

# THE GENERAL SOLUTION FOR RELATIVISTIC SPHERICAL SHELLS

J. KIJOWSKI, G. MAGLI, AND D. MALAFARINA

ABSTRACT. The general exact solution of the Einstein-matter field equations describing spherically symmetric shells satisfying an equation of state in closed form is discussed under general assumptions of physical reasonableness. The solutions split into two classes: a class of “astro-physically interesting” solutions describing “ordinary” matter with positive density and pressure, and a class of “phantom-like” solutions with positive density but negative active gravitational mass, which can also be of interest in several “very strong fields” regimes. Known results on linear-barotropic equations of state are recovered as particular cases.

## 1. INTRODUCTION

Relativistic shells have found applications in an impressive list of different scenarios, such as sources of vacuum gravitational fields [1]-[4], cosmological models [5] and [6], gravitational collapse [7], canonical quantization of gravity [8] and [9], higher dimensional models in brane world, quantum gravitational collapse [10], entropy of black holes, motion of thin clouds surrounding an exploding star [11], sources of “phantom” fields and dark energy [12]. In spite of the wide range of their applications, very few analytical solutions in closed form are known even in spherical symmetry. Exceptions are the dust (i.e. zero pressure) shells, which were completely analyzed already in the pioneering work by Israel [13], [14], and solutions with linear-barotropic equations of state of the form  $P = \Gamma_0 \sigma$  with a constant  $\Gamma_0$ . In the present paper we discuss the general exact solution for shells satisfying an equation of state in closed form  $P = \Gamma(\sigma)$  with some general assumption of physical reasonableness on the behavior of the function  $\Gamma$ . Our solutions include as particular cases the linear e.o.s. cited above but also, for instance, non-linear polytropic equations of state of the form  $\Gamma(\sigma) = \Gamma_0 \sigma^\nu$  [15]. It turns out that the qualitative behavior of spherical shells always presents two “branches”, an “astro-physically relevant” branch in which the weak energy condition is strictly satisfied, and a “phantom” branch in which the density remains positive but the active gravitational mass  $\sigma + P$  is strictly negative.

## 2. THE FIELD EQUATIONS

We consider a spherical shell in vacuum. The shell “separates” a portion of Minkowski space (“interior”) from Schwarzschild space of arbitrarily

fixed mass parameter  $M$  (“exterior”). The field equations relate the jump of the exterior curvature with the matter content of the shell, i.e. a surface distribution of stress-energy. It can be shown [16] that such equations can be written as follows:

$$(2.1) \quad \dot{\sigma} = -2\frac{\dot{R}}{R}(\sigma + P)$$

$$(2.2) \quad \dot{R}^2 = V(R, \sigma)$$

where  $R(t)$  is the “radius” of the shell as a function of the proper time on the shell,  $\sigma$  and  $P$  are the surface (rest-frame) energy-density and the surface pressure (due to spherical symmetry, the two eigenvalues of the stress are equal, i.e. the material is necessarily a “two-dimensional perfect fluid”) and the “effective potential”  $V$  is given by

$$V(R, \sigma) := \frac{M^2}{16\pi^2 R^4 \sigma^2} + 4\pi^2 R^2 \sigma^2 - 1 + \frac{M}{R}.$$

The equations (2.1) and (2.2) contain three unknowns, and become a closed set once a further relation has been given. This has been achieved in various ways in the literature; for instance, one can assign the pressure profile as a “a priori” known function [11] and calculate the corresponding density profile. However, in this way, the resulting “equation of state”, i.e. the relationship between density and pressure which holds during the dynamics may turn out to be unphysical. Here, we are going to consider the case of a closed equation of state of the barotropic type, namely  $P = \Gamma(\sigma)$ . In this case, as we shall see, the equations above decouple and the dynamics can be qualitatively analyzed.

The function  $\Gamma$  has to satisfy a certain set of assumptions which assure the physical reasonability of the model. It seems that a reasonable compromise between generality and physical reasonableness can be obtained assuming that  $\Gamma(\sigma)$  is a continuous, monotonic function, differentiable except perhaps at  $\sigma = 0$ , where its value vanishes. These assumptions comprise a wide range of equations of state and, in particular, the linear-barotropic ( $P = \Gamma_0 \sigma$  with a constant  $\Gamma_0$ ) that has been widely studied in the literature [18], [17], and the polytropic case  $P = \Gamma_0 \sigma^\nu$  (both these e.o.s. reduce to the newtonian polytropes in the weak field regime [19], [20]).

If the weak energy condition is required, then solutions have to be selected in such a way that  $\sigma > 0$ , and on such solutions the effective gravitational mass

$$\mu(\sigma) := \sigma + \Gamma(\sigma)$$

must be non negative. Therefore the requirement  $\Gamma(\sigma) \geq -\sigma$  has to be added. However, shells with positive density but negative effective mass are of interest on their own, for instance as toy models for sources of dark energy (“phantom” fields, see e.g. [12], exotic matter and wormholes, see

[22]), thus we shall consider also this possibility in what follows.

### 3. THE STRUCTURE OF THE SOLUTIONS

We take initial data  $\sigma(R_0) = \sigma_0 > 0$  and consider the behavior of the solutions of the ordinary differential equation

$$(3.1) \quad \dot{\sigma} = -2 \frac{\dot{R}}{R} \mu(\sigma).$$

First of all, we notice that it can be reduced to a quadrature

$$(3.2) \quad \int_{\sigma_0}^{\sigma(R)} \frac{d\sigma}{\mu(\sigma)} = -2 \log(R/R_0).$$

Thus, if the function  $1/\mu(\sigma)$  is integrable in neighborhood of  $+\infty$ , the density diverges then this occurs at a finite non zero value  $R_s$  (with  $R_0 > R_s > 0$ ) (“Case A”). If instead the function  $1/\mu(\sigma)$  is not integrable in neighborhood of  $+\infty$ , the density can diverge only as  $R$  approaches zero (“Case B”). Further, a straightforward analysis shows that the following cases have to be distinguished:

Case I):  $\mu(\sigma) > 0$  for any  $\sigma$ :

In this case each solution is strictly decreasing from  $+\infty$  to zero (one among behaviors A or B occurs). Weak energy condition is always satisfied. Since  $\sigma = 0$  is a solution, if  $d\Gamma/d\sigma$  is continuous in  $\sigma = 0$  the standard uniqueness theorem for ODE gives  $\sigma \rightarrow 0$  as  $R \rightarrow +\infty$ . For completeness we notice, however, that one could also conceive models in which  $d\Gamma/d\sigma$  has a finite jump in  $\sigma = 0$ . In this case a loss of uniqueness occurs at a finite value  $R_l$  (at which the solution goes to zero and can be prolonged to be zero, thus giving a sort of “evaporation” of the shell). We thus have four qualitatively different cases reported in figures 1-4.

Case II): There exist a zero (at  $\sigma_p$ ) of the function  $\mu(\sigma)$ :

Clearly  $\sigma = \sigma_p$  is a regular solution at which no loss of uniqueness can occur. Thus the behavior depends on the initial datum for  $\sigma$ :

(IIa) If  $\sigma_0 > \sigma_p$  then all solutions are strictly decreasing and approach from above  $\sigma_p$  as  $R \rightarrow +\infty$ ; one among cases A and B occurs. Weak energy condition is satisfied everywhere.

(IIb) If  $\sigma_0 < \sigma_p$  then all solutions are strictly increasing and approach  $\sigma_p$  from below as  $R \rightarrow +\infty$ ; their behavior “in the past” intersects  $\sigma = 0$  where a loss of uniqueness occurs. Weak energy condition is violated everywhere because the density is positive but  $\mu$  is strictly negative. Two behaviors must still be distinguished: if the derivative  $\sigma'' = -\sigma'/R(3 + d\Gamma/d\sigma)$  has no zeroes, the solutions come out from zero with infinite tangent, if it has a zero the solutions have a flex point, and the matching at  $\sigma = 0$  is

smooth in this case. All in all we have the four qualitatively different cases reported in figures 5-8.

#### 4. DISCUSSION AND CONCLUSION

Once the behavior of the energy function  $\sigma$  is known the corresponding qualitative behavior of the shell radius  $R(t)$  can be read off from the “effective potential” equation (2.2).

First of all, we notice that, due to the presence of the  $M/R + 4\pi^2 R^2 \sigma^2$  term, the potential always diverges at the left end side of the interval of definition, in all the eight qualitatively different cases discussed above. The behavior at the other extreme is divergent as well in the cases I without loss of uniqueness and in all cases II (where  $V$  diverges as  $R$  goes to infinity). Thus in all such cases the potential is a continuous curve positively diverging at the extremes, and it follows that it has an absolute minimum. The dynamics is allowed in both the neighborhoods of the extremes, and it is allowed everywhere if  $V$  is positive at the minimum. Bouncing or also oscillating behaviors can appear as well, if the minimum is negative and therefore there is at least one region in which the dynamics is not allowed.

Shells with general equation of state can thus be viewed as very useful “toy” models for astrophysical objects, in order to investigate the effect of the equation of state on the dynamics in several physically interesting scenarios. Work in this direction is in progress.

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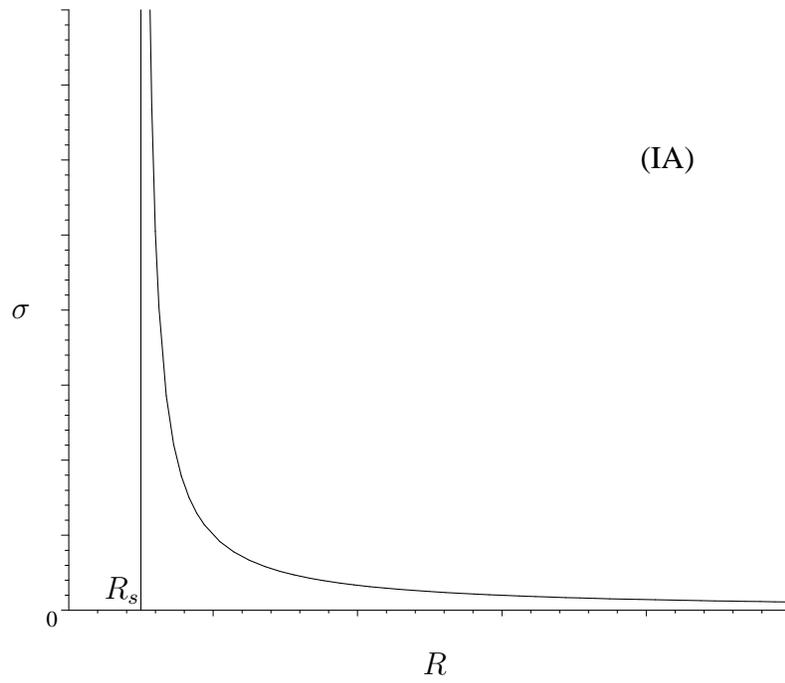


FIGURE 1.  $\sigma(R)$ : Case IA when  $\frac{d\Gamma}{d\sigma}$  is continuous in  $\sigma = 0$ .

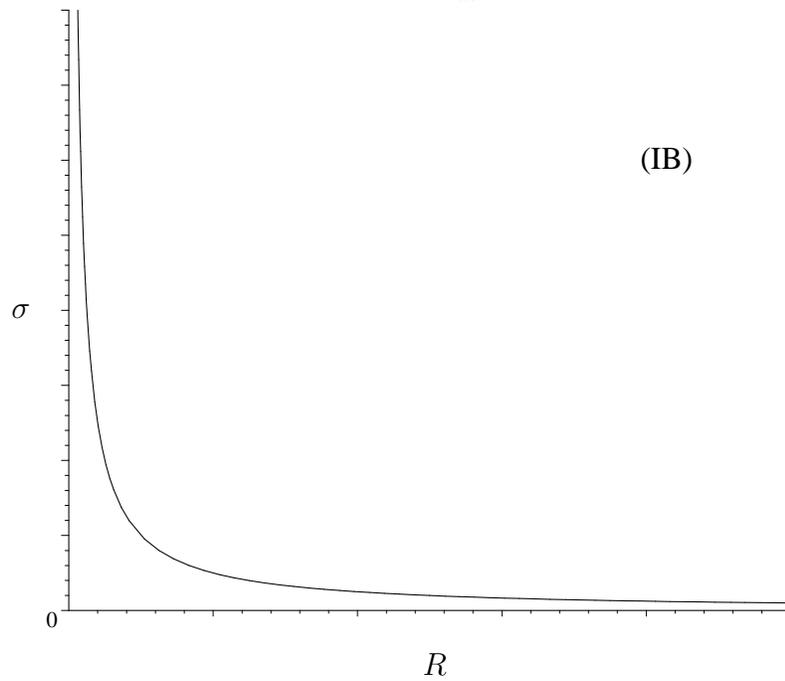


FIGURE 2.  $\sigma(R)$ : Case IB when  $\frac{d\Gamma}{d\sigma}$  is continuous in  $\sigma = 0$ .

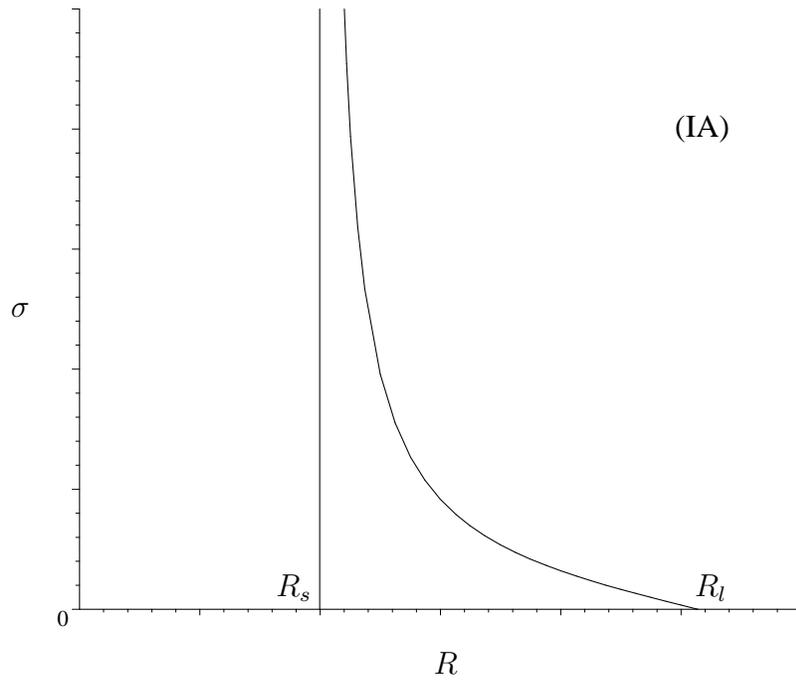


FIGURE 3.  $\sigma(R)$ : Case IA when  $\frac{d\Gamma}{d\sigma}$  is discontinuous in  $\sigma = 0$ .

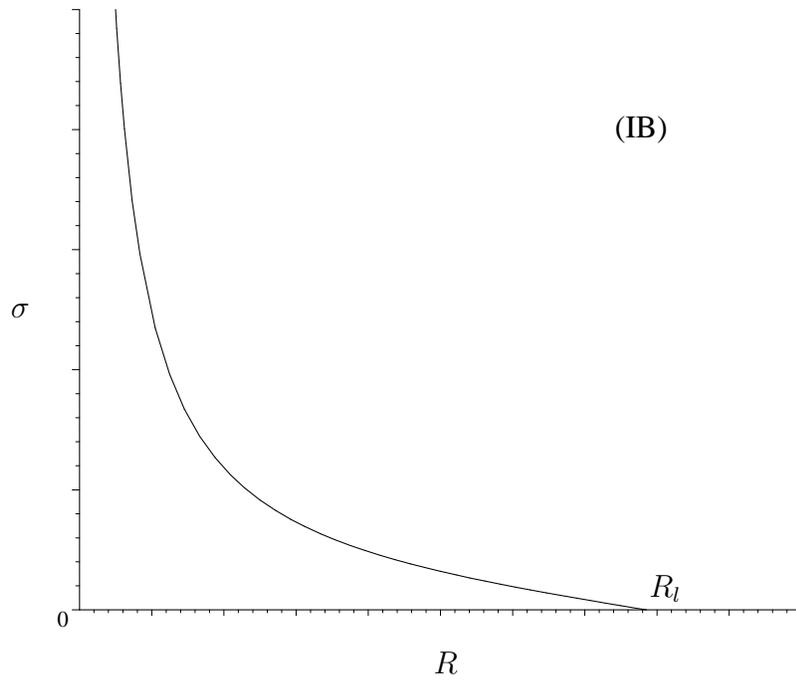
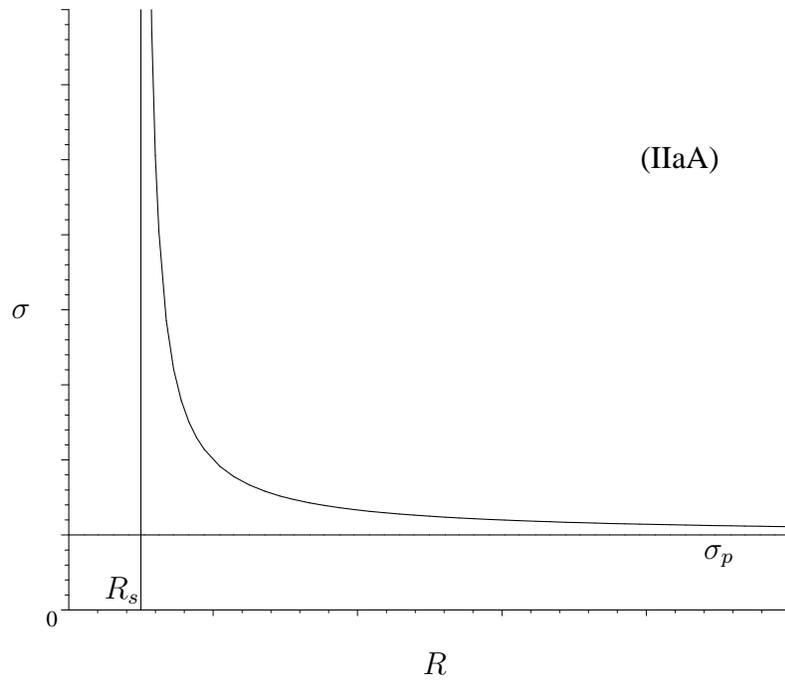
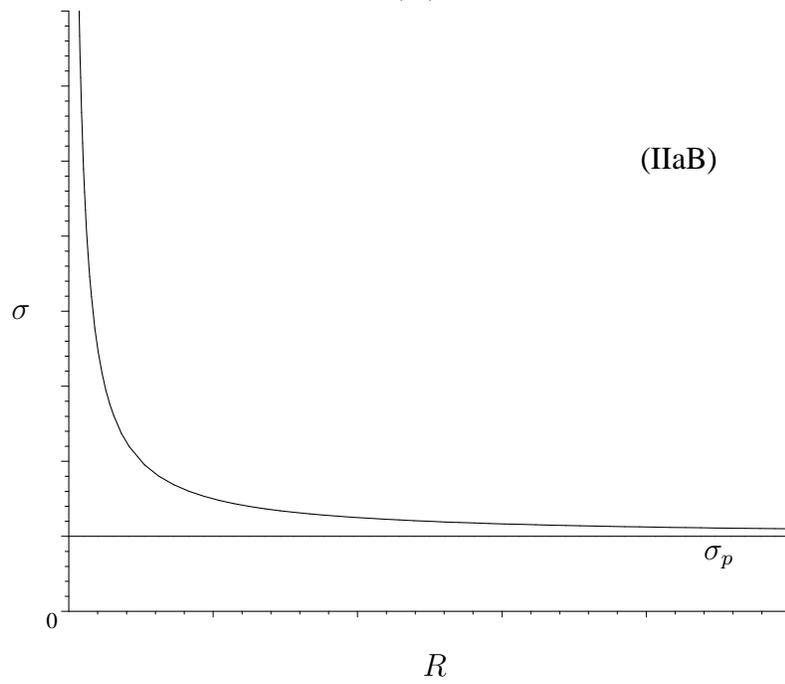
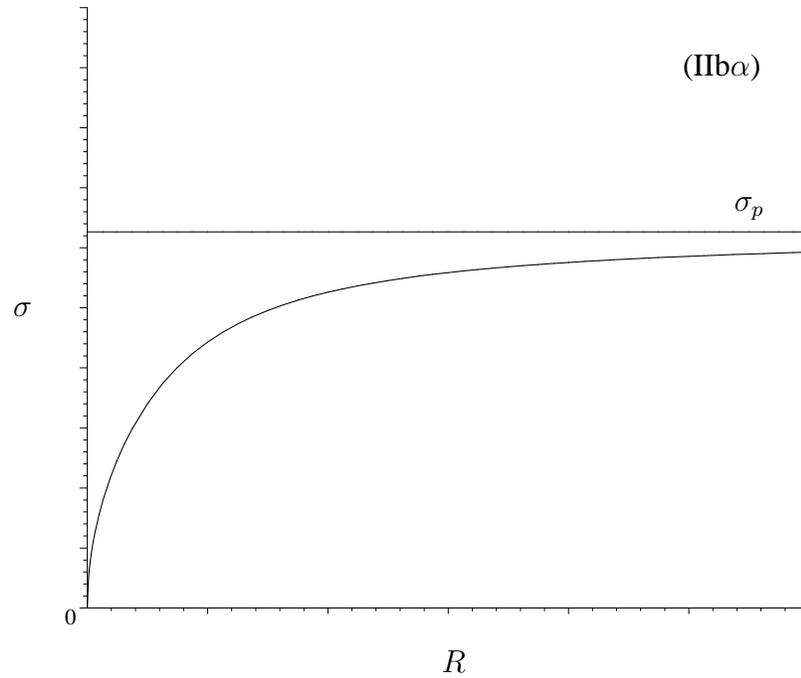
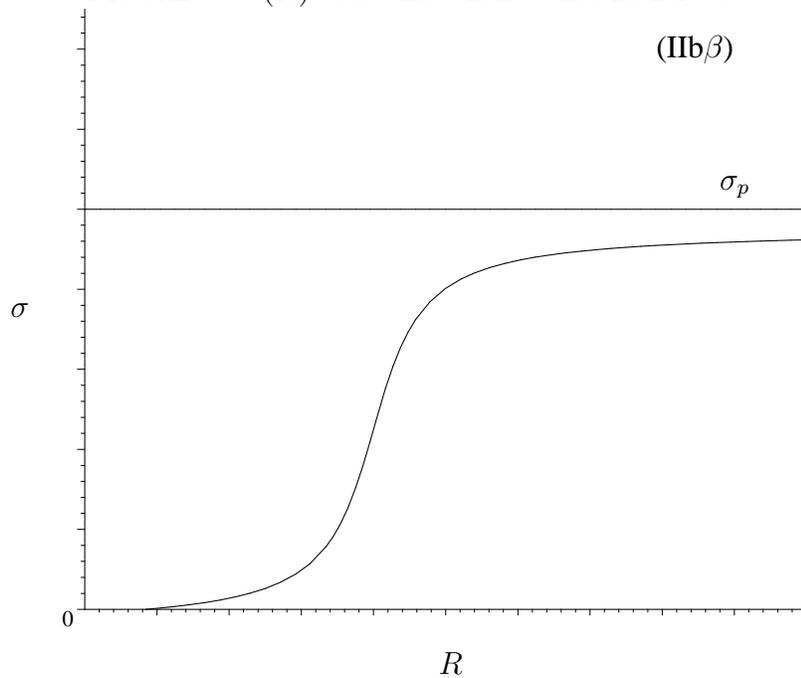


FIGURE 4.  $\sigma(R)$ : Case IB when  $\frac{d\Gamma}{d\sigma}$  is discontinuous in  $\sigma = 0$ .

FIGURE 5.  $\sigma(R)$ : Case IIaA.FIGURE 6.  $\sigma(R)$ : Case IIaB.

FIGURE 7.  $\sigma(R)$ : Case IIb when  $\sigma''$  has no zeroes.FIGURE 8.  $\sigma(R)$ : Case IIb when  $\sigma''$  has one zero.

INSTITUTE FOR THEORETICAL PHYSICS,  
 POLISH ACADEMY OF SCIENCES, WARSAW, POLAND AND  
 DEPARTMENT OF MATHEMATICAL METHODS IN PHYSICS,  
 UNIVERSITY OF WARSAW, POLAND  
*E-mail address:* kijowski@thetal.cft.edu.pl

DIPARTIMENTO DI MATEMATICA,  
 POLITECNICO DI MILANO, ITALY  
*E-mail address:* magli@mate.polimi.it

DIPARTIMENTO DI MATEMATICA,  
 POLITECNICO DI MILANO, ITALY AND  
 INSTITUTE FOR THEORETICAL PHYSICS,