

# Note on the effect of a massive accretion disk in the measurements of black hole spins

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The spin measurement of black holes has important implications in physics and astrophysics. Regardless of the specific technique to estimate the black hole spin, all the current approaches assume that the space-time geometry around the compact object is exactly described by the Kerr solution. This is clearly an approximation, because the Kerr metric is a stationary solution of the vacuum Einstein equations. In this paper, we estimate the effect of a massive accretion disk in the measurement of the black hole spin with a simple analytical model. For typical accretion disks, the mass of the disk is completely negligible, even for future more accurate measurements. However, for systems with very massive disks the effect may not be ignored.

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## I. INTRODUCTION

In 4-dimensional general relativity, an uncharged black hole (BH) is described by the Kerr solution and it is characterized by two parameters, associated respectively with the mass  $M$  and the spin angular momentum  $J$  of the object [1]. Astrophysical BHs are thought to be well described by the Kerr solution. Initial deviations from the Kerr background can be quickly radiated away through the emission of gravitational waves [2]. An initially non-vanishing electric charge can be soon neutralized in the highly ionized environment of the BH [3]. The presence of the accretion disk is typically completely negligible.

The BH mass can be measured with dynamical methods, by studying the orbital motion of individual stars. This kind of measurement is robust and one can use Newtonian mechanics, because the stars are far from the BH. The estimate of the spin is much more challenging. The spin has no gravitational effects in the Newtonian theory, and therefore it can be measured only probing the space-time geometry close to the BH. At present, there are two relatively reliable techniques to measure BH spins: the continuum-fitting method [4] and the analysis of the  $K\alpha$  iron line [5]. The continuum-fitting method is based on the study of the thermal spectrum of geometrically thin and optically thick accretion disks. It can be used only for stellar-mass BHs because the disk temperature is proportional to  $M^{-1/4}$  and the spectrum turns out to be in the X-ray range for stellar-mass BHs and in the UV/optical range for super-massive BHs. In the latter case, absorption from dust makes an accurate measurement impossible. The analysis of the  $K\alpha$  iron line can be applied to both stellar-mass and super-massive BHs and is based on the study of the profile of this line: the latter is intrinsically narrow in frequency, while the one in the spectrum of BHs is affected by special and general relativistic effects.

Both the continuum-fitting method and the analysis of the iron line strongly rely on the assumption that the inner edge

of the accretion disk is at the innermost stable circular orbit (ISCO) of the background metric. In an exact Kerr metric, the ISCO radius in Boyer-Lindquist coordinates  $\{t, r, \theta, \phi\}$  is given by [6]

$$r_{\text{ISCO}} = \left[ 3 + Z_2 \mp \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)} \right] M, \quad (1)$$

$$Z_1 = 1 + (1 - a_*^2)^{1/3} \left[ (1 + a_*)^{1/3} + (1 - a_*)^{1/3} \right],$$

$$Z_2 = \sqrt{3a_*^2 + Z_1^2},$$

where  $a_* = a/M = J/M^2$  is the spin parameter and the sign  $- (+)$  is for corotating (counterrotating) orbits. For a Schwarzschild BH ( $a_* = 0$ ), the ISCO radius is at  $r_{\text{ISCO}} = 6M$  and its value decreases (increases) for a rotating BH and a corotating (counterrotating) disk.

In general relativity, the radial coordinate has not really any physical meaning, being determined by the coordinate system, which is arbitrary. In the analysis of the thermal spectrum of thin disks, one can see that the key-quantity is the radiative efficiency in the Novikov-Thorne model [7]. In the case of the iron line profile, the picture is more complicated, but the value of the radiative efficiency in the Novikov-Thorne model can still be used as a crude estimate to compare different spacetimes with similar iron lines [8]. In the Kerr metric, the specific energy of a test-particle on the equatorial circular orbit of radius  $r$  is

$$E = \frac{r^{3/2} - 2Mr^{1/2} \pm aM^{1/2}}{r^{3/4} \sqrt{r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2}}}. \quad (2)$$

The Novikov-Thorne radiative efficiency is

$$\eta_{\text{NT}} = 1 - E_{\text{ISCO}}, \quad (3)$$

where  $E_{\text{ISCO}}$  is the energy in Eq. (2) evaluated at  $r = r_{\text{ISCO}}$ . In the slow-rotating case ( $a \ll M$ ), the specific conserved energy of a test-particle at the ISCO radius is

$$E_{\text{ISCO}} = \frac{2\sqrt{2}}{3} \mp \frac{\sqrt{3}}{54} \frac{a}{M}. \quad (4)$$

Here we attempt to estimate how the mass of the accretion disk can affect the above measurement. We make use of an analytical model for the BH plus disk system to show that

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there exists a degeneracy in the measurement of the BH spin angular momentum when the disk is sufficiently massive with respect to the mass of the central object. The effect of massive disks on gravitational wave measurements was previously studied in [9].

## II. AXIALLY SYMMETRIC SPACETIMES

In the following we review the formalism to obtain an exact solution describing a system composed by a non-rotating BH plus a thin disk of matter. In Weyl coordinates  $\{t, \rho, z, \phi\}$ , the general form of a static and axially symmetric metric depends only upon two functions  $\lambda(\rho, z)$  and  $\nu(\rho, z)$  and is given by

$$ds^2 = -e^{2\lambda} dt^2 + e^{2(\nu-\lambda)}(d\rho^2 + dz^2) + \rho^2 e^{-2\lambda} d\phi^2. \quad (5)$$

With the line element in Eq. (5), the Einstein equations become

$$4\pi e^{2\nu-2\lambda}(T_\phi^\phi - T_t^t) = \nabla^2 \lambda = \lambda_{,\rho\rho} + \frac{\lambda_{,\rho}}{\rho} + \lambda_{,zz}, \quad (6)$$

$$4\pi(T_{\rho\rho} - T_{zz}) = \frac{\nu_{,\rho}}{\rho} - (\lambda_{,\rho})^2 + (\lambda_{,z})^2, \quad (7)$$

$$4\pi T_{\rho z} = \frac{\nu_{,z}}{2\rho} - \lambda_{,\rho}\lambda_{,z} \quad (8)$$

$$4\pi e^{2\nu-2\lambda}(T_\phi^\phi + T_t^t) = (\lambda_{,\rho})^2 + (\lambda_{,z})^2 + \nu_{,\rho\rho} + \nu_{,zz} - \nabla^2 \lambda, \quad (9)$$

and in vacuum they reduce to

$$\nabla^2 \lambda = \lambda_{,\rho\rho} + \frac{\lambda_{,\rho}}{\rho} + \lambda_{,zz} = 0, \quad (10)$$

$$\nu_{,z} = 2\rho\lambda_{,\rho}\lambda_{,z} \quad \nu_{,\rho} = \rho [(\lambda_{,\rho})^2 - (\lambda_{,z})^2]. \quad (11)$$

Notice that Eq. (10) is just the Laplace equation in the flat two-dimensional space. Therefore, in principle, every vacuum solution is known. In fact, once we choose a solution  $\lambda$  of Eq. (10) for the exterior region of a certain Newtonian density distribution, we use Eqs. (11) to find  $\nu$  and obtain the corresponding solution for the vacuum Einstein equations. Moreover, since Eq. (10) is linear, if  $\lambda_1$  and  $\lambda_2$  are two solutions then also  $\lambda = \lambda_1 + \lambda_2$  must be a solution. The non linearity of Einstein equations comes from the second function  $\nu$  which is then given by

$$\nu = \nu_1 + \nu_2 + 2 \int \rho [(\lambda_{1,\rho}\lambda_{2,\rho} - \lambda_{1,z}\lambda_{2,z})d\rho + (\lambda_{1,\rho}\lambda_{2,z} + \lambda_{1,z}\lambda_{2,\rho})dz], \quad (12)$$

where  $\nu_1$  and  $\nu_2$  are obtained from Eq. (11), respectively for  $\lambda_1$  and  $\lambda_2$ .

Spherical symmetry is a subclass of axial symmetry and therefore it must be possible to obtain the Schwarzschild BH in the above formalism. In Weyl coordinates, the Schwarzschild solution is given by a function  $\lambda_{\text{BH}}$  corresponding to a Newtonian source of constant density  $\omega = 1$

distributed along the  $z$  axis, from  $z = -M$  to  $z = M$ . Then  $\lambda_{\text{BH}}$  and  $\nu_{\text{BH}}$  are given by

$$\lambda_{\text{BH}} = \frac{1}{2} \ln \left( \frac{K_+ + K_- - 2M}{K_+ + K_- + 2M} \right), \quad (13)$$

$$\nu_{\text{BH}} = \frac{1}{2} \ln \frac{(K_- + K_+ - 2M)(K_- + K_+ + 2M)}{4K_- K_+}, \quad (14)$$

where  $K_\pm = \sqrt{\rho^2 + (z \pm M)^2}$ . For  $z = 0$ ,  $\nu_{\text{BH}}$  reduced to

$$\nu_{\text{BH}} = \frac{1}{2} \ln \frac{\rho^2}{\rho^2 + M^2}. \quad (15)$$

The Schwarzschild solution in the usual Schwarzschild coordinates  $\{t, r, \theta, \phi\}$  is recovered after the following change of coordinates

$$\rho = \sqrt{r^2 - 2Mr} \sin \theta, \quad z = (r - M) \cos \theta. \quad (16)$$

A realistic thin disk composed of ordinary matter was obtained by Lemos and Letelier in [10] by making an inversion of a thin disk of the Morgan-Morgan family [11]. The disk has an inner edge at  $b$  and the metric function  $\lambda_{\text{D}}$  solution of Eq. (10) is given by a Newtonian density distribution  $\omega$  concentrated on the equatorial plane as

$$\omega(\rho, z) = \frac{2mb}{\pi^2 \rho^4} \sqrt{\rho^2 - b^2} \delta(z). \quad (17)$$

The disk is located on the equatorial plane  $z = 0$  and it can be described in a distributional sense with the thin shell formalism that was developed by Israel in [12]. Then the full four dimensional metric is divided into three parts: (i) a vacuum 4-dimensional part above the disk,  $ds_+^2$ , for  $z > 0$ , (ii) a vacuum 4-dimensional part below the disk,  $ds_-^2$ , for  $z < 0$  (the same as the above with negative  $z$ ), (iii) a 3-dimensional part with non-vanishing energy-momentum tensor,  $ds_\Sigma^2$ , restricted to the hypersurface  $\Sigma$  given by  $z = 0$ . The continuity of the first fundamental form ensures that the metric is continuous across  $z = 0$ . The jump of the second fundamental form across  $\Sigma$  is related to the matter content of the disk. The energy-momentum tensor is restricted to the equatorial plane and can be understood as a delta-like distribution on  $z = 0$ . It is given by

$$\epsilon = -T_t^t = 4e^{2\lambda_{\text{D}} - 2\nu_{\text{D}}} (1 - \rho\lambda_{\text{D},\rho})\lambda_{\text{D},z}\delta(z), \quad (18)$$

$$p_{\phi\phi} = T_\phi^\phi = 4e^{2\lambda_{\text{D}} - 2\nu_{\text{D}}} \rho\lambda_{\text{D},\rho}\lambda_{\text{D},z}\delta(z), \quad (19)$$

$$T_\rho^\rho = T_z^z = 0. \quad (20)$$

We can introduce oblate spheroidal coordinates  $\{t, R, \theta, \phi\}$  that are related to the Weyl coordinates by

$$\rho = \sqrt{R^2 + b^2} \sin \theta, \quad z = R \cos \theta. \quad (21)$$

Then  $\lambda_{\text{D}}$  is given by (see [13] for details)

$$\lambda_D = -\frac{m}{\pi(R^2 + b^2 \sin^2 \theta)^{3/2}} \left\{ \frac{(2R^2 - b^2 \sin^2 \theta)b |\cos \theta|}{\sqrt{R^2 + b^2 \sin^2 \theta}} \left[ \frac{b |\cos \theta|}{\sqrt{R^2 + b^2 \sin^2 \theta}} \cot^{-1} \left( \frac{b |\cos \theta|}{\sqrt{R^2 + b^2 \sin^2 \theta}} \right) - 1 \right] \right. \\ \left. + (2R^2 + b^2 \sin^2 \theta) \cot^{-1} \left( \frac{b |\cos \theta|}{\sqrt{R^2 + b^2 \sin^2 \theta}} \right) \right\}. \quad (22)$$

A system composed by a BH surrounded by a thin accretion disk can be described by an exact solution of the Weyl class where the line element in Eq. (5) has  $\lambda = \lambda_{\text{BH}} + \lambda_D$ , while  $\nu$  is solution of Eq. (12). The metric of the space-time depends

upon three parameters: the BH mass  $M$ , the disk mass  $m$ , and the disk's inner radius  $b$ . The proper mass of the disk within the radial coordinate  $\rho$  is given by

$$m_p(\rho) = 32mb \int_b^\rho \frac{\sqrt{\rho'^2 - b^2} (\sqrt{\rho'^2 + M^2} + M)^2}{\rho'^5} \left[ 1 - \frac{M}{\sqrt{\rho'^2 + M^2}} - \frac{(2\rho'^2 - 3b^2)m}{2\rho'^3} \right] \exp \left[ \frac{(2\rho'^2 - b^2)m}{2\rho'^3} \right] d\rho'. \quad (23)$$

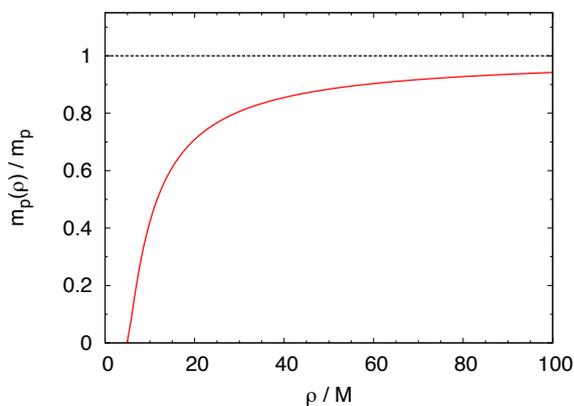


FIG. 1. Fraction of the proper mass of the disk within the radius  $\rho$ .

The proper mass  $m_p(\rho)$  does depend on the BH mass  $M$  because it includes the gravitational binding energy. Fig. 1 shows the fraction of the disk mass,  $m_p(\rho)/m_p$  where  $m_p = m_p(\infty)$ , as a function of the radial coordinate  $\rho$ . In the Lemos-Leteilier disk, most of the mass is concentrated at small radii, not far from the BH.

### III. ISCO RADIUS IN PRESENCE OF A MASSIVE DISK AND IMPLICATIONS ON THE MEASUREMENT OF THE BLACK HOLE SPIN

If we plug the functions  $\lambda_{\text{BH}}$ ,  $\nu_{\text{BH}}$ ,  $\lambda_D$ , and  $\nu_D$  into Eq. (12), we get the  $\nu$  function for the spacetime of the BH surrounded by the accretion disk. On the equatorial plane  $z = 0$ , one finds

$$\nu = \frac{1}{2} \ln \frac{\rho^2}{\rho^2 + M^2} + \frac{m}{M} \left[ 2 - \frac{2\sqrt{\rho^2 + M^2}}{\rho} + \frac{2b^2}{M^2} - \frac{3b^2\sqrt{\rho^2 + M^2}}{M^2\rho} + \frac{b^2(\rho^2 + M^2)^{3/2}}{M^2\rho^3} \right] \\ + m^2 \left[ -\frac{1}{2\rho^2} + \left( \frac{3}{4} + \frac{4}{\pi^2} \right) \frac{b^2}{\rho^4} - \left( \frac{3}{8} + \frac{8}{3\pi^2} \right) \frac{b^4}{\rho^6} \right]. \quad (24)$$

From the conservation of the rest-mass, the motion of a test particle on the equatorial plane is governed by the equation

$$\dot{\rho}^2 + V_{\text{eff}} = 0, \quad (25)$$

where

$$V_{\text{eff}} = \frac{1}{g_{\rho\rho}} \left( 1 + \frac{E^2}{g_{tt}} + \frac{L^2}{g_{\phi\phi}} \right), \quad (26)$$

is the effective potential,  $E = -g_{tt}\dot{t}$  is the specific conserved energy, and  $L = g_{\phi\phi}\dot{\phi}$  is the specific angular momentum of

the particle. Here, a dot indicates the derivative with respect to the proper time for the particle. By definition, circular orbits are located at the zeros and the turning points of the effective potential:  $\dot{\rho} = 0$ , which implies  $V_{\text{eff}} = 0$ , and  $\ddot{\rho} = 0$ , requiring  $V_{\text{eff},\rho} = 0$ . From these conditions, one finds

$$E = \sqrt{\frac{e^{2\lambda}(1 - \lambda_{,\rho}\rho)}{1 - 2\lambda_{,\rho}\rho}}, \quad L = \sqrt{\frac{\lambda_{,\rho}\rho^3}{e^{2\lambda}(1 - 2\lambda_{,\rho}\rho)}}. \quad (27)$$

The ISCO radius is at the minimum of  $E$  and  $L$  and can be inferred from

$$\lambda_{,\rho\rho}\rho + \lambda_{,\rho} [4(\lambda_{,\rho})^2\rho^2 - 6\lambda_{,\rho}\rho + 3] = 0. \quad (28)$$

For  $m \ll M$  and  $mb^2 \ll M^3$ , the radius of the ISCO is

$$\rho_{\text{ISCO}} \approx 2\sqrt{6}M - \frac{25M}{1536} \frac{73b^2 - 368M^2}{M^2} \frac{m}{M}. \quad (29)$$

The corresponding specific energy for a test-particle at the ISCO radius is

$$E_{\text{ISCO}} \approx \frac{2\sqrt{2}}{3} \left( 1 - \frac{\sqrt{6}}{4608} \frac{17b^2 - 16M^2}{M^2} \frac{m}{M} \right). \quad (30)$$

Assuming that the inner edge of the disk  $b$  is at the ISCO radius  $\rho_{\text{ISCO}}$ , one finds

$$\rho_{\text{ISCO}} \approx 2\sqrt{6}M - \frac{4325M}{192} \frac{m}{M}, \quad (31)$$

and the Novikov-Thorne radiative efficiency in presence of a massive disk is

$$\eta_{\text{NT}} \approx 1 - \frac{2\sqrt{2}}{3} \left( 1 - \frac{49\sqrt{6}}{576} \frac{m}{M} \right). \quad (32)$$

Eq. (32) has to be compared with Eq. (4): the effects of a massive disk can mimic the ones of a rotating BH with a massless corotating disk. In particular,

$$\frac{a}{M} \approx \frac{49}{8} \frac{m}{M}. \quad (33)$$

Fig. 2 shows  $\eta_{\text{NT}}$  as a function of  $m/M$  (left panel) and  $a/M$  (right panel). Typical accretion disks around stellar-mass BHs have  $m/M \sim 10^{-9} - 10^{-10}$  and their mass is not concentrated as close as to the compact object as shown in Fig. 1. Considering that, even in the most favorable conditions, a spin measurement can be at the level of  $a/M = 0 \pm 0.05$ , the effect of the mass of the accretion disk can be definitively neglected, even in the case of future more accurate measurements. However, BHs with very massive disks may exist, and when  $m/M$  approaches 0.01, the effect may not be ignored.

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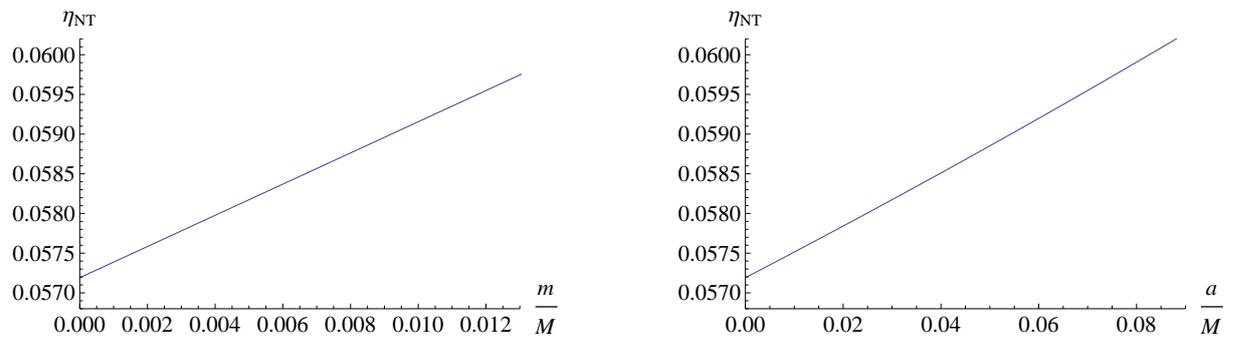


FIG. 2. Novikov-Thorne radiative efficiency  $\eta_{NT} = 1 - E_{ISCO}$  as a function of the disk to BH mass ratio  $m/M$  in the case of a massive disk around a non-rotating BH (left panel) and as a function of the spin parameter  $a/M$  in the case of zero-mass disk around a rotating BH (right panel).