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ON LOGISTIC-NORMAL DISTRIBUTION

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April, 2016

Abstract:

Existing distributions do not always provide an adequate fit to the complex real world data. Hence, the interest in developing more flexible statistical distributions remains strong in statistics profession. In this project, we present a family of generalized normal distributions, the T-normal family. We study in some details a member of the proposed family namely, the logistic-normal (LN) distribution. Some properties of the LN distribution including moments, tail behavior, and modes are examined. The distribution is symmetric and can be unimodal or bimodal. The tail of the LN distribution can be heavier or lighter than the tail of the normal distribution. The performance of the maximum likelihood estimators is evaluated through small simulation study. Two bimodal data sets are used to show the applicability of the LN distribution.

1. Introduction

Since real world data are usually complex and can take a variety of shapes, existing distributions do not always provide an adequate fit. Hence, generalizing distributions and studying their flexibility are of interest of researchers for last decades. One of the earliest works on generating distributions was done by Pearson (1895), who proposed a method of differential equation as fundamental approach to generate statistical distributions. Burr (1942) also made a contribution on this category and developed another method based on differential equation. Later on method of transformation (Johnson, 1949) and method of quantile function (Hastings et al., 1947; Tukey, 1960) were developed. More recent techniques emerged after 1980s were summarized into five major categories (Lee et al., 2013): method of generating skew distributions, method of adding parameters, beta generated method, transformed-transformer method, and composite method.

The beta generated (BG) method grasped the interest of modern researchers. Eugene et al. (2002)

introduced the beta-generated family of distributions with cumulative distribution function (CDF) given by

$$G(x) = \int_0^{F(x)} b(t)dt, \tag{1.1}$$

where $b(t)$ is the probability density function (PDF) of the beta random variable and $F(x)$ is the CDF of any random variable. The PDF corresponding to (1.1) is given by

$$g(x) = \frac{1}{B(\alpha,\beta)} f(x)F^{\alpha-1}(x)(1-F(x))^{\beta-1},\ \alpha,\beta > 0;\ x \in Supp(F), \tag{1.2}$$

where $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/(\Gamma(\alpha+\beta))$.

Several members of BG family of distributions were investigated in recent literature, for example, beta-normal (Eugene et al., 2002; Famoye et al., 2004; Gupta and Nadarajah, 2004; Rego et al., 2012), beta-Gumbel (Nadarajah and Kotz, 2004), beta-Frechet (Baretto-Souza et al., 2011), beta-Weibull (Famoye et al., 2005; Lee et al., 2007; Wahed et al., 2009; Cordeiro et al., 2011), beta-Pareto (Akinsete et al., 2008), beta generalized logistic of type IV (de Morais, 2009) and beta-Burr XII (Paranaiba et al., 2011). Some extensions of BG family such as Kw-G distribution (Jones, 2009; Cordeiro and de Castro, 2011), beta type I generalization (Alexander et al., 2012), and generalized gamma-generated family (Zografos and Balakrishnan, 2009) were recently introduced.

The beta-generated family of distributions is formed by using the beta distribution in (1.2) with support between 0 and 1 as a generator. Alzaatreh et al. (2013), in turn, were interested whether other distributions with different support can be used as a generator. They extended the family of BG distributions and defined the so called $T-X$ family. In the $T-X$ family, the generator $b(t)$ was replaced by the generator $r_T(t)$, where T is any random variable with support (a,b) and $W[0,1] \to \mathbb{R}$ is a link function that is absolutely continuous and satisfies $W(0) \to a$ and $W(1) \to b$. The CDF of the $T-X$ family is given by

$$G(x) = \int_a^{W(F(x))} r(t)dt. \tag{1.3}$$

Aljarrah et al. (2014) studied a special case of the $T - X$ family where the link function, $W(.)$, is a quantile function of a random variable Y. The proposed CDF is defined as

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = P\left[T \le Q_Y(F_R(x))\right] = F_T(Q_Y(F_R(x))), \tag{1.4}$$

where T, R and Y are random variables with CDF $F_T(x) = P(T \le x)$, $F_R(x) = P(R \le x)$ and $F_Y(x) = P(Y \le x)$. The corresponding quantile functions are $Q_T(p)$, $Q_R(p)$ and $Q_Y(p)$, where the quantile function is defined as $Q_Z(p) = \inf\{z : F_Z(z) \ge p\}, 0 < p < 1.$ If densities exist, we denote them by $f_T(x)$, $f_R(x)$ and $f_Y(x)$. Now assume the random variable $T \in (a,b)$ and $Y \in (c,d)$, for $-\infty \le a < b \le \infty$ and $-\infty \le c < d \le \infty$, then the corresponding PDF to (1.4) is

$$f_X(x) = f_R(x) \times \frac{f_T(Q_Y(F_R(x)))}{f_Y(Q_Y(F_R(x)))}. \tag{1.5}$$

If R follows the normal distribution with mean μ and variance σ^2 with PDF $f_R(x) = \phi(x)$ and CDF $F_R(x) = \Phi(x)$, then (1.4) reduces to the T-normal family of distributions with PDF given by

$$f_x(x) = \phi(x) \times \frac{f_T(Q_Y(\Phi(x)))}{f_Y(Q_Y(\Phi(x)))}. \tag{1.6}$$

The T-normal family is a general base for generating many different generalizations of the normal distribution. The distributions generated from the T-normal family can be symmetric, skewed to right, skewed to the left, or bimodal. Some of the existing generalizations of normal distributions can be obtained using this framework. In particular, such generalizations of normal distribution are beta-normal (Eugene et al., 2002) and Kumaraswamy normal (Cordeiro and de Castro, 2011). These are special cases of T-normal family of distributions where the link function $W(x) = x.$

Another generalization of the normal distribution, the gamma-normal distribution, was investigated by Alzaatreh et al. (2014a). It is a member of the *T*-normal family, where $W(x) = -\log(1-x)$. The distribution can be right skewed, left skewed, or symmetric. According to Alzaatreh et al. (2014a), there are special cases in which gamma-normal distribution can provide a more accurate fit to the data compared to normal distribution. It was shown that if the data is skewed, one should fit a gamma-normal distribution instead of a normal distribution.

2. The logistic-normal distribution

If *Y* follows the standard logistic distribution and T follows the logistic distribution with PDF $f_T(x) = \lambda e^{-\lambda x}(1+e^{-\lambda x})^{-2},\ \lambda > 0,$ then equation (1.4) reduces to

$$F_X(x) = \frac{G^{\lambda}(x)}{G^{\lambda}(x) + (1-G(x))^{\lambda}},\ \ \lambda > 0;\ x \in Supp(G). \tag{2.1}$$

where $G(x)$ is CDF of any distribution. It is interesting to see that the family in (2.1) preserves the symmetry property. Now, if we use the normal distribution as the generator in (2.1), we get

$$F_X(x) = \frac{\Phi^{\lambda}(x)}{\Phi^{\lambda}(x) + (1-\Phi(x))^{\lambda}}, \qquad x \in \mathbb{R}, \tag{2.2}$$

where $\lambda > 0,\ \sigma > 0$ and $-\infty < \mu < \infty.$

When $\lambda = 1,$ the logistic-normal (LN for short) in (2.2) reduces to the normal distribution. Thus LN distribution is a generalization of the normal distribution.

The corresponding PDF to (2.2) is

$$f_X(x) = \frac{\lambda \phi(x)\Phi^{\lambda-1}(x)(1-\Phi(x))^{\lambda-1}}{[\Phi^{\lambda}(x) + (1-\Phi(x))^{\lambda}]^2},\ x \in \mathbb{R}. \tag{2.3}$$

In Figure 1, various graphs of $f(x)$ when $\mu = 0$, $\sigma = 1$ and for various values of λ are provided. Figure 1 indicates that the logistic-normal PDF can be unimodal or bimodal. It appears that the bimodality occurs when λ is less than 0.5.

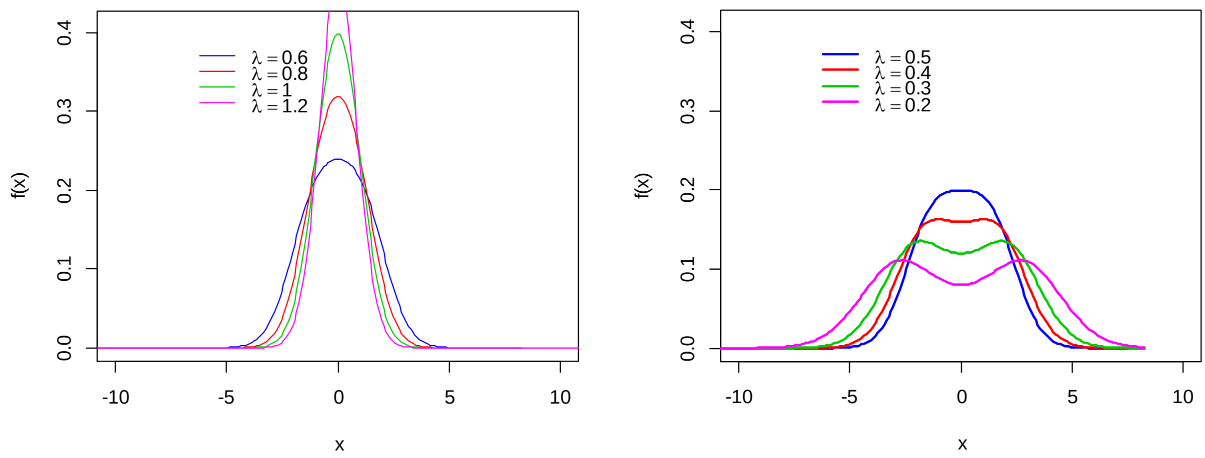

Figure 1. The LN PDF for various values of λ.

3. Some properties of LN distribution

Remark 1.

If a random variable T follows the logistic distribution with PDF $f_T(x) = \lambda e^{-\lambda x}(1+e^{-\lambda x})^{-2},\ \lambda > 0$, then the random variable $X = \Phi^{-1}(e^T/(1+e^T))$ follows the LN distribution with parameters λ, μ and σ.

Remark 1 can be used to simulate random sample from the LN distribution by first simulating random sample, $t_i,\ i = 1, \cdots, n$, from logistic(λ) distribution and then computing $x_i = \Phi^{-1}(e^{t_i}/(1+e^{t_i}))$ which follows the LN distribution.

Remark 2.

i. The $LN(\lambda, \mu, \sigma)$ is symmetric about μ.

ii. The mean and median of LN distribution are equal to μ which comes from normal distribution.

Proof. The symmetricity of normal distribution implies $\phi(x-\mu) = \phi(\mu - x)$ and $\Phi(x-\mu) = 1 - \Phi(\mu - x)$. Hence,

$$f_X(x-\mu) = \frac{\lambda\phi(x-\mu)\Phi^{\lambda-1}(x-\mu)(1-\Phi(x-\mu))^{\lambda-1}}{[\Phi^{\lambda}(x-\mu)+(1-\Phi(x-\mu))^{\lambda}]^2}$$

$$= \frac{\lambda\phi(\mu-x)(1-\Phi(\mu-x))^{\lambda-1}\Phi^{\lambda-1}(\mu-x)}{[(1-\Phi(\mu-x))^{\lambda}+\Phi^{\lambda}(\mu-x)]^2} = f_X(\mu-x). \quad \square$$

Lemma 1.

The mode of the LN distribution is the solution of the equation

$$x = \mu + \lambda\sigma^2 \frac{h(x)}{\Phi(x)}\left(\frac{\Phi^{\lambda}(x)-(1-\Phi(x))^{\lambda}}{\Phi^{\lambda}(x)+(1-\Phi(x))^{\lambda}} + 2\Phi(x) - 1\right), \tag{3.1}$$

where $h(x) = \phi(x)/[1-\Phi(x)]$ is the hazard function of the normal distribution.

Proof. Using the fact that $\phi'(x) = -\sigma^{-2}(x-\mu)\phi(x)$ and setting the derivative of $\log f(x)$ in (2.3) to 0, one can get the result in (3.1). $\square$

It is clear that μ satisfies the equation (3.1). Also, it appears that for $\lambda > 0.5$ the distribution is always unimodal. A simulation study using Mathematica was conducted and it supported this claim. Therefore, $x = \mu$ is the unique mode for $\lambda > 0.5$. Since LN distribution remains symmetric about μ for all λ, for bimodal case if $x = a < \mu$ is a mode, then the other mode is $x = 2\mu - a.$

Remark 3.

If $Q(p),\ 0 < p < 1$ denotes the quantile function for the LN distribution, then

$$Q(p) = \Phi^{-1}\left[(1+(p^{-1}-1)^{1/\lambda})^{-1}\right]. \tag{3.2}$$

The quantile function can also be used to simulate a random sample from LN distribution, by first simulating a random sample $u_i, i = 1,\ldots,n,$ from uniform [0, 1] distribution and then computing $Q(u_i)$.

3.1. The tail behavior of LN distribution

Lemma 2. As $x \to \infty$,

$$f_X(x) \sim \frac{e^{-\lambda x^2/2}}{x^{\lambda-1}}, \ \lambda > 0. \tag{3.3}$$

Proof. It is known that as $x \to \infty$, $\phi(x) \sim e^{-x^2/2}$, $1-\Phi(x) \sim \phi(x)/x$ (Patel and Read, 1982). Therefore as $x \to \infty$,

$$f_X(x) \sim \frac{\lambda\phi(x)\Phi^{\lambda-1}(x)(1-\Phi(x))^{\lambda-1}}{[\Phi^{\lambda}(x)+(1-\Phi(x))^{\lambda}]^2} \sim \phi(x)(\phi(x)/x)^{\lambda-1}$$

$$\sim e^{-x^2/2}(e^{-x^2/2}/x)^{\lambda-1} \sim \frac{e^{-\lambda x^2/2}}{x^{\lambda-1}}. \quad \square$$

Similarly, as $x \to -\infty$, $f_X(x) \sim e^{-\lambda x^2/2}/|x|^{\lambda-1}$. This implies that as $x \to \pm\infty$, the tails of the LN distribution behave in similar way as the right tail of the function $e^{-\lambda x^2/2}/x^{\lambda-1}$.

Note that when $0<\lambda<1$, $f_X(x)$ converges to 0 slowly, while for $\lambda>1$, $f_X(x)$ approaches 0 faster, meaning that the tail weight increases for higher λ .

The graphical representation of the connection between tail weight and λ can be made using the measure of Kurtosis defined by Moore (1988) as

$$\gamma_M = \frac{Q(7/8)-Q(5/8)+Q(3/8)+Q(1/8)}{Q(6/8)-Q(2/8)}. \tag{3.4}$$

Figure 2 shows the plot of the Moore's kurtosis versus λ. It indicates that as λ increases the Moore's kurtosis increases. For $0 < \lambda < 1$ there is a sharp change in the kurtosis, while for $\lambda > 1$ the change becomes gradual. Figure 3 provides a clear comparison between the tails of LN and normal distributions. Figure 3 indicates that for $\lambda < 1$, the tail of LN distribution is lighter than that of the normal distribution, while for $\lambda > 1$ the tail of LN distribution is heavier than that of the normal distribution. Also, for $\lambda > 1$ the LN distribution is leptokurtic with more cone-shaped higher peak. And for $\lambda < 1$, the LN distribution is platykurtic with more flat-shaped lower peak (see Figures 1 and 3).

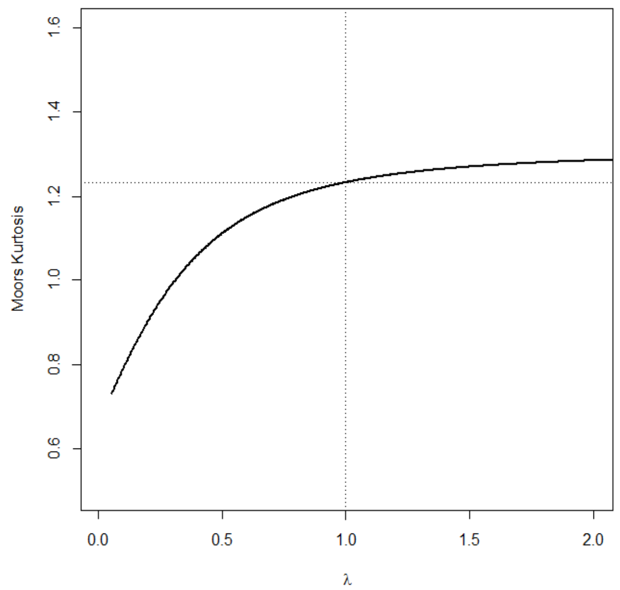

Figure 2. Moore's measure of kurtosis vs λ .

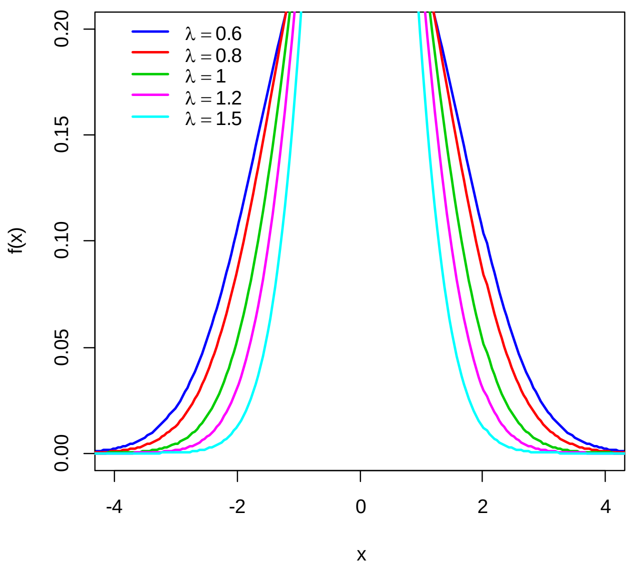

Figure 3. The tails of LN PDF for various λ .

3.2. Moments of LN distribution

Using Remark 1, the moments for the LN distribution in (2.3) can be written as

$E(X^r) = E\left(\Phi^{-1}(e^T / (1+e^T))\right)^r$ where T follows the logistic distribution with parameter λ .

Therefore,

$$E(X^r)=\lambda\int_{-\infty}^{\infty}\left(\Phi^{-1}(e^t/(1+e^t))\right)^r e^{\lambda t}(1+e^{\lambda t})^{-2}\,dt,$$

Since $\Phi^{-1}=\sqrt{2}\,\sigma\,\mathrm{erf}^{-1}(2F-1)+\mu$, where $\mathrm{erf(x)}=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$, it is easy to see that

$\Phi^{-1}(e^T/(1+e^T)=\sqrt{2}\,\sigma\mathrm{erf}^{-1}\{(e^T-1)/(e^T+1)\}+\mu=\sqrt{2}\,\sigma\mathrm{erf}^{-1}(1-2(1+e^T)^{-1})+\mu$. Hence,

$$E(X^r)=E\left(\sqrt{2}\,\sigma\mathrm{erf}^{-1}(1-2(1+e^T)^{-1})+\mu\right)^r=\sum_{j=0}^{r}\binom{r}{j}2^{j/2}\sigma^j\,\mu^{r-j}\xi_j,$$

where $\xi_j=\int_{-\infty}^{\infty}[\mathrm{erf}^{-1}(1-2(1+e^T)^{-1})]^j e^{\lambda t}(1+e^{\lambda t})^{-2}dt$. As far as we know, no closed form value for ξ_j exist. However, ξ_j can be evaluated using numerical integration from available software such as Mathematica.

Lemma 3.

If $X\sim LN(\lambda,0,1)$, then $E(X^r)=0$ for odd positive integers *r*.

Proof. From Remark 2 (i), it follows that PDF $f(x)$ of $LN(\lambda,0,1)$ is symmetric around 0. Hence, for all odd positive integers r, the function $x^r f(x)$ is an odd function and, hence, $E(x^r)=\int_{-\infty}^{\infty}x^r f(x)dx=0$. □

Lemma 4.

If $X\sim LN(\lambda,\mu,\sigma)$, then

$$E(X^r)=\sum_{\text{even }k}^{r}\binom{r}{k}\sigma^k\mu^{r-k}E(Z^k),\;\; Z\sim LN(\lambda,0,1)$$

Proof. Since $Z=(X-\mu)/\sigma$, it is true that $X=\sigma Z+\mu$. Rising X to the power *r* and taking its expectation gives

$$E(X^r)\;=E(\sigma Z+\mu)^r=\;E\sum_{k=0}^{r}\binom{r}{k}\sigma^k\mu^{r-k}Z^k\;=\;\sum_{k=0}^{r}\binom{r}{k}\sigma^k\mu^{r-k}\,E(Z^k)=\sum_{\text{even }k}^{r}\binom{r}{k}\sigma^k\mu^{r-k}E(Z^k).$$

Only even cases are considered due to Lemma 3. □

4. Parameter estimation and simulation for LN distribution

Let $x_1, x_2, \cdots, x_n$ be a random sample of size *n* taken from LN distribution. Then the log-likelihood function is given by

$$\ell(\lambda, \mu, \sigma) = n \log(\lambda) + \sum_{i=1}^{n} \log \phi(x_i) + (\lambda - 1) \sum_{i=1}^{n} \log \Phi(x_i)$$

$$+(\lambda-1)\sum_{i=1}^{n}\log(1-\Phi(x_i)) - 2\sum_{i=1}^{n}\log\{\Phi^{\lambda}(x_i)+(1-\Phi(x_i))^{\lambda}\}. \tag{4.1}$$

Using the facts that $\dfrac{\partial\Phi(x)}{\partial\mu} = -\phi(x)$, $\dfrac{\partial\Phi(x)}{\partial\sigma} = \dfrac{-(x-\mu)\phi(x)}{\sigma}$, $\dfrac{\partial\phi(x)}{\partial\mu} = \dfrac{-(x-\mu)}{\sigma^2}\phi(x)$, and $\dfrac{\partial\phi(x)}{\partial\sigma} = \dfrac{(x-\mu)^2-\sigma^2}{\sigma^3}\phi(x)$, the derivatives of (4.1) with respect to λ, μ and σ respectively, are given by

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\log\Phi(x_i) + \sum_{i=1}^{n}\log(1-\Phi(x_i))$$

$$-2\sum_{i=1}^{n}\frac{g^{\lambda}(x_i)\log\Phi(x_i)+\log(1-\Phi(x_i))}{g^{\lambda}(x_i)+1},\ g(x_i)=\frac{\Phi(x_i)}{1-\Phi(x_i)} \tag{4.2}$$

$$\frac{\partial\ell}{\partial\mu} = -\frac{\bar{x}-\mu}{\sigma^2} - (\lambda-1)\sum_{i=1}^{n}\frac{\phi(x_i)}{\Phi(x_i)} + (\lambda-1)\sum_{i=1}^{n}\frac{\phi(x_i)}{1-\Phi(x_i)}$$

$$+ 2\lambda\sum_{i=1}^{n}\phi(x_i)\frac{\Phi^{\lambda-1}(x_i) - (1-\Phi(x_i))^{\lambda-1}}{\Phi^{\lambda}(x_i) + (1-\Phi(x_i))^{\lambda}} \tag{4.3}$$

$$\frac{\partial\ell}{\partial\sigma} = \frac{1}{\sigma^3}\sum_{i=1}^{n}((x_i-\mu)^2-\sigma^2) - \frac{\lambda-1}{\sigma}\sum_{i=1}^{n}\frac{(x_i-\mu)\phi(x_i)}{\Phi(x_i)} + \frac{\lambda-1}{\sigma}\sum_{i=1}^{n}\frac{(x_i-\mu)\phi(x_i)}{1-\Phi(x_i)}$$

$$+ \frac{2\lambda}{\sigma}\sum_{i=1}^{n}(x_i-\mu)\phi(x_i)\frac{\Phi^{\lambda-1}(x_i) - (1-\Phi(x_i))^{\lambda-1}}{\Phi^{\lambda}(x_i) + (1-\Phi(x_i))^{\lambda}}. \tag{4.4}$$

The MLE, $\hat{\lambda}$, $\hat{\mu}$ and $\hat{\sigma}$, of the parameters λ, μ and σ can be obtained by setting equations (4.2), (4.3), and (4.4) to zero and solving them numerically.

The initial value for μ is taken to be the moment estimates $\bar{x}$. The initial value for σ is taken to be the sample standard deviation, *s*. To obtain the initial value for the parameter λ we use Remark 1 as follows; assume the random sample $t_i = \log\left(\dfrac{\Phi(x_i, \bar{x}, s^2)}{1-\Phi(x_i, \bar{x}, s^2))}\right)$, $i = 1, \ldots, n$ is taken from the logistic distribution with parameter λ. By equating the population variance $\pi^2 / (3\lambda^2)$ of logistic distribution with the sample variance s_T^2 of the random sample t_i and solving it for λ, we obtain $\lambda_0 = \sqrt{1/3}\,\pi / s_T$.

We used the trust-region optimization routine in SAS (PROC IML and CALL NLPTR) in order to maximize the likelihood function in (4.1). The trust-region optimization routine is a powerful technique that can optimize complicated function. It outputs the iteration details including parameter estimates, their standard errors, and the value of a gradient at which iteration stops. In order to evaluate the performance of maximum likelihood estimators, we conducted a small simulation study with sample sizes $n = 30,\ 50, 70$ and three different parameter combinations. The study involved computing and analyzing the relative bias of the estimators [(Estimate-Actual)/Actual] and standard deviation. The results of the study are reported in Table 1. Based on the small simulation study in Table 1, it is evident that the MLE for the parameter μ is overestimated. Also, when $\lambda < 1$, the MLEs for λ and σ are overestimated while when $\lambda > 1$ they are underestimated. It can be seen that for small sample size (*n*=30) and $\lambda < 1$, MLE does not perform well. However, the results for $\lambda > 1$ show that the MLE method performs quite well in estimating the model parameters.

Table 1: Relative bias and standard deviation for the MLE estimates

n	λ	μ	σ	Relative bias			Standard deviation		
30	0.5	2	1	1.2698	0.0276	0.8064	1.5928	0.2797	1.8965
50				0.6606	0.0256	0.4205	0.4651	0.2749	0.5992
70				0.3290	0.0140	0.1422	0.4005	0.2013	0.4485
30	1.5	2	1	-0.1422	0.0101	-0.1210	0.7309	0.1224	0.4959
50				-0.0692	0.0339	-0.1074	0.5927	0.1005	0.3494
70				-0.0671	0.0087	-0.0898	0.3460	0.0773	0.2153
30	2	3	1	-0.3089	0.0113	-0.3005	0.8190	0.0978	0.3418
50				-0.3247	0.0083	-0.2915	0.8695	0.0827	0.2212
70				-0.3162	0.0076	-0.2990	0.8007	0.0575	0.2379

5. Application

In this section the LN distribution is fitted to two bimodal data sets. The results of the maximum likelihood estimates, the log-likelihood value, the AIC (Akaike Information Criterion), the Kolmogorov-Smirnov (K-S) test statistic, and the *p*-value for the K-S statistic for the fitted distributions are reported in Tables 2 and 3. Figures 4-6 display the empirical and the fitted cumulative distribution as well as the probability density functions for the fitted distributions.

The first data is obtained from National Data Buoy Center (NDBC). It represents the number of buoys situated in the North East Pacific: Buoy 46005 (46 N, 131 W).The time period January 1, 1983, to December 31, 2003, was investigated. For each calendar year, the maximum observation was extracted; hence, for each buoy 21 yearly maxima were found. The data is available from Persson et al. (2010). Histogram in Figure 4 shows that the data is bimodal. Hence, the data is fitted to the LN and mixture normal distributions. The K-S values in Table 2 indicate that the LN distribution provides an adequate fit and performs much better than the mixture normal distribution. In fact, the CDF in Figure 4 shows that the mixture normal distribution does not provide an adequate fit. The fact that the LN distribution has only three parameters adds an extra advantage to the distribution over the mixture normal distribution.

Table 2: Parameter estimates for the buoys data

Distribution	LN	Mixture Normal
Parameter Estimates	$\hat{\lambda}$=0.2734 (0.3304) $\hat{\mu}$ =10.5700 (0.3145) $\hat{\sigma}$ =0.6507 (0.5051)	$\hat{\lambda}$=0.5515 (0.2920) $\hat{\mu}_1$ =8.6051 (0.6836) $\hat{\mu}_2$ =11.4634 (0.6930) $\hat{\sigma}_1$ =1.4994 (0.7293) $\hat{\sigma}_2$ =1.0750 (0.3770)
Log-likelihood	80.7	109.5
AIC	86.7	119.5
K-S	0.2273	0.6901

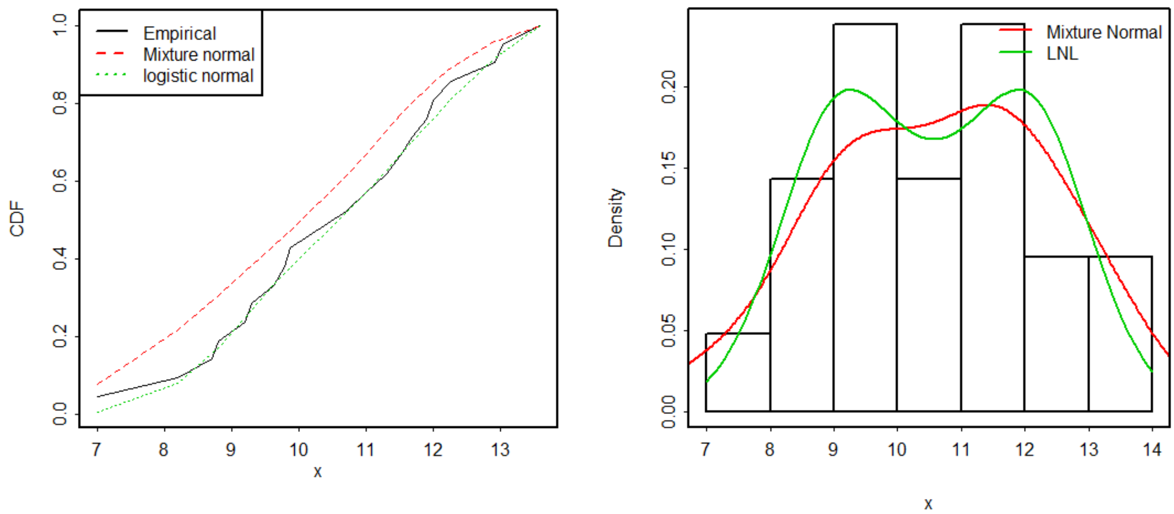

Figure 4: CDF and PDF for the fitted distributions for the buoys data

The second application is from Emlet et al. (1987). It represents the asteroid and echinoid egg size. The data consists of 88 asteroid species divided into three types; 35 planktotrophic larvae, 36 lecithotrophic larvae, and 17 brooding larvae. Since the logarithm of the egg diameters of the asteroids data has a bimodal shape, Famoye et al. (2004) applied the beta-normal distribution to the logarithm of the data set. The results in Table 3 show that both the LN and beta-normal distributions provide an adequate fit to the data. However, the K-S values indicate that the LN distribution provides a better fit. This is also evident from Figures 5 and 6. The fact that the LN distribution involves less number of parameters also adds an advantage over the beta-normal distribution.

Table 3: Parameter estimates for the asteroids data

Distribution	LN	Beta-normal
Parameter Estimates	$\hat{\lambda}$ = 0.1498 (0.0185) $\hat{\mu}$ = 6.0348 (0.0685) $\hat{\sigma}$ = 0.2604 (0.010)	$\hat{\alpha}$ = 0.0129 $\hat{\beta}$ = 0.0070 $\hat{\mu}$ = 5.7466 $\hat{\sigma}$ = 0.0675
Log-likelihood	-111.4287	-109.4800
AIC	228.4974	226.9600
K-S	0.0988	0.1233

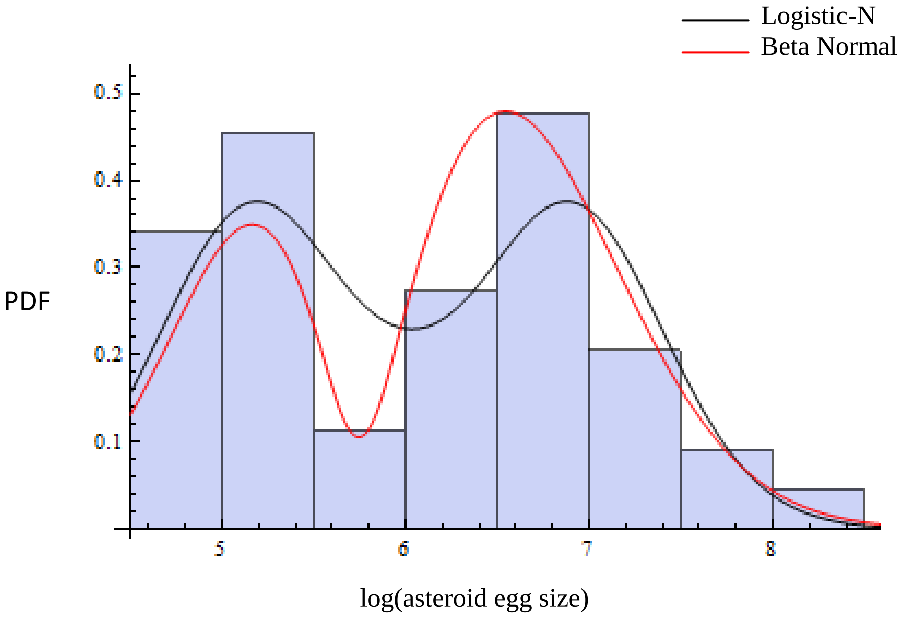

Figure 5: PDF for the fitted distributions for the asteroids data

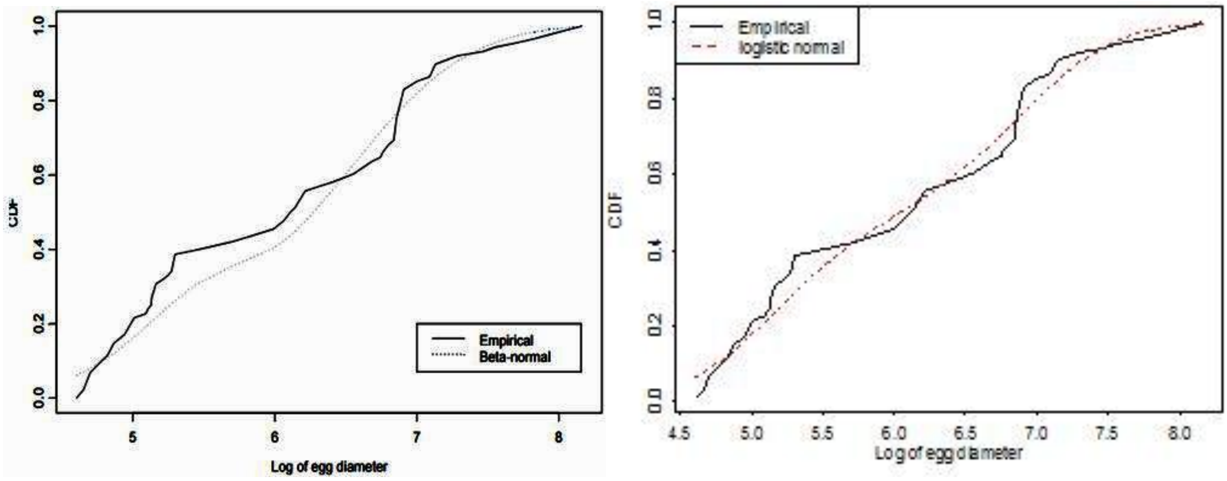

Figure 6: CDF for the fitted distributions for the asteroids data

Conclusion

In this project the generalization of the normal distribution, the logistic normal (LN), is introduced. We study the LN distribution in some detail. Some properties of the LN distribution are investigated, including moments, modes and tail behavior. The LN distribution is a symmetric distribution, which can be unimodal or bimodal. A small simulation study showed that maximum likelihood estimators perform well. It is noteworthy to mention that we fitted the LN distribution to several unimodal data sets with approximately symmetric characteristic. The results showed that the LN provides excellent fit to most of these data. The results are available from the author upon request. In this project, we showed the applicability of fitting the LN distribution to two bimodal datasets. The LN distribution provided a good fit for each data.

For skewed type of data one can generate a skewed LN distribution by exponentiating the CDF of the LN distribution as

$$F_X(x) = \left(\frac{\Phi^{\lambda}(x)}{\Phi^{\lambda}(x) + (1-\Phi(x))^{\lambda}} \right)^{\alpha}, \ \alpha, \lambda > 0; x \in \mathbb{R}.$$

To analyze the skewness and kurtosis regions of the distribution, the Galton's skewness *S* (1883) and Moore's kurtosis *K* (1988) measures were plotted against the parameters α and λ. Figure 7 shows that the distribution is right skewed for $\alpha, \lambda < 1$ and left skewed for $\alpha > 1, \lambda < 1$ and $\alpha < 1, \lambda > 1$. The plot of kurtosis in Figure 7 demonstrates the flexibility of the proposed distribution. For $\lambda < 1$, the tails of the skewed LN can be heavier or lighter than that tail of the normal distribution, while for $\lambda > 1$ the kurtosis is always higher than that of the normal distribution. The skewed LN distribution was also fitted to different skewed unimodal and bimodal real data sets. For most of the cases, the distribution provided a very good fit.

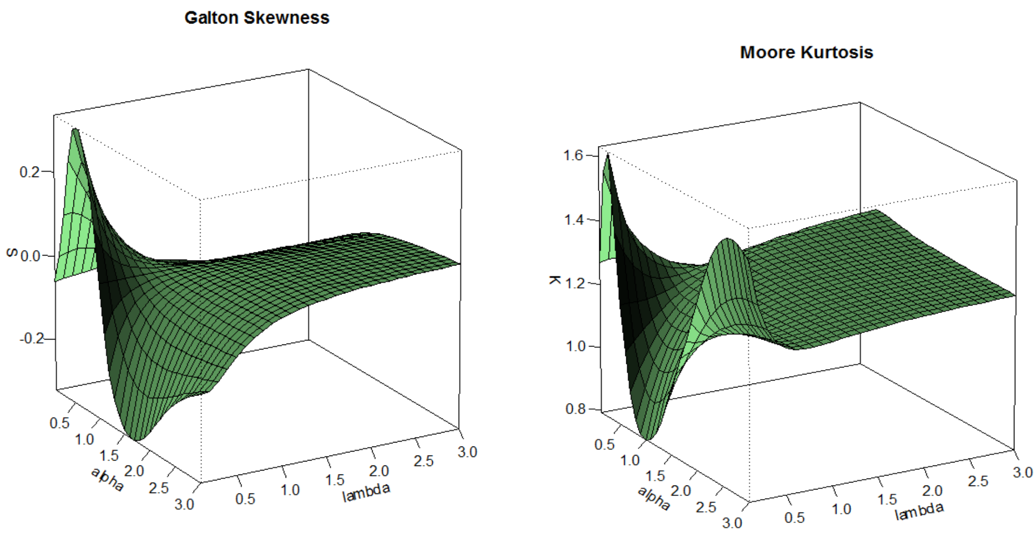

Figure 7. 3D plots of Galton's skewness and Moore's kurtosis for various values of λ and α.

A detailed investigation of the skewed LN distribution and general properties of the LN distribution, such as moments and bimodality-unimodality regions can be studied in future research. SAS and R codes for simulation study and goodness of fit are provided in the appendix.

Appendix

SAS code for the simulation study

```
proc iml;
n=50; p={2 3 1}; ld=p[1]; mu=p[2]; s=p[3]; nsim=100; m=j(nsim,3);
pi=constant("pi");
do k=1 to nsim;

            u=j(n,1);
```

```
            call randseed(12345);
again:      call randgen(u, "Uniform",0,1);
            q=1/(1+(1/u-1)##(1/ld));
            y=quantile("Normal",q, mu,s);

            start f_ln(x) global(y);
                  p=nrow(y);
                  sum1=0.; sum2=0.;  sum3=0.;  sum4=0;
                  thet=t(y);
                  /* x[1]=ld; x[2]=mu; x[3]=sigma;*/
                  pnorm=pdf('normal', thet, x[2], x[3]);
                  cnorm=cdf('normal', thet, x[2], x[3]);

                  do i=1 to p;
                        surv[i]=1-cnorm[i];
                        if (pnorm[i]>0 & cnorm[i]>0 & surv[i]^=0) then
                        do;
                         sum1=sum1+log(pnorm[i]);
                         sum2=sum2+log(cnorm[i]);
                         sum3=sum3+log(surv[i]);
                         sum4=sum4+log((cnorm[i])##x[1]+(surv[i])##x[1]);
                        end;
                  end;
                  f=p*log(x[1])+sum1+(x[1]-1)*sum2+(x[1]-1)*sum3-2*sum4;
                  return(f);
            finish f_ln;

            thet=t(y);
            mu0=mean(y);
            s0=std(y);

            /*building initial for lambda*/
            cnorm_t=cdf('normal', thet, mu0, s0);
            t_i=log(cnorm_t/(1-cnorm_t));
            st=std(t(t_i));
            ld0=((1/3)##(1/2))*(pi/st);

            x={1 1 1};
            x[1]=ld0; x[2]=mu0; x[3]=s0;
            optn={1 0};
            con={0.001 . 0.001,
                   . . .};

            call nlptr(rc, xres, "f_ln", x, optn, con);
            if (rc<0) then goto again;
            xopt=t(xres);
            fopt=f_ln(xopt);

            call nlpfdd (f,g,hess,"f_ln",xopt);
            gld=g[1];
            gmu=g[2];
            gs=g[3];
            *print gld gmu gs;
            if (abs(gld)>0.0001|abs(gmu)>0.0001|abs(gs)>0.0001) then
goto    again;
```

```
/*bias calculations*/
          m[k,1]=ld-xopt[1];
          m[k,2]=mu-xopt[2];
          m[k,3]=s-xopt[3];

end;

/* average and std of bias for ld, mu, s*/
          av=mean(m);
          st=std(m);

print n nsim ld mu s av st;

quit;
```

SAS code for estimating the parameters for the first data

```
data one;
input y @@;
datalines;

10.70 10.70 7.00 11.30 13.60 11.70 8.20
12.00 9.30 8.80 11.00 11.90 9.20 8.71
9.63 9.87 13.04 9.79 12.26 11.52 12.92;

proc means data=one; run;

/* L-N */
proc nlmixed data=one tech=trureg;
title 'logistic normal';
a=1;

bounds th g > 0;
parms th 1 m 10.6257143 g 1.7430736;
pd=pdf('normal', y, m, g);
cd=cdf('normal', y, m, g);
cc=log(th)+log(a);
f1=log(cd); f2=log(1-cd); f3=log(pd);
f4=log(((cd)**th)+((1-cd)**th));
ll=cc+f3+(a*th-1)*f1+(th-1)*f2-(a+1)*f4;

model y ~ general(ll);

run;

/* Mixture normal */
proc nlmixed data=one tech=trureg;
title 'mixture normal';
```

```
bounds th1 g1 g2 > 0;
parms th1 0.3 m1 9.5  m2 11.5 g1 1.7430736 g2 1.7430736;
pd1=pdf('normal', y, m1, g1);
pd2=pdf('normal', y, m2, g2);
cd1=cdf('normal', y, m1, g1);
cd2=cdf('normal', y, m2, g2);

ll=log(th1*pd1*cd1+(1-th1)*pd2*cd2);
model y ~ general(ll);

run;
```

R code for the calculations of KS and p-values

```
y=c(10.70,10.70,7.00,11.30,13.60,11.70,8.20,12.00,9.30,8.80,
11.00,11.90,9.20,8.71,9.63,9.87,13.04,9.79,12.26,11.52,12.92);
yd = sort(y)

# Distributions

# The following are the parameter estimate for mixture normal density
ld1=0.5515
ld2=1-ld1

m1=8.6051
m2= 11.4634
sg1= 1.4994
sg2=1.0750

#The cdf of mixture normal distribution
bbs.cd=ld1*pnorm(yd,m1,sg1)+ld2*pnorm(yd,m2,sg2)

# The following are the parameter estimate for logistic normal density
ld=0.2734
mu=10.5700
g=0.6507

aa=pnorm(yd,mu,g)

#The cdf of logistic normal distribution
lnl.cd=aa^ld/(aa^ld+(1-aa)^ld)

y.sort = yd
n=length(yd)
ecdf = rep(n, 0.0)
bbs.cd[n] = 1.0
lnl.cd[n]=1.0
```

```
for (j in 1:n) {ecdf[j] = sum(y.sort <= y.sort[j])/n}
bbs.di= abs(bbs.cd - ecdf)
lnl.di= abs(lnl.cd - ecdf)
# K-S mixture normal
bbs.ks= sqrt(n)*max(bbs.di)
# K-S logistic-normal
lnl.ks= sqrt(n)*max(lnl.di)
cbind(n, max(bbs.di),max(lnl.di))

nk=40
pv.bbs = rep(nk, 0.0)
pv.lnl = rep(nk, 0.0)
for (i in 1:nk)
{
pv.bbs[i] = ((-1)^(i-1))*exp(-2*(bbs.ks*i)^2);
pv.lnl[i] = ((-1)^(i-1))*exp(-2*(lnl.ks*i)^2);
}
bbs.pv = 2.0*sum(pv.bbs)

lnl.pv = 2.0*sum(pv.lnl)
cbind(n, bbs.pv,lnl.pv)

new0 = cbind(ecdf,bbs.cd,lnl.cd)
par(lwd=2.5, font=1, tck=-0.01, mgp=c(3,0.5,0))
matplot(y.sort, new0, type='l', col=c(1:3), lty=1:3, xlab=" ", ylab="
")
legend("topleft", c("Empirical", "Mixture normal","logistic normal"),
col=c(1,2,3), lty=1:3)
mtext(text="x", side=1, line=1.5)
mtext(text="CDF", side=2, line=2.5)
box()

lnl.ks
bbs.ks
```


Acknowledgments

We express our gratitude to our supervisor Dr. Ayman Alzaatreh for his valuable guidance and strong support. His assistance was a major contribution to the quality of the project. Special thanks to Dr. Natalya Pya for her comments and suggestions. Her comments were very helpful and greatly improved the project.

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