collapse models as well as the FLRW class belong. By integrating the equation (8) we obtain the function $t(r,v)$ that describes the time at which the shell labeled by r reaches the 'epoch' v. Then we can retrieve the singularity curve $t_s(r)$, namely the curve describing the time at which each shell becomes singular, by setting $t_s(r) = t(r, 0)$.

For all sufficiently regular solutions $M(r,v)$ the singularity curve is also regular and can be written near the center as,

$$t_s(r) = t_0 + \chi_1 r + \chi_2 r^2 + o(r^3) \ , \tag{10}$$

where $t_0 = t(0,0)$ is the time at which the central shell becomes singular and $\chi_i = \frac{1}{i!}\frac{d^i t}{dr^i}|_{r=0}$. Here χ_1 vanishes due to regularity requirements. The tangent to the singularity curve close to the center is then determined by the coefficient χ_2 and it is then possible to show that it is this coefficient that determines the local visibility of the central singularity or otherwise ([11], [13]). Specifically, if the singularity curve is an increasing function of the coordinate r at the center, that is $\chi_2 > 0$, we then have an outgoing null geodesic family coming out from the central singularity. In that case the singularity is locally naked. On the other hand, the singularity is covered if $\chi_2 \leq 0$.

The FLRW collapse is obtained when we take $M = m_0/v^3$, for which $v = v(t)$ and G becomes $\frac{6m_0}{f^2}$ (from which we can see that imposing the regularity condition $f^2(0) = 6m_0$ ensures $G = 1$ at the center). The singularity curve is constant in this case, all shells fall into the singularity at the same time and the event horizon forms before the time of formation of the singularity, thus giving rise to a black hole final state for collapse. Going to more general case, solving the complete second order PDE (9) is in general unattainable. Nevertheless our purpose here is to extract the crucial information regarding the local visibility or otherwise of the central singularity, and this can be achieved by considering a close neighborhood of the center. In order to do so we expand Einstein equations and all relevant functions in powers of r close to the origin.

We therefore studied here a class of spacetimes obtained by this procedure and the two-dimensional metric can be given as,

$$ds^2 = -\left(1+\frac{m_{,v}}{v^2}r^2\right)dt^2+v^2\left(1+\frac{m_{,v}v^4-5mv^3}{3m_0}r^2\right)dr^2. \tag{11}$$

This line element is a solution of the Einstein equations to second order in r which is valid in a small neighborhood of $r=0$. The metric corresponds to a mass function $M(r,v) = \frac{m_0}{v^3}+m(v)r^2$, a velocity profile $f^2=6m_0$, and $\dot{\phi}(t(v,r))^2 = \frac{6m_0}{v^6}$. Here $m(v)$ is a solution to the corresponding ordinary differential equation coming from the expansion of equation (9) to second order in r, which now becomes

$$0 = \left(\frac{5}{3}mv^9+m_0^2v^2\right)m_{,vv}+5(mv^8+2m_0^2v)m_{,v}+ \\ +\frac{50}{3}m^2v^7+24m_0^2m-m_{,v}^2v^9\ , \tag{12}$$

note that the zeroth order gives the FLRW differential equation.

Once (12) is solved we obtain $m(v)$ from which we can evaluate the singularity curve as in equation (10) to determine the coefficient χ_2, that eventually decides the local visibility or otherwise of the singularity. For the above class of solutions χ_2 is given by

$$\chi_2 = -\frac{1}{6}\int_0^1 \frac{\frac{3m}{v}+m_{,v}+\frac{5mm_{,v}v^7}{3m_0^2}+\frac{(5mv^3+m_{,v}v^4)^2}{12m_0^2}}{\left(\frac{m_0}{v^4}+\frac{5mv^3}{3m_0}\right)^{\frac{3}{2}}}dv\ . \tag{13}$$

We note that the 'reality condition' in this case takes the form $\frac{m_0}{v^4}+\frac{5mv^3}{3m_0}>0$ and appears at the denominator in χ_2. Therefore those mass profiles m for which it is violated will have complex values for χ_2. Hence we can easily see that all possible local behaviours of the central singularity are entirely determined by the value of χ_2. The central singularity is locally naked if χ_2 is positive, is covered if it is negative, and there is a dispersal of the collapsing central shell if χ_2 is complex.

Given a certain initial data in the form $\{m(1), m_{,v}(1)\}$, a solution of equation (12) exists and it is possible to solve it numerically and obtain the value for χ_2. If we choose initial data sufficiently close to FLRW, meaning with $|m(1)|<m_0$, and require $m_{,v}(1)$ to be small to satisfy the 'reality condition', we can then evaluate χ_2 as above and its sign will then decide the visibility of the central singularity for the specific massless scalar field collapse given by that $m(v)$.

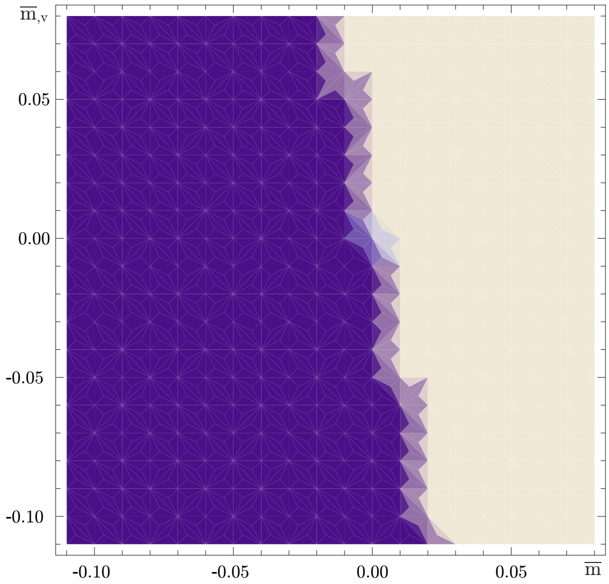

FIG. 1. The final outcome of collapse of a massless scalar field depending on the initial data $\{\bar{m}=m(1), \bar{m}_{,v}=m_{,v}(1)\}$ in a neighborhood of the initial data for FLRW collapse. For every point in the phase space of initial data, the sign of χ_2 was evaluated numerically. The thick curve represents the boundary at which $\chi_2=0$ that separates the initial data leading to a black hole outcome (on the left) from the initial data leading to a naked singularity (on the right). Note that $m(1)=m_{,v}(1)=0$ corresponds to initial data for the FLRW solution.

What we find then is, for initial data close to FLRW, both the collapse outcomes namely the black hole and naked singularity are equally possible and the space of initial data leading to either of them within this specific class is evenly divided (see Fig. 1). The general behaviour found here for the initial data sets close to FLRW resembles the results that were found in similar scenarios for collapse of perfect fluids and fluids sustained only by tangential pressure [14]. The main virtue of the study of scalar field collapse comes from the fact that these are fundamental matter fields obeying the Klein-Gordon equation and therefore are possibly valid at all late stages of collapse, if quantum gravity effects are ignored. Also, such a 'stiff fluid' is a fully closed system where there are no global free functions, as in the case of perfect fluids or tangential pressures when no equations of state are specified. So the physical meaning of the matter model comes out in a clear and straightforward manner.

The above result provides intriguing perspective on the genericity aspect of occurrence of black holes and naked singularity final states in gravitational collapse. The definition of genericity in general relativity is a delicate matter that requires a proper and deeper understanding of both the measure and topology of the parent space than is available presently. No precisely well-defined criteria are available for the same in the gravitation theory today. What we have showed here is that there exist classes of solutions for which initial data arbitrarily close to that of the FLRW models lead the collapse to a naked singularity. Furthermore, the set of initial data leading to a naked singularity within this particular space of solutions does not reduce to a set of zero measure, and the same is true for the black hole outcomes. In this sense, within this class, the naked singularity formation can be considered somehow generic.

Finally we would like to emphasize that what we deduced here is the occurrence of a locally naked singularity that disproves the stronger version of the censorship conjecture. It is possible in principle that the singularity is only locally naked and trajectories do come out, but then they eventually all fall back into the horizon, without being globally visible, and the weaker version of the censorship conjecture may still be obeyed. This needs to be examined separately.

Acknowledgement: The authors would like to thank Mandar Patil for the help with Fig 1. RG would like to thank Claude Leon Foundation Postdoctoral Fellowship, South Africa.

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