Trinomial Method for Option Pricing with Transaction Costs for Supply Chain Financing

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Abstract

In this study, we investigate the trinomial method for option pricing, incorporating transaction costs into a discrete-time framework. Taking the binomial option pricing model in Cox et al. (1979) as a foundation, we further extend it to construct the trinomial model as referenced in Bjorefeldt et al. (2016). Our research examines the integration of transaction costs into three option pricing models – Black-Scholes, binomial, and trinomial – through the comparative analysis of the results. We also explore the application of the trinomial model as a pricing tool for supply chain financial products, aiming to address financial challenges faced by small and medium-sized businesses. Building upon the case studies outlined by Yunzhang et al. (2021), we use the framework of American call options. Despite our efforts to integrate our model into the supply chain financing context, we have encountered challenges. Our current model, while proficient in handling fixed parameters, lacks the flexibility required to incorporate the variables needed in supply chain financing scenarios.

Keywords: option, binomial model, trinomial model, transaction costs

1 Introduction

An option is a financial instrument that grants its holder the right, but not the obligation, to trade a fixed quantity of shares of a specific stock at a predetermined price within a specified time frame. Executing this trade is called "exercising the option." The predetermined price is known as the exercise price or strike price, K, and the specified date is the expiration date or maturity, t. A call option gives the right to buy the underlying asset; a put option gives the right to sell the underlying asset (Cox et al., 1979). The paper first focuses on a European call option, which can be exercised only at maturity. Later, in the supply chain financing section, American call option pricing model is introduced, where American options can be exercised at any time up to the expiration date.

For centuries, options have been exchanged, yet they stayed relatively unknown financial tools until the introduction of a listed options exchange in 1973. Since then option pricing models went through many modifications. The first completely satisfactory equilibrium option pricing model was presented by Fischer Black and Myron Scholes in 1973. In the same year, Robert Merton extended their model (Cox et al., 1979). In 1979, Cox, Ross and Rubinstein found a way to simplify the Black-Scholes-Merton model and proposed a new valuation method of options using basic mathematical tools. Eventually, in 1986, the trinomial model was introduced by Phelim

Boyle, as an extension to the Cox-Ross-Rubinstein (CRR) model. It is said to be more flexible and accurate as it offers an additional possible evolution of the stock price (Bjorefeldt et al., 2016).

1.1 The Black-Scholes model

The Black-Scholes (BS) model stands as a revolutionary approach in the field of financial economics, providing a theoretical framework for valuing European options on stocks. It was presented by Fischer Black, Myron Scholes, and Robert Merton in 1973 and offers an analytical tool for estimating the fair price of options based on key parameters: the current underlying asset price, the option's strike price, time to maturity, risk-free interest rate, and the underlying asset's volatility (Hull, 2021). The partial differential equation (PDE) formulation for European options without transaction costs in a complete market is (Al–Zhour et al., 2019):

$$C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} + (r-q)SC_S - rC = 0 \tag{1}$$

where σ is the volatility constant, C is the call price, S is the price of the underlying asset, C_{SS} denotes the second derivative of C with respect to S, r is the risk-free interest rate, q is the dividend yield. The initial conditions fot the call option is:

$$V(S,T) = \max\{S-K, 0\}, \tag{2}$$

where K is the strike price. The BS PDE formulation (1) has the following analytical solution:

$$C(S,t) = S_0 N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{3}$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right], \quad d_2 = d_1 - \sigma\sqrt{T-t} \tag{4}$$

Here, $C(S,t)$ is the price of the call option at time t, T is the time to maturity of the option, and N is the cumulative distribution function of the standard normal distribution. where K is the strike price.

The initial cost of the hedge portfolio which employs a dynamic strategy to replicate a call option at maturity being self-financing with the transaction costs included uses the modified volatility in equation (1) (Boyle and Vorst, 1992):

$$\sigma^2(1 + \frac{2k\sqrt{n}}{\sigma\sqrt{T}}), \tag{5}$$

where k is the transaction cost rate on stocks.

1.2 Binomial model

According to the CRR model, the stock price follows a discrete-time multiplicative binomial process. There are two possible return rates on the stock each period: u with probability q, when the stock price goes up, and d with probability $1-q$ when the stock price goes down.

The up and down rates are defined by:

$$u = e^{\sigma\sqrt{h}}, \quad d = e^{-\sigma\sqrt{h}} \tag{6}$$

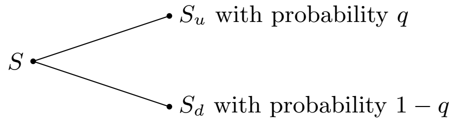

Figure 1: The binomial movement process

where

$$h = T/n$$

To see the valuation of a call option on the stock, let's consider the case when the expiration date is just one period. C is the current value of the call option, C_u and C_d be its the values at the end of the period if the stock price goes up to S_u goes down to S_d, respectively. Since the call has only one period left, the exercise policy is the following (Cox et al., 1979):

$$C_u = max[0, S_u - K] \quad \text{and} \quad C_d = max[0, S_d - K] \tag{7}$$

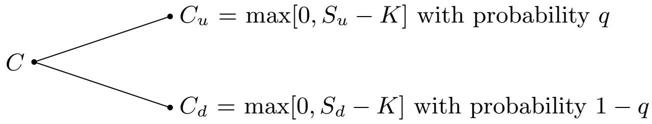

Figure 2: Call price in one period

We also assume that the interest rate is constant. People have the freedom to borrow or invest any amount at this fixed rate.

1.3 Trinomial model

The trinomial model integrates three potential outcomes for the value of an underlying asset within a single time period, where the possible values will be greater than, the same as, or less than the current value. Trinomial method is more complex due to the additional possible outcome but can offer a more accurate and nuanced representation of the market movements. This leads to increase in the number of scenarios exponentially with more time steps, which demands more computational power (Bjorefeldt et al., 2016).

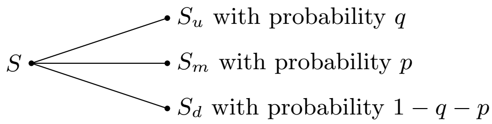

Figure 3: The trinomial movement process

The up, down and middle rates are

$$u = e^{\sigma\sqrt{h}}, \quad d = e^{-\sigma\sqrt{h}}, \quad m = 1, \tag{8}$$

where

$$h = T/n \tag{9}$$

The value of a call option at expiration date for trinomial method is (Bjorefeldt et al., 2016):

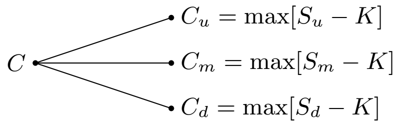

Figure 4: Call price in one period

2 Methodology

2.1 Incorporating transaction costs: Boyle and Vorst's assumption

Transaction costs in option pricing have often been neglected in academic research. The Black-Scholes model operates only under perfect frictionless markets (Boyle and Vorst, 1992). Subsequently, the CRR binomial option pricing model, introduced in 1979, has also not covered the case of transaction costs. Avoiding the transaction costs implies that these models are only applicable in perfect market conditions, where the rebalancing is costless. In 1992, Boyle and Vorst were the first to analyze how transaction costs affect option prices and replication.

Definition. Transaction costs - the cost of carrying out a trade (commissions plus the difference between the price obtained and the midpoint of the bid–offer spread).

When incorporating transaction costs, besides the values of replicating portfolio at each node, we also have to know the allocation between investment in the risky asset and borrowing. The following $\triangle$ represents the number of shares, and B represents the number of bonds.

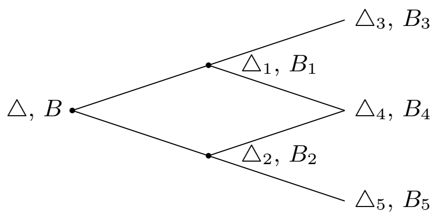

Figure 5: $\triangle$'s and B's at each node

Let k be the transaction costs rate. We need to choose $\triangle$ and B such that we can purchase the portfolio ($\triangle_1$, B_1) if the stock price goes up, and ($\triangle_2$, B_2) if it goes down. This results in the following two equations:

$$\triangle S_u + BR = \triangle_1 S_u + B_1 + k|\triangle - \triangle_1|S_u \tag{10}$$

$$\triangle S_d + BR = \triangle_2 S_d + B_2 + k|\triangle - \triangle_2|S_d \tag{11}$$

According to equation (1), the value of the up-state portfolio is precisely sufficient to purchase the matching replicating portfolio for this state to pay for the transaction costs associated with the rebalancing. For the down-state, equation (2) can be interpreted similarly. We use the absolute value of $\triangle - \triangle_1$ and $\triangle - \triangle_2$, since we are uncertain regarding whether a sale or purchase of the risky asset would be necessary in the portfolio.

Theorem 1: In the construction of a long European call option by dynamic hedging, equations (1) and (2) have a unique solution ($\triangle$, B). Furthermore for this solution the following inequality holds:

$$\triangle_2 \leq \triangle \leq \triangle_1 \tag{12}$$

This theorem enables us to rewrite equations (1) and (2) in the following form

$$\triangle S\bar{u} + BR = \triangle_1 S\bar{u} + B_1 \tag{13}$$

$$\triangle S\bar{d} + BR = \triangle_2 S\bar{d} + B_2 \tag{14}$$

where

$$\bar{u} = u(1+k) \quad \text{and} \quad \bar{d} = d(1-k) \tag{15}$$

$$R = e^{rh}, \quad h = T/n$$

This way Boyle and Vorst introduced an assumption that simplifies the system of equations, leading to a reduction in transaction costs. This assumption revolves around the idea that market participants can bypass the initial transaction, which involves setting up the replicating portfolio at time zero (Tichý, 2005). The simplification is particularly beneficial in multi-period models where there are multiple time periods and transactions involved.

2.2 n-period model

Pricing of the option and hedging of its payoff is usually done not by a single-step. Thus, we now use the standard backward recursive procedure to find the option price at time zero starting with the terminal payoff up to the time one (Luenberger, 1998). In this subsection we will look more closely on the multi-period model. The solution process now evolves to employing a step-by-step backward approach in multi-period options.

Binomial multi-period option pricing:

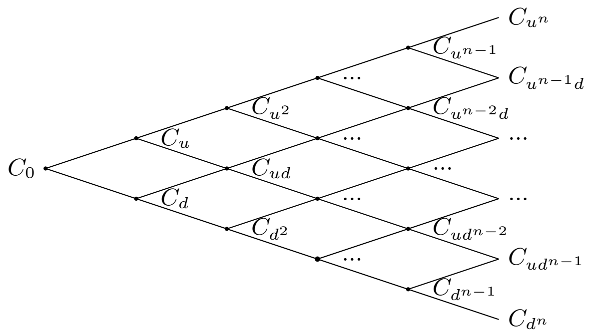

Figure 6: n-period option pricing for binomial method

The figure above illustrates the movements of the stock and their call options, where the payoffs of lattice note call options are calculated using equation (7). Specifically,

$$C_{u^n} = \max[0, u^n S - K]$$

$$C_{u^{n-1}d} = \max[0, u^{n-1} dS - K]$$

$$...$$

$$C_{ud^{n-1}} = \max[0, ud^{n-1} S - K]$$

$$C_{d^n} = \max[0, d^n S - K]$$

Option pricing formula: The price of a call option for a single period on a stock structured by a binomial lattice is (Luenberger, 1998):

$$C = \frac{1}{R}\left[q \cdot C_u + (1 - q) \cdot C_d\right] \quad (16)$$

Now we define the risk-neutral probability:

$$q = \frac{R - d}{u - d} \quad (17)$$

where R is the one-period return on the risk-free asset.

Thus, the values of the call prices will be constructed backwards from the expiration date, treating each node as a single step tree, until getting to initial time period to find the value of the call option. For example, call prices for $n - 1$ can be calculated as

$$C_{u^{n-1}} = \frac{1}{R}\left[q \cdot C_{u^n} + (1 - q) \cdot C_{u^{n-1}d}\right] \quad (18)$$

$$C_{u^{n-2}d} = \frac{1}{R}\left[q \cdot C_{u^{n-1}d} + (1 - q) \cdot C_{u^{n-2}d^2}\right]$$

$$...$$

$$C_{ud^{n-2}} = \frac{1}{R}\left[q \cdot C_{u^2 d^{n-2}} + (1 - q) \cdot C_{ud^{n-1}}\right]$$

$$C_{d^{n-1}} = \frac{1}{R}\left[q \cdot C_{ud^{n-1}} + (1 - q) \cdot C_{d^n}\right] \quad (19)$$

Trinomial multi-period option pricing:

The tree above illustrates the stock movements in trinomial method in case of multi-period option pricing, where the values of the call options at the terminal nodes are calculated as in figure (4), which is a single period tree.

Now, as before, we apply standard recursive algorithm to compute the option price at time zero, starting with the payoff at the maturity date.

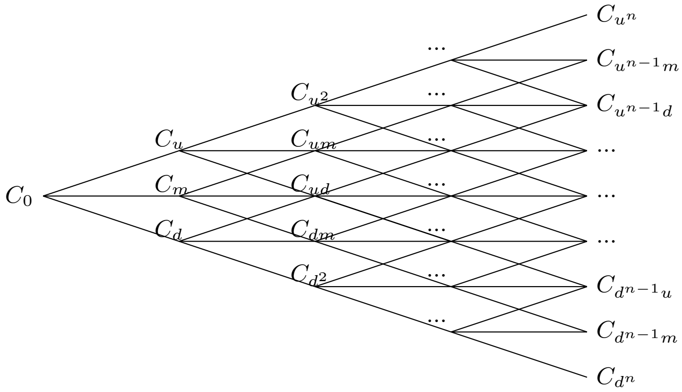

Figure 7: n-period option pricing for trinomial method

Option pricing formula for trinomial method: The price of a call option for a single period trinomial tree on a stock is (Rouah and Vainberg, 2007):

$$C = \frac{1}{R}[p_u \cdot C_u + p_d \cdot C_d + p_m \cdot C_m], \tag{20}$$

where the probability of the up movement is

$$p_u = \left(\frac{\exp\left(\frac{1}{2}r\triangle(t)\right) - \exp\left(-\sigma\sqrt{\frac{1}{2}\triangle(t)}\right)}{\exp\left(\sigma\sqrt{\frac{1}{2}\triangle(t)}\right) - \exp\left(-\sigma\sqrt{\frac{1}{2}\triangle(t)}\right)} \right)^2; \tag{21}$$

the probability of a down movement:

$$p_d = \left(\frac{\exp\left(\sigma\sqrt{\frac{1}{2}\triangle(t)}\right) - \exp\left(\frac{1}{2}r\triangle(t)\right)}{\exp\left(\sigma\sqrt{\frac{1}{2}\triangle(t)}\right) - \exp\left(-\sigma\sqrt{\frac{1}{2}\triangle(t)}\right)} \right)^2; \tag{22}$$

and the probability of a lateral move is:

$$p_m = 1 - p_u - p_d \tag{23}$$

Again, we find all the other call prices of the trinomial tree by moving backwards in time periods until valuing the initial call price. Specifically, call prices for time period (n-1):

$$C_{u^{n-1}} = \frac{1}{R}[p_u \cdot C_{u^n} + p_d \cdot C_{u^{n-1}d} + p_m \cdot C_{u^{n-1}m}] \tag{24}$$

...

$$C_{d^{n-1}} = \frac{1}{R}[p_u \cdot C_{d^{n-1}u} + p_d \cdot C_{d^n} + p_m \cdot C_{d^{n-1}m}] \tag{25}$$

Further, we will use the estimated initial call prices for models comparison purposes and to see how transaction cost affects the option pricing.

3 Numerical implementations

Constant parameters for the numerical representations: S = 100, K = 80, n = 6, interest rate = 10%, standard deviation = 20%, time to expiry = 1 year.

Binomial method without transaction cost, k = 0%

Time		0	1	2	3	4	5	6
	6							83.215
	5						71.740	
	4					61.247		58.624
	3				51.657		49.078	
	2			42.898		40.362		37.739
	1		34.996		32.409		29.830	
State	0	28.019		25.383		22.623		20
	-1		19.415		16.606		13.482	
	-2			11.874		8.883		4.934
	-3				5.752		2.826	
	-4					1.619		0
	-5						0	
	-6							0

Table 1: Binomial method without transaction cost

The table employs a matrix format for the binomial method in options pricing, specifically for call options. It outlines the evolution of option prices over discrete time intervals, ranging from 0 to 6 periods, while considering various possible states of the underlying asset's price movements (up and down), labeled from 6 to -6. Each cell in the matrix contains the calculated option call price corresponding to a specific combination of time and asset price state. The call price in coordinates *Time*=0 and *State* = 0 , 28.019, is the initial call option value, which we aimed to calculate.

Binomial method with transaction cost, k = 2%

Time		0	1	2	3	4	5	6
	6							103.807
	5						87.396	
	4					72.674		69.992
	3				59.477		56.843	
	2			47.655		45.069		42.398
	1		37.283		34.535		31.911	
State	0	28.475		25.601		22.543		19.880
	-1		18.432		15.289		11.566	
	-2			10.078		6.708		1.505
	-3				3.879		0.826	
	-4					0.453		0
	-5						0	
	-6							0

Table 2: Binomial method with transaction cost

Here, we can see that transaction cost of 2% affected the call option value (see the call option in coordinates time and state equal to zero) increasing it to 1.916 comparing with the previous table, when the transaction cost was not considered in option pricing.

Trinomial method without transaction cost, k = 0%

Time		0	1	2	3	4	5	6
	6							83.215
	5						70.907	70.418
	4					59.715	59.179	58.624
	3				49.545	48.966	48.370	47.756
	2			40.310	39.694	39.060	38.409	37.739
	1		31.939	31.284	30.615	29.931	29.229	28.508
State	0	24.425	23.718	22.996	22.261	21.518	20.768	20
	-1		16.337	15.525	14.686	13.827	12.971	12.159
	-2			9.181	8.242	7.230	6.117	4.934
	-3				3.567	2.606	1.511	0
	-4					0.463	0	0
	-5						0	0
	-6							0

Table 3: Trinomial method without transaction cost

According to the table above, we can see that the call option value equals *24.425* using trinomial option pricing method with the parameters mentioned in the beginning of the section.

Trinomial method with transaction cost k = 2%:

Time		0	1	2	3	4	5	6
	6							103.806
	5						86.991	86.074
	4					71.929	70.979	69.991
	3				58.462	57.498	56.513	55.521
	2			46.430	45.456	44.462	43.446	42.397
	1		35.714	34.717	33.707	32.685	31.642	30.589
State	0	26.341	25.295	24.226	23.139	22.047	20.979	19.880
	-1		16.311	15.154	13.943	12.672	11.346	10.244
	-2			8.020	6.789	5.439	3.867	1.505
	-3				2.376	1.408	0.461	0
	-4					0.141	0	0
	-5						0	0
	-6							0

Table 4: Trinomial method with transaction cost

Here we can see that transaction costs of 2% affected the trinomial method slightlty, specifically to *1.916*, again as in binomial method.

Comparison of models without transaction costs

The numerical representations maintain constant parameters as follows: S = 100, n = 6, k = 0%, interest rate = 10%, standard deviation = 20%, time to expiry = 1 year. However, K (the strike price) varies from 80 to 120 for each model.

Strike Price	Black-Scholes	Binomial	Trinomial
80	27.675	28.019	24.425
90	19.675	20.129	15.981
100	12.993	12.930	8.851
110	7.966	8.335	4.182
120	4.555	4.355	1.609

Table 5: Comparison of models without transaction costs

Each row represents a different strike price, and the columns show the corresponding call prices for the Black-Scholes, Binomial, and Trinomial models. The values obtained from the Binomial and Trinomial models are closer to each other compared to the Black-Scholes model. This indicates that the Binomial and Trinomial models produce more consistent results with each other.

Comparison of models with transaction costs

Parameters: S = 100, k = 2%, interest rate = 10%, standard deviation = 20%, time to expiry = 1 year. K (the strike price) varies from 80 to 120 for each model.

Strike Price	LeLand's model	Binomial	Trinomial
80	28.207	28.464	26.341
90	20.724	21.373	18.114
100	14.513	14.547	11.251
110	9.715	10.403	6.278
120	6.246	6.258	3.222

Table 6: Comparison of models with transaction costs

This table compares option prices calculated using LeLand's model, the Binomial model, and the Trinomial model, all with transaction costs included. Similar to the comparison without transaction costs, the Binomial and Trinomial models produce prices closer to each other compared to LeLand's model. This indicates that the Binomial and Trinomial models remain consistent in their pricing even with transaction costs included.

4 Supply Chain Financing

In recent years, due to the rapid growth of supply chain finance, cutting financing costs has become a top priority for corporations among many financial issue. Supply chain financing is a financial strategy that aims to optimize capital availability and cost within a supply chain network. In order to meet the financial demands of businesses involved in a supply chain, financial institutions — typically large commercial banks — seek to innovate traditional finance models (Yunzhang et al., 2021).

By using complex option pricing models, such as the binomial and trinomial tree models, for various scenarios and possible outcomes, stakeholders can successfully manage financial risk control and pricing challenges. The expected return of investment projects over a certain period of time under the binary tree pricing model only allows two alternative states, which causes significant errors in numerical computations. In contrast to the binomial tree model, the trinomial tree model has an extra state, which gives it greater modeling flexibility when it comes to the price movements of the underlying asset. The behavior of the asset can be more accurately represented by this additional state, which can also result in more accurate pricing of financial instruments in supply chain financing (Yunzhang et al., 2021).

Major firms and small and medium-sized enterprises (SMEs) have implemented three different financing approaches: *advance payment financing, inventory financing, and accounts receivable financing. The advance payment financing* is a model using a company gets funding from a commercial bank by pledging warehouse receipts it has received using prepaid accounts. *Inventory financing* involves banks offering funds to businesses with their inventory serving as collateral. Firms may use *the accounts receivable model*, which involves pledging receivables based on actual trading relationships or through direct transfer to the bank, to get financial support from a bank (Yunzhang et al., 2021).

Accounts receivable financing stands out as the most prevalent and prominent among the three financing approaches, also serving as a primary model within supply chain financing. Contrasted with other financing models, accounts receivable financing has distinct advantages including high quality, rapid realization, and minimal risk (Yunzhang et al., 2021). Since this model is more practical, Yunzhang et al. (2021) has chosen the accounts receivable model to analyze and elaborate it using a trinomial option pricing method.

In supply chain financing, accounts receivable serve as the underlying asset for financial products. Due to factors like early repayment, timely repayment, or customer defaults in the accounts receivable financing scenarios, the American call option framework proves to be particularly fitting. Unlike European options, which can only be exercised at maturity, American call options grant the holder the right to exercise the option at any point before the expiration date. This flexibility aligns with the dynamic character of supply chain financing, where conditions and circumstances can change rapidly, requiring immediate decision-making.

To establish a trinomial pricing model for American options, we first need to create a trinomial tree model for European options (Yunzhang et al., 2021). Since a trinomial option pricing model has already been established in thee previous sections, we further continue establishing American call option pricing model.

4.1 American call option

Definition: American option is an option that can be exercised at any time during its lifetime.

The up and down rates for the American option pricing are the same as for European option pricing, see equation (4), and the probability being same as equation (8).

The payoff of the American call options stays also the same as in European call option: equations (5) and (6). However, in applying the backwards approach to American call option, we use the following formulas to find the call prices at node $n-1$, respectively (Hull, 2021):

$$C_u^{n-1} = \max[u^{n-1}S_0 - K, e^{-r\cdot\frac{t}{n}} \cdot (qC_n + (1-q)C_{u^{n-1}d})] \tag{26}$$

...

$$C_d^{n-1} = \max[d^{n-1}S_0 - K, e^{-r\cdot\frac{t}{n}} \cdot (qC_{ud^{n-1}} + (1-q)C_{d^n})] \tag{27}$$

Settings	Values
The accounts receivable	14454459.09 (RMB)
The company's bad debt accrual ratio	5%, 10%, 30%
The coupon rate	3.56%
The interest rate of medium and long-term loans for 1–3 years	4.75%
Risk-free rate	2.415%

Table 7: The value of each parameter. The table was taken from Yunzhang et al. (2021)

4.2 Case application taken from Yunzhang et al. (2021)

Yunzhang et al. (2021) have focused on examining the relationship between Tesla, the core enterprise, and its supplier, NINGBOYOSUN AUTO-PARTS CO., LTD, which serves as the financing enterprise. All the values provided above were taken from the the supplier's 2019 annual report (Yunzhang et al., 2021). The total accounts receivable is 14454459.09 RMB, and the company's bad debt accrual ratio is detailed: 5% within one year, 10% within 1-2 years, and 30% within 2-3 years.

It can be seen that the supply chain financing case requires more variables to be considered, while our current model only accounts for certain fixed parameters. Therefore, we leave room for further enhancements in the research, requiring revisions to the model to better align with the complexities of supply chain financing.

5 Conclusion

Transaction costs are significant factors in pricing, hedging, and replicating financial derivatives. While the impact on the price of extensive options portfolios may be minimal due to position netting, the additional capital required to replicate or hedge specific assets can be substantial.

This paper demonstrates the process of pricing and replicating option payoffs using both binomial and trinomial model. We have been able to incorporate transaction cost not only into binomial model, which has already been studied in Boyle and Vorst (1992), but also have integrated transaction cost into the trinomial option pricing model. We have derived all basic equations for multi-period model of long European and American options by generating payoffs and further using backward recursive formula.

We have conducted our calculations for both binomial and trinomial models, with and without transaction costs, for a given parameter value of $n = 6$. This allows us to compare our findings with those derived from the Black-Scholes and LeLand's (which includes the transaction cost) models.

We have made several attempts to apply our model into supply chain financing case outlined in Yunzhang et al. (2021). This involved constructing a trinomial model for American call options and conducting in-depth research into the accounts receivable model. However, our current model only accommodates certain fixed parameters, whereas the supply chain financing case demands consideration of additional variables. Consequently, we need to refine our model to better address the complexities of supply chain financing.

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7 Appendix

Proof of Theorem 1. We prove Theorem 1 by backward induction. By induction, we may assume that $\triangle_4 \geq \triangle_1 \geq \triangle_3$ and $\triangle_5 \geq \triangle_2 \geq \triangle_4$. Thus, $\triangle_2 \geq \triangle_1$.

Subtracting (2) from (1), transferring everything to the right-hand side, and introducing the function $f(\triangle)$, we get:

$$\begin{aligned} f(\triangle) := \triangle S(u-d) - \triangle_1 Su + \triangle_2 Sd - B_1 + B_2 \\ - k(\triangle - \triangle_1)Su + k(\triangle - \triangle_1)Sd = 0 \end{aligned} \tag{28}$$

The function $f(\triangle)$ is continuous and piecewise linear; i.e., it is a linear function on $(-\infty, \triangle_2)$, $(\triangle_2, \triangle_1)$, and $(\triangle_1, \infty)$ with constant derivatives on each interval with values $[(1+k)u-(1+k)d]S$, $[(1+k)u-(1-k)d]S$ and $[(1-k)u-(1-k)d]S$, respectively. Since all these derivatives are strictly positive, $f(\triangle)$ is a monotonically increasing piecewise linear function. Hence, it has a unique zero. This proves the first claim of Theorem 1. For the second part it is enough to show that

$f(\triangle_2 \leq 0)$ and $f(\triangle_1) \geq 0$

since this implies that $\triangle \in [\triangle_2, \triangle_1]$. Now

$$\begin{aligned} f(\triangle_2) = (\triangle_2 - \triangle_1)Su(1+k) - B_1 + B_2 \\ f(\triangle_1) = (\triangle_2 - \triangle_1)Sd(1-k) - B_1 + B_2 \end{aligned}$$

Since by induction $\triangle_4 \leq \triangle_1 \leq \triangle_3$, we know that one of the equations from which $\triangle_1$ has been deduced reads as follows:

$$\triangle_1 Sud + B_1 R = \triangle_4 Sud + B_4 + k(\triangle_1 - \triangle_4)Sud \tag{29}$$

Similarly, since $\triangle_5 \leq \triangle_2 \leq \triangle_4$ we have:

$$\triangle_2 Sud + B_2 R = \triangle_4 Sud + B_4 + k(\triangle_4 - \triangle_2)Sdu \tag{30}$$

Subtracting the second equation from the first and dividing by R gives:

$$(\triangle_2 - \triangle_1)Sdu/R + B_2 - B_1 = k[(\triangle_4 - \triangle_2) - (\triangle_1 - \triangle_4)]Sdu/R$$

Using this, we derive:

$$\begin{aligned} f(\triangle_2) &= (\triangle_2 - \triangle_1)Su(1+k) - B_1 + B_2 \\ &\leq (\triangle_2 - \triangle_1)Sdu(1+k)/R - B_1 + B_2 \\ &= k[(\triangle_4 - \triangle_2) - (\triangle_1 - \triangle_4)]Sdu/R + k(\triangle_2 - \triangle_1)Sdu/R \\ &= 2k(\triangle_4 - \triangle_1)Sdu/R \leq 0 \end{aligned}$$

Similarly, $f(\triangle_1) \geq 0$, and thus we have proved the second part of Theorem 1. To start the induction, we consider the option at maturity. At maturity, there are two possible portfolios: $\triangle = 1$ and $B = -K$, if the asset price is above the exercise price, and $A = 0$ and $B = 0$, otherwise. Hence, at maturity, we always have $\triangle_1 \geq \triangle_2$ in the notation of this appendix.

One period before maturity, there are three different cases: First, if $\triangle_1 = \triangle_2 = 1$ in which case $\triangle = \triangle_1$ and $B = -K/R$ is the unique solution, which indeed has $\triangle_2 \geq \triangle \geq \triangle_1$. Second, if $\triangle_1 = \triangle_2 = 0$ in which case $\triangle = 0$ and $B = 0$ is the unique solution, which indeed has $\triangle_2 \leq \triangle \leq \triangle_1$. Finally, if $\triangle_1 = 1$ and $\triangle_2 = 0$, in this case, the unique solution is:

$$\triangle = \frac{(S_{\bar{u}} - K)}{(S_{\bar{u}} - S_{\bar{d}})}.$$

Hence, $\triangle_2 = 0 \leq \triangle \leq \triangle_1 = \triangle_1$.